Adavnced Engineering Mathematics
Lecture 57

1. **Singular point:** If $f'(z)$ exists at every point of a domain D except for a finite number of points. Then, $f(z)$ is said to be an analytic function on the domain D and those exceptional/non-analytic points are called singular points or singularities of f.

   Ex. $f(z) = \frac{1}{z-2}$, 2 is a singular point.

   In other words, a point $z_0 \in D$ is said to be a singular point if $f'(z_0)$ does not exists.

Theorem 1 (Cauchy-Riemann equation/CR-equation)**.** *A complex function $f : \mathbb{C} \to \mathbb{C}$ is given by*

$$f(z) = f(x, y) = u(x, y) + iv(x, y),$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$. The necessary condition that f is analytic on a domain D is that both u and v satisfy

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \textit{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \textit{on } D.$$

If the partial derivatives of u and v are also continuous on D, then the CR-equation become sufficient condition for the f to be analytic on D.

Example. Verify whether the function $f : \mathbb{C} \to \mathbb{C}$ satisfies CR-equation or not at $z = 0$.

$$f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0. \end{cases}$$

Solution: For $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$, where $u(x, y) = \frac{x^3-y^3}{x^2+y^2}$, and $v(x, y) = \frac{x^3+y^3}{x^2+y^2}$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

for $z \neq 0$. To verify the CR-equation, we need to only show that the relation also hold for $z = 0$. Now,

$$\begin{aligned}
\left.\frac{\partial u}{\partial x}\right|_{(0,0)} &= \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x\to 0} \frac{x^3}{x^3} = 1. \\
\left.\frac{\partial v}{\partial y}\right|_{(0,0)} &= \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y - 0} = \lim_{x\to 0} \frac{y^3}{y^3} = 1. \\
\left.\frac{\partial u}{\partial y}\right|_{(0,0)} &= \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y - 0} = \lim_{x\to 0} \frac{-y^3}{y^3} = -1. \\
\left.\frac{\partial v}{\partial x}\right|_{(0,0)} &= \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x - 0} = \lim_{x\to 0} \frac{x^3}{x^3} = 1.
\end{aligned}$$

Hence, f satisfies CR-equation at $z = 0$. As, the partial derivatives at $z = 0$ are continuous at $z = 0$. The function f is analytic at $z = 0$.

Example. Find the analytic function $f = u + iv$, where $u(x, y) = e^x(x\cos y - y \sin y)$.

Solution: If f is analytic, then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. This implies

$$\frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x \cos y, \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -e^x(x\sin y - y\cos y) - e^x \sin y.$$

To determine v, we write

$$\begin{aligned}
dv =& \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy \\
=& -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \qquad \text{(by CR-equation)} \\
=& e^x(x\sin y - y\cos y + \sin y)\,dx + e^x(x\cos y - y\sin y + \cos y)\,dy \\
=& d(e^x x\sin y + e^x y\cos y) \\
v(x,y) =& e^x(x\sin y + y\cos y) + C
\end{aligned}$$

The required analytic function:

$$f(z) = u + iv = e^x(x\cos y - y\sin y) + i\big[e^x(x\sin y + y\cos y) + C\big].$$

CR-equation in polar form: $x = r\cos\theta$, $y = r\sin\theta$. Then the CR-equation reduces to

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \text{and} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Example. Verify whether $f(z) = z^m$, $m \in \mathbb{N}$ satisfies CR-equations.
Solution: $f(z) = z^m = r^m e^{im\theta} = \underbrace{r^m\cos m\theta}_{u} + i\,\underbrace{r^m\sin m\theta}_{v}$. Now,

$$\begin{aligned}
\frac{\partial u}{\partial r} =& mr^{m-1}\cos m\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}. \\
\frac{\partial u}{\partial \theta} =& -mr^m\sin m\theta = -r\frac{\partial v}{\partial r}.
\end{aligned}$$

Therefore, $f(z) = z^m$ satisfies CR-equation.