Course Name: Essentials of Topology
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Welcome to Lecture 25 on Essentials of Topology.

In this lecture, we will study the concept of Countability Axioms. Specifically, we will study topological spaces where countable sets play a key role. Let us have a look at what we will study under the countability axioms.

We have already seen the Euclidean topology on the set of real numbers. We know that $cl(\mathbb{Q}) = \mathbb{R}$; that is, $\mathbb{Q}$ is a dense set, and this $\mathbb{Q}$ is also a countable set. At the same time, we have studied a basis for this topology that was given by $\{(a, b) : a, b \in \mathbb{Q}\}$, and note that this basis is a countable basis. A question comes, as countability is playing a key role here, and also, in this basis, a countable set is playing the role. Is there any relationship in the topological spaces if they are having countable dense sets, or they are having countable basis? The answer is yes. First, the spaces with countable dense sets are known as separable spaces. The spaces having a countable basis are known as second countable spaces. A question arises when we say that this is the second countable. Is there anything first countable, and how does the countability appear here? The answer is yes, and such spaces are also defined in terms of a basis that is a countable basis, but that basis is a weaker version of this basis. So, what we will study in this lecture, that is the concept of first countable spaces, and we will continue in the next lecture with second countable spaces and separable spaces, and, finally, we will try to explore the relationship between these spaces, if any.

Begin with the concept of a first countable space. A topological space $(X, \mathcal{T})$ is said to satisfy the first countability axiom, or to be first countable if for all $x \in X$, there exists a countable collection $\mathcal{B}_x$ of open sets containing x such that for all open sets G containing x there exists $B \in \mathcal{B}_x$ satisfying $x \in B \subseteq G$. What are we looking here? Actually, if the topological space $(X, \mathcal{T})$ is given to us, for any $x \in X$, we are trying to find some $\mathcal{B}_x$, and this $\mathcal{B}_x$ is a collection of some B containing this x. Note that, this B itself is an open set, and what is our requirement, that is, if we are taking any open set, that is, $G \in \mathcal{T}$, and if $x \in G$, that is, G is an open set containing x, there exists some $B \in \mathcal{B}_x$ such that $x \in B \subseteq G$. Note that this $\mathcal{B}_x$ should be a countable set. We call this countable collection $\mathcal{B}_x$, a countable basis at x.

Let us take some of the examples. The first example is, an uncountable set with the discrete topology is first countable. Let us see the justification. If we are taking a topological space $(X, \mathcal{T})$, note that here, X is an uncountable set, and the topology we have considered here is discrete. If we are taking any $x \in X$, let us take the collection $\mathcal{B}_x = \{\{x\}\}$. Note that because the topology is discrete, singleton set $\{x\}$ is a member of the topology. So, singleton set $\{x\}$ is an open set in this topology. This singleton set $\{x\}$ contains x. So, this $\mathcal{B}_x$ is a countable collection of open sets containing x. Now, if we are taking another open set, that is, let us take $G \in \mathcal{T}$, which is containing x, then we always write that $x \in \{x\} \subseteq G$. Thus, $\mathcal{B}_x$ is a countable basis at x. Therefore, uncountable set with discrete topology is first countable.

Moving to the next example, every metrizable space is first countable. Let us see it. Begin with a metric space (X, d). Let us take the corresponding metric topology as $\mathcal{T}_d$. For all $x \in X$, let us take this $\mathcal{B}_x = \{B(x, 1/n) : n \in \mathbb{N}\}$. Note that from the definition itself, we can say that $\mathcal{B}_x$ is a countable set. Also, we know that open balls are members of this metric topology; therefore, this is a countable collection of open sets with respect to this topology, and these sets contain x. Now, if we are taking any open set, that is $G \in \mathcal{T}_d$ which is containing x, then by the definition of metric topology, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq G$. As $\epsilon > 0$, if we are using Archimedean property which says that for every $\epsilon > 0$ there exists a natural number n such that $1/n < \epsilon$. So, what we can do is we can construct an open ball $B(x, 1/n)$ inside the ball $B(x, \epsilon)$, that is a subset of G. Thus, we have shown that $x \in B(x, 1/n) \subseteq G$, that is corresponding to every open set G containing x there exists some $B \in \mathcal{B}_x$ such that $x \in B(x, 1/n) \subseteq G$. Therefore, every metrizable space is first countable.

Moving ahead, let us take the lower limit topology on the set of real numbers. We can show that this $(\mathbb{R}, \mathcal{T}_l)$ is also first countable. Let us take any real number x and construct a set $\mathcal{B}_x = \{[x, x + 1/n) : n \in \mathbb{N}\}$. Note that the semi-open intervals of the form $[x, x + 1/n) \in \mathcal{T}_l$. Therefore, this is an open set, and this open set contains x. Also, from the definition of $\mathcal{B}_x$, it is clear that $\mathcal{B}_x$ is countable. In order to justify that this $\mathcal{B}_x$ is a countable basis at x, let us take an open set, that is, $G \in \mathcal{T}_l$, where G is containing x. By definition of the lower limit topology, there exist two real numbers, a and b, $a < b$ such that $x \in [a, b) \subseteq G$. From here, we can conclude that $a \leq x < b$, or we can say that this $b - x > 0$. Now, use Archimedean property here. By using the Archimedean property, there exists a natural number n such that this $b - x > 1/n$, or we can say that $b > x + 1/n$. So, we can say that this $x \in [x, x + 1/n) \subseteq [a, b) \subseteq G$, or that $x \in [x, x + 1/n) \subseteq G$. So, corresponding to every open set G containing x, we have shown that there exists a member in this $\mathcal{B}_x = \{[x, x + 1/n) : n \in \mathbb{N}\}$ satisfying $x \in [x, x + 1/n) \subseteq G$. Therefore, $\mathcal{B}_x = \{[x, x + 1/n) : n \in \mathbb{N}\}$ is a countable basis at x. Thus, the lower limit topological space is first countable.

Moving ahead, let us see an example of topological space, which is not first countable, and this is well known to us; that is the co-finite topological space $(X, \mathcal{T})$. Note that this X is an uncountable set. We will prove this by contradiction. If possible, let us assume that this topological space is first countable, and if this is first countable, then it has a countable basis at each $x \in X$. Now, let us take $x \in X$ and a countable basis $\mathcal{B}_x$ here; this is a countable basis at x. Now, if we are taking any $y \in X$, y is not equal to x. Then we can talk about $\{y\}^c$. Note that $x \in \{y\}^c$. Also, because the topology we have considered here is co-finite, $\{y\}^c$ is finite. Therefore, this $\{y\}^c$ is a member of the topology, and $\{y\}^c$ is containing x. Thus, there exists some $B \in \mathcal{B}_x$ such that $x \in B \subseteq \{y\}^c$. From here, we can conclude that for all $y \in X$ such that y is not equal to x, we can find $B \in \mathcal{B}_x$ such that $x \in B \subseteq \{y\}^c$. Note that this B will not contain this y. So, from here, we can conclude that $\cap\{B : B \in \mathcal{B}_x\} = \{x\}$ because no other y can be a member of at least one B. Therefore, $(\cap\{B : B \in \mathcal{B}_x\})^c = \{x\}^c$, or by De Morgan's law, $\cup\{B^c : B \in \mathcal{B}_x\} = \{x\}^c$. If we see the right-hand side of this expression, note that this is a complement of a singleton set, and X is an uncountable set. Therefore, $\{x\}^c$ is uncountable. If we see the left-hand side of this expression, this is $\cup\{B^c : B \in \mathcal{B}_x\}$. Note that this is a countable union of finite sets, and if this is a countable union of finite sets, this union will be countable. Thus, we have shown that the countable set is equal to an uncountable set, which is not possible. Meaning is, we reached to a contradiction. So, our assumption is wrong. What we have assumed is that this co-finite topological space is first countable and therefore uncountable set with co-finite topology is not a first countable space.

Moving ahead, let us see a theorem regarding the subspace of first countable spaces. We can justify that a subspace of a first countable space is also first countable. In order to justify or in order to prove this theorem, let a topological space $(X, \mathcal{T})$ and this space is first countable. Also, let us take a subset of subset Y of X, and we have to justify that the topological space $(Y, \mathcal{T}_Y)$ is also a first countable space. For which, we have to justify that for all $y \in Y$, we have to construct a countable basis at y. In order to achieve this target, begin with $y \in Y$ and note that $Y \subseteq X$, so $y \in X$, and the space $(X, \mathcal{T})$ is a first countable space. So, corresponding to this y, we can get a countable basis $\mathcal{B}_y$ at y with respect to the topology $\mathcal{T}$, that is, the members of this $\mathcal{B}_y$ are $\mathcal{T}$-open sets. Now, let us construct a set $\mathcal{B}_y^*$, this is given by $\mathcal{B}_y^* = \{Y \cap B : B \in \mathcal{B}_y\}$. From here, it is clear that this $\mathcal{B}_y^*$ is countable because $\mathcal{B}_y$ is countable. Also, $Y \cap B$ is a $\mathcal{T}_Y$-open set, and this $Y \cap B$ also contains y. Now, we are taking $H \in \mathcal{T}_Y$, which is containing this y. Note that this H can be written as $Y \cap G$, and what about this G? $G \in \mathcal{T}$, that is, y is a member of this G, and $G \in \mathcal{T}$. So, there is some $B \in \mathcal{B}_y$ because this is a basis at y such that $y \in B \subseteq G$. Therefore, $y \in Y \cap B \subseteq Y \cap G$. But note that $Y \cap G$ is nothing but H, or that $y \in Y \cap B \subseteq H$. So, what we have shown here is that for all open sets with respect to the relative topology on Y containing y, there exists B' in this set, that is, $\mathcal{B}_y^*$, such that $y \in B' \subseteq H$. Therefore, this $\mathcal{B}_y^*$ is a countable basis with respect to topology $\mathcal{T}_Y$ at y. Thus, this $(Y, \mathcal{T}_Y)$ has a countable basis at each of its points, and therefore, the subspace is first countable; that's all about this result.

These are the references.

That's all from this lecture. Thank you.