Adavnced Engineering Mathematics
Lecture 58

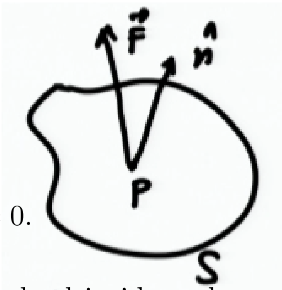
Complex integration: $\int_C f(z)\,dz$, where C is called the contour.

Contour: It means that C is composed of a continuous chain of finite number of regular arc.

Example. Let $f(z) = z^n$, $n \in \mathbb{N}$ and C is a closed curve $|z| = r$, $r > 0$. Evaluate $\int_C f(z)\,dz$.

Solution: The contour C is a circle of radius r centered at $(0, 0)$, i.e., $x^2 + y^2 = r^2$.

$$\begin{aligned}\int_C f(z)\,dz &= \int_C z^n\,dz \\ &= \int_0^{2\pi} r^n e^{in\theta}\, ire^{i\theta}\,d\theta \\ &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)\theta}\,d\theta = 0.\end{aligned}$$



Cauchy's Theorem: If a function f is analytic and single valued inside and on a simple closed curve or, contour C, then

$$\int_C f(z)\,dz = 0.$$

Example. Let $f(z) = \frac{1}{(z-1)^2}$. Then find the value of $\int_C f(z)\,dz$, where C is the curve/contour $|z - 1| = 1$.

Solution: Given contour is $|z - 5| = 1$ implies the circle of radius 1 centered at $(5, 0)$. Clearly, $f(z)$ is not analytic at $z = 1$ which does not belong to region R bounded by the curve C. In other words, $f(z)$ is always analytic in R. Therefore, by Cauchy's theorem,

$$\int_C f(z)\,dz = 0.$$

Cauchy's integral formula: If f is a analytic function within and on a closed curve C and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - a}\,dz.$$

Higher order derivatives: If $f(z)$ is analytic within and on a closed contour C and a is any point within C, then the derivatives of all order are analytic and they are given by

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z - a)^{n+1}}\,dz.$$