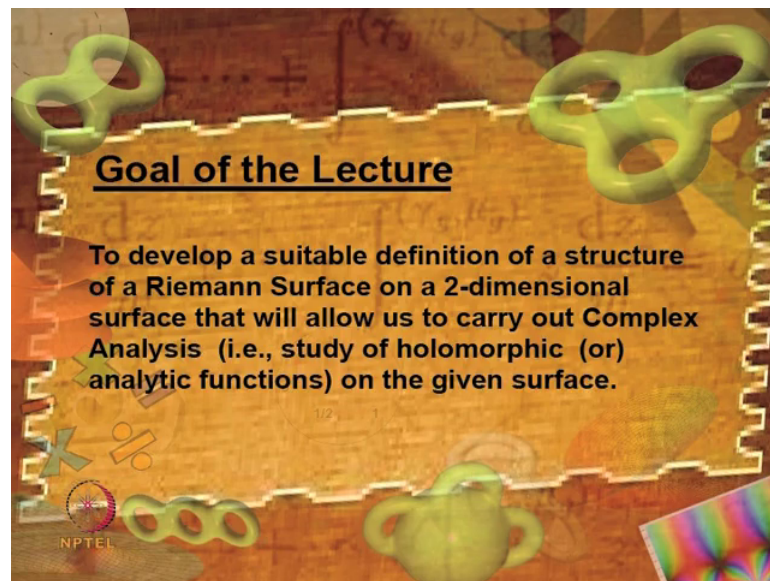**An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimentional Tori and Elliptic Curves**
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Lecture – 01
The Idea of a Riemann Surface
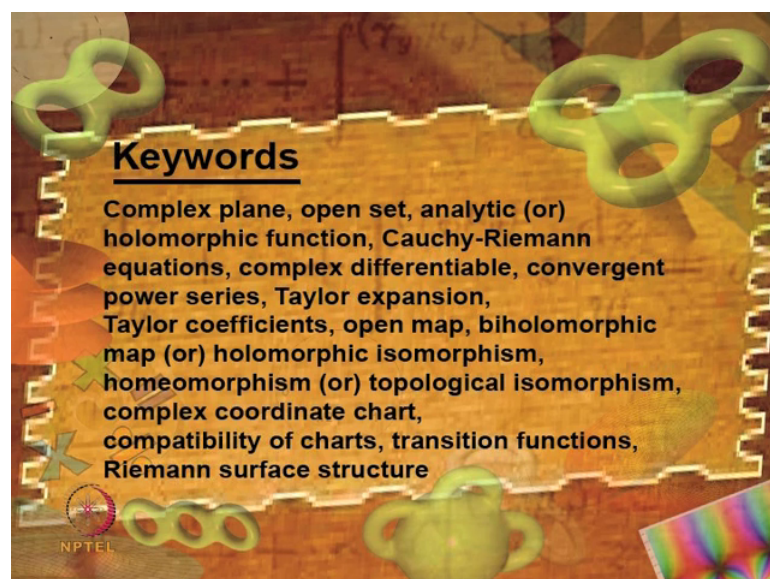
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Welcome to this first lecture in a series of lectures for the course titled algebraic curves and Riemann surfaces.

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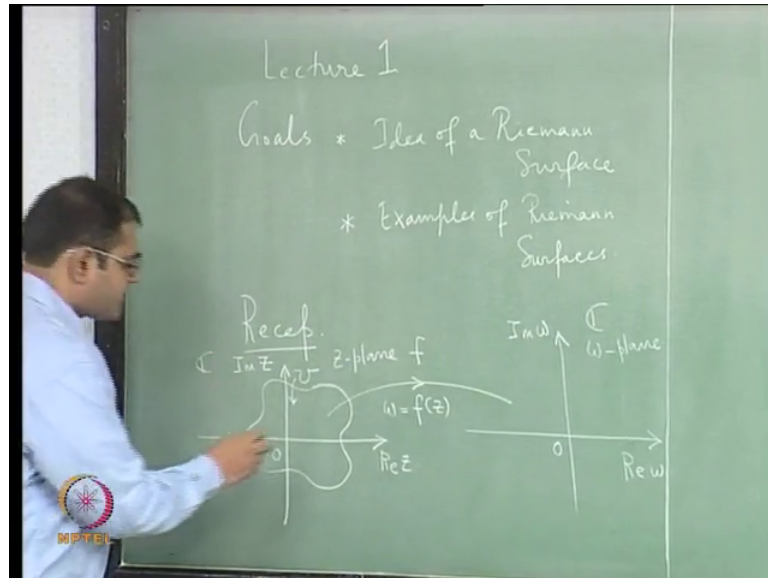
So, let me quickly tell you to begin with what the goals of the present lecture are. So, in this lecture, we will first try to understand the idea of a Riemann surface, and second we will try to look at some examples. So, before we begin let me try to recollect some basic ideas from complex analysis that is functions of one complex variable. So, we look at a function. So, here is a complex plane which we call as a z-plane. And suppose we have an open set u in the complex plane and we have a function f which is defined on this open set u, you can think of u as the interior of this, this amoeba like region that I have drawn here and the function takes complex values again.

So, there is another copy of the complex plane and we call this the omega plane where omega is the variable is the image of the variable z under f. Of course, this is the origin in on both planes and this is the real axis, this is the imaginary axis. And likewise we here have the real axis and we have the imaginary axis. And the idea is to recall what it means for a function to be holomorphic or analytic at a point z naught in this open set u.

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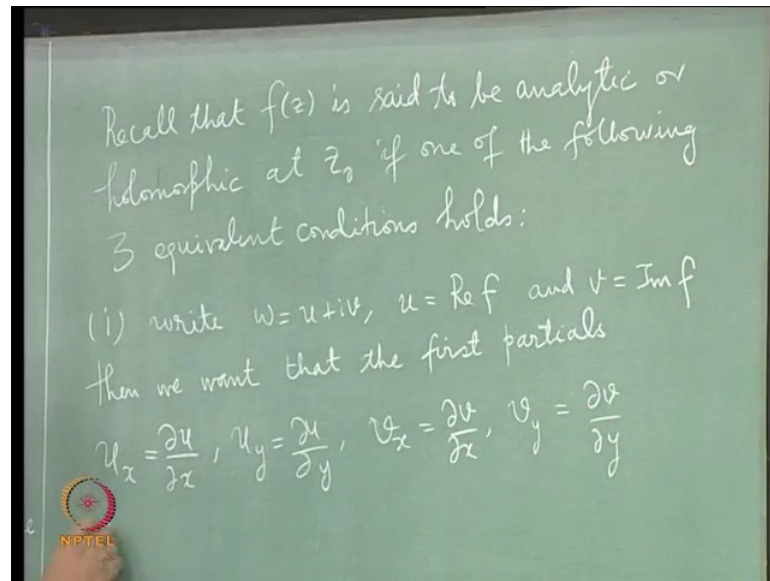
So, recall that f of z is said to be analytic or holomorphic at z naught, if one of the following three equivalent conditions holds. So, the first condition is that if you write omega as u plus iv, so that u becomes the real part of f and v becomes the imaginary part of f. Then we want that the first partial derivatives u x which is dou u by dou x the first partial derivative of u with respect to x, and then u y similarly dou u by dou y; and then v x dou v by dou x, v y is dou v by dou y exists.

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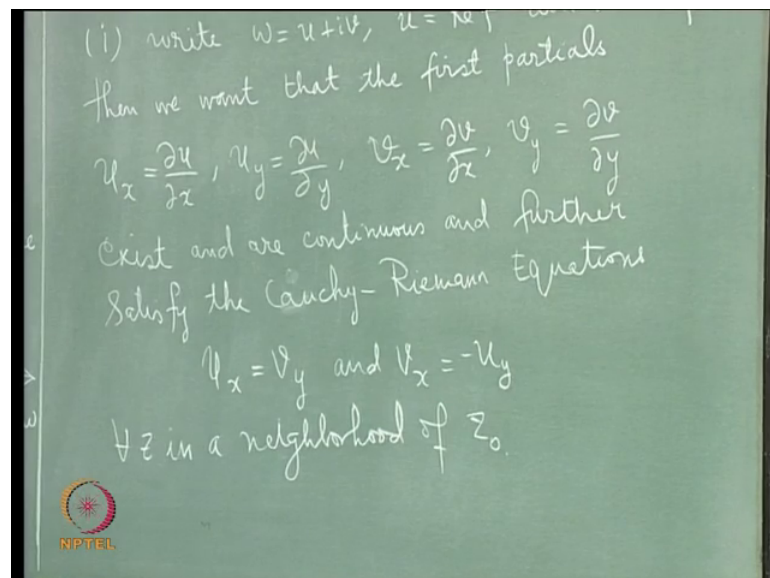
And or continuous and further satisfy the Cauchy-Riemann equations u x is equal to v y and v x is equal to minus u y for all z in a neighborhood of observe of the point z naught. So, this is one condition that would define f to be analytic or holomorphic at the point z naught. There are and usually this is the condition that you come across in a first course in complex analysis, which I think all of you have done. The next condition that is used to define the holomorphicity or analyticity of a function at a point is the usual definition the down to a definition that the function is differentiable not only at that point, but at every point in a neighborhood of that point. So, it is a straightforward definition.

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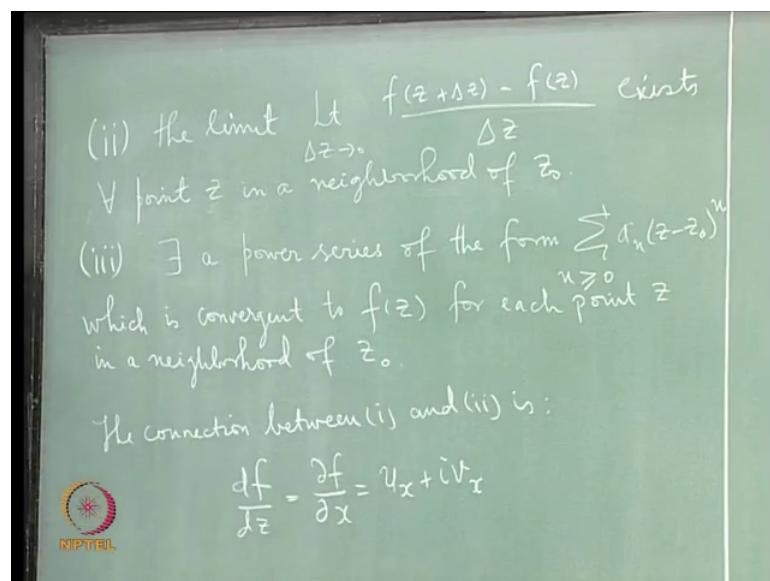
So, it is that is what I am going to write down the second definition the limit as delta is z tends to 0 of f of z plus delta z minus f of z by delta z exists for every point z in a neighborhood of z naught. So, this is the condition that the function is not only differentiable at that point, but it is also differentiable at every point in a neighborhood of that point. And then the third condition, which is a condition which is also often adopted is that the function is represented by a convergent power series in a neighborhood of the given point z naught.

So, let me write that down there exists a power series of the form sigma n greater than or equal to 0 a n z minus z naught to the power of n which is convergent to f of z for each point z in a neighborhood of z naught. So, these are the three equivalent definitions of a function being analytic or holomorphic at a point. Now, let me make some remarks about

these definitions. The connection between one and two is that the derivative can be expressed in terms of u and v. So, let me write that former down, the connection between one and two is the derivative of f is just the partial derivative of f with respect to x namely it is u x plus i v x.

And the connection between two and three is what I would call as most spectacular it is this amazing thing which says that if a function is differentiable not only at a point. But in a neighborhood of that point then the function is actually infinitely differentiable that is because you see you would have studied that a power series a convergent power series it allows you to differentiate it term by term and the differentiated series also is a convergent power series with the same radius of convergence. And since you can do this ad infinitum it amounts to saying that a convergent power series is infinitely differentiable and see requiring that such a convergent power series converges point wise to the function f of z also therefore, requires that f is infinitely differentiable.

So, what is spectacular about two that two and three are equivalent is this amazing fact that differentiability once at all points in a neighborhood of a given point gives you infinite differentiability for all points in a neighborhood of that point. And this is something that I hope all of you would have realized when you did a first course in complex analysis that is the major distinguishing feature between functions of one real variable and functions of one complex variable.

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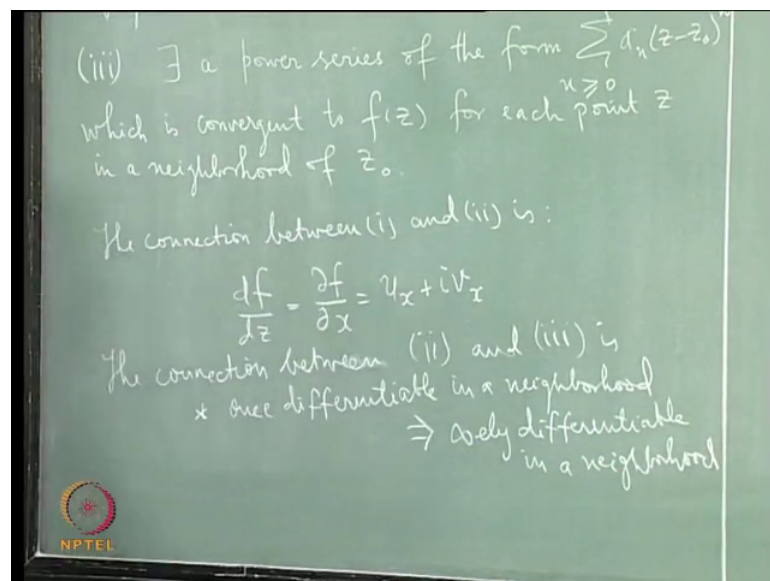
So, the connection between the a ns is that they are actually the Taylor coefficients. So, let me write that down the connection between, so let me write it two and three is. So, the first thing is once differentiable in a neighborhood implies infinitely differentiable in a neighborhood.

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$$* \quad a_n = \frac{f^{(n)}(z_0)}{\underline{|n}}, \qquad f^{(n)}(z) = \frac{d^n f}{dz^n}$$

so that $\sum_{n \geq 0} a_n (z - z_0)^n$ is just the Taylor Expansion of f around z_0.

Recall:

An injective holomorphic map is a holomorphic isomorphism.
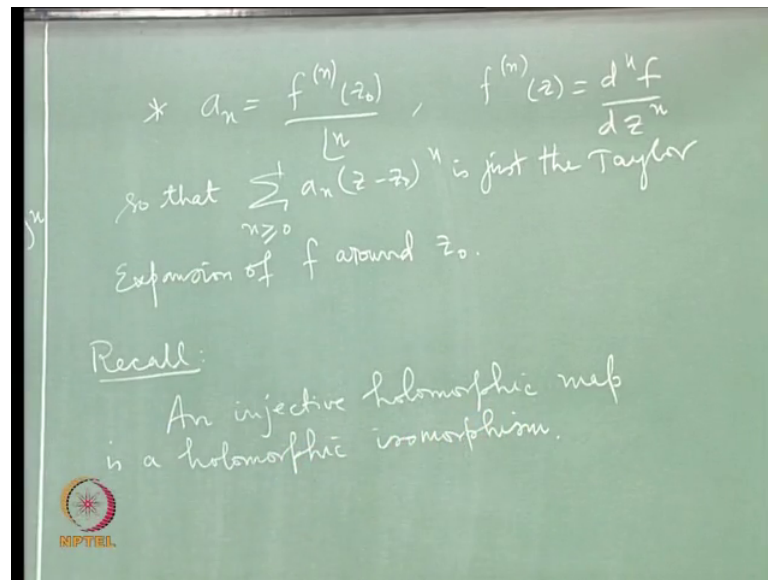
And then the next thing is that a ns are actually given by the nth derivative of f at z naught by factorial n where the nth derivative of f is just differentiating f n times with respect to z. So, that this power series to which converges to f is nothing but the Taylor expansion of the function f around the point z naught is just the Taylor expansion. So, this is just recall the idea of an analytic function. So, we will proceed to some further properties of analytic functions which I will require in the in the sequel. Now, I also want to recall another important fact, which is the following. So, recall again an injective holomorphic map is a holomorphic isomorphism, so this is again a deep fact.

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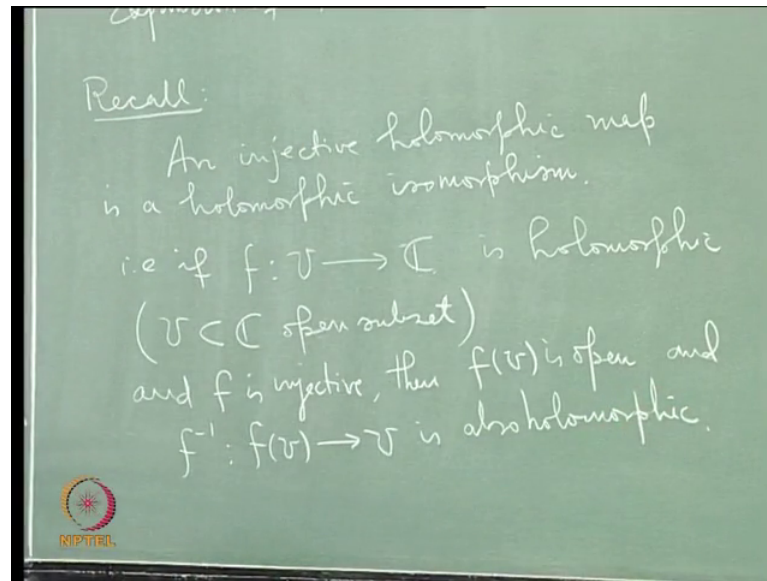
So, if you have a holomorphic map say f is holomorphic map from u to c where u is an open subset. And if f is holomorphic on u and f is injective that is if f is holomorphic and f is injective then f of u is open, in fact, f is what is called an open map it will take open sets to open sets. And since f from u to f u is a bijective map you can make sense of the set theoretic inverse from f u to u and it is a deep fact that f in that set theoretic inverse is also holomorphic. And f inverse from f of u to u is also holomorphic.

So, therefore, the important thing about this is that if you have a holomorphic map, and you know that it is injective, then you do not have to put the extra condition that the image of the open set on which its defined is open that comes automatically. Because it is an open map which means it takes open sets to open sets. And you also have the condition that the inverse map is not only continuous it is actually holomorphic. So, what this tells you is that whenever you have an injective holomorphic map that is actually giving you a holomorphic isomorphism with the image of the source with the image. So, this is another thing that we would use. So, now let us try to go and try to understand the idea of a Riemann surface. So, let me rub this half.
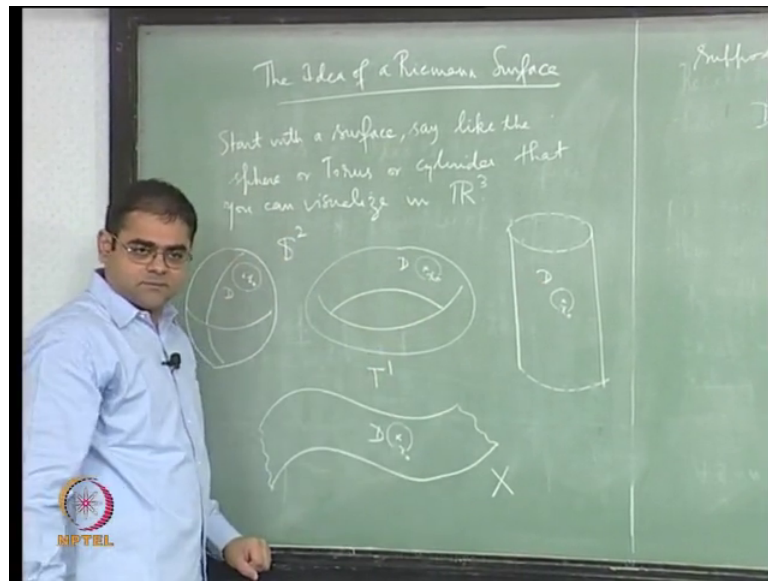
The idea of a Riemann surface, so what I am thinking of to begin with, so that we are down to earth and we have the concrete gasps grasp of kind of ideas that we want to formulate is the following. So, we start with the surface start with a surface say like this sphere or torus or cylinder that you can visualize in three space. You start with the surface. So, let me draw pictures of these. So, here is the here is this sphere S 2 and then here is a torus which we call as T 1. And then you can also think of a cylinder of course, is I will make this dotted line, so that I just wanted to understand that this is not the mount your cylinder, but this is an infinite cylinder. So, these are all surfaces these are all surfaces that you can imagine in three space and more generally you can also imagine some surface like this in three space more generally any surface like this.

And what is it that I want to do is the following. Suppose I am suppose I am given a point x naught on that surface, and suppose I am given a small neighborhood of that point which looks like a disc. So, I can think of a similar situation on each of these surfaces, I have a point x naught and a small discs surrounding it. So, when I say a small disc surrounding it the disc is not flat you know because the surface is curved. So, it is some kind of a curved disc, but topologically you can flatten it and think of it as a disc in the complex plane. So, when I say I have a point in the disc like neighborhood surrounding it I mean that there is small neighborhood surrounding the point which topologically looks like a disc in the complex plane. And well on a general surface also here is my point and here is my disc a disc like neighborhood. And suppose that so I let

me call this more generally as x. So, x could be any one of the three or even other things that you can think of.

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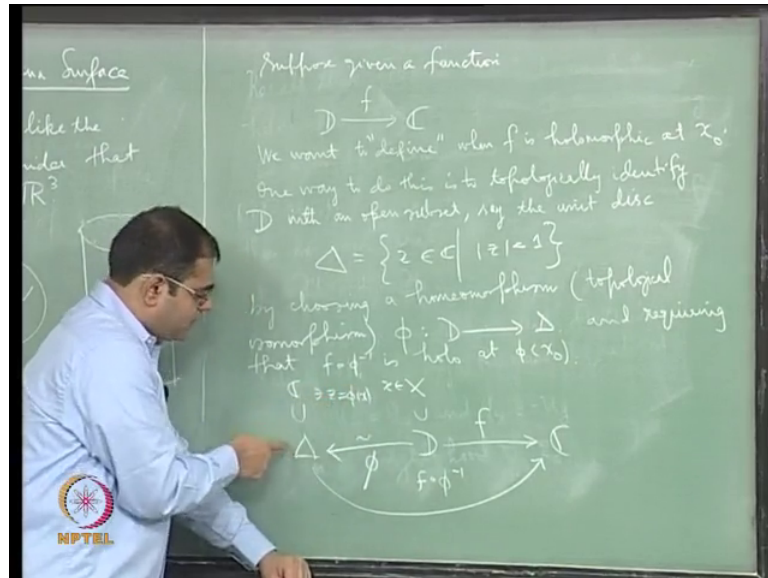
And what is it that I want to do? So, suppose you are given, so from D to C, suppose you are given a function f from this disc like neighborhood to C. So, it is a complex valid function. So, I have a surface I have a point of the surface and I have a small disc like neighborhood D, and I have a function. And this function is defined for every point on this disc which means it is also defined at x naught and this function takes complex values. And what is it that I want to do, what I want to do is I want to formulate a definition by giving a set of conditions as to when the function f is holomorphic at the point z naught, at the point x naught.

So, we want to define when f is holomorphic at x naught this is what you want to do. So, this is the point; I have a surface, I have a point and I have a small disc surrounding that point disc like neighborhood surrounding that point and I have function is defined on this disc. And I want to say I want to be able to say in a nice way in a clear way without any ambiguity when the function f is holomorphic at that point x naught. So, what is it that I want, I want to actually do the complex analysis that I do on the plane, I want to do that same kind of complex analysis I want to do that on a surface. So, I want to be able to do complex analysis on a sphere or on a torus on a cylinder wherever it is possible that is the whole idea and it is because of this that Riemann surfaces are being considered. So,

the main use of Riemann surfaces the idea of Riemann surface is to be able to do complex analysis on a surface.

So, well how do you define f to be holomorphic at x naught or more generally how do you define f to be holomorphic at every point on D if you want. How do you do that there is a there is a very easy way of doing it in the following sense, which is also very, very natural very, very intuitive it is the following. So, what we do one way to do this do this is to identify D with an open subset say the unit disc delta in the set of all complex numbers with modulus one, modulus less than one this is open unit disc by choosing a homeomorphism. And by the way let me remind you that a homeomorphism is a topological isomorphism topological isomorphism phi from D to delta.

So, I have my D here which is sitting inside surface and which contains the point x naught and I have this function f which is defined on D and which is taking complex values. And my aim is to be able to say that f is holomorphic at a point of D. So, what I do is I take an isomorphism topological isomorphism phi from D into the unit disc in the complex plane, so this is subset of C.

And then I take this composition what is this composition this is first I apply phi inverse which is correct because phi is a topological isomorphism. So, phi inverse is also continuous map phi inverse is also in fact a topological isomorphism. So, I apply phi inverse and then I compose it with f. So, I will get a map from the unit disc in the complex plane to complex numbers. So, it is a function from an open subset of the complex plane taking complex values; and for such a function it is very easy to define when it is holomorphic at a point.

So, what I do is that I require that this function is holomorphic at the point here, which is the image of the point x naught. So, and requiring that f circle phi inverse is holomorphic at phi of x naught. So, I have just used the intuitive idea that the neighborhood of the point that I have been given on the surface really looks like a disc. So, I identify that neighborhood with the disc in the complex say the unit disc in the complex plane and then using this identification I am able to get a function from that disc in the complex plane into the complex numbers for which it is easy for me to say when it is holomorphic at a point. So, it is a very intuitive definition. So, this a pair like this a pair so in fact, let me also say that I can now say I can extend this definition not only to the point x naught

of D, but I can extend it to all points of D. So, I can say f is holomorphic on D if f circle phi inverse is holomorphic on delta, because that way I have covered every point of D.

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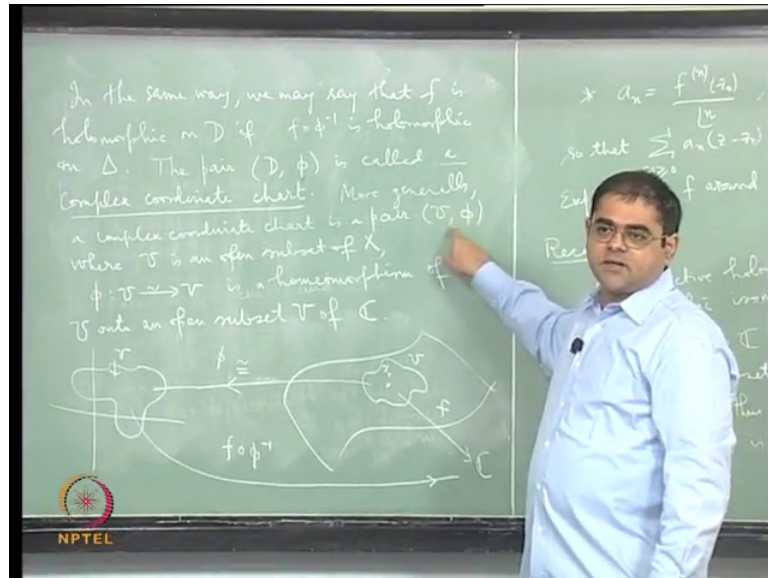
So, you see in the same way we may say that f is holomorphic on D if f circle phi inverse is holomorphic on delta. So, therefore, you see what you can understand is that this choice of this identification phi of this disc like neighborhood with a real disc, this identification and this disc like neighborhood is a pair of data which is what is called a chart, it is called a complex coordinate chart. So, the pair D comma phi is called a complex coordinate chart.

So, I am now starting with a very intuitive point of view that a Riemann surface comes up as trying to be able to do complex analysis on surfaces that we see in everyday life the surfaces that you can really imagine concretely. So, I am starting with that intuitive definition. So, for the moment my surface x is I have not defined what a surface x is formally, but I will do that formally in the succeeding lectures for the moment I am taking something very concrete that you can really see.

So, well a pair like this is called a complex coordinate chart and it is called a complex coordinate chart because it allows you to do complex analysis on this disc D, this disc like neighborhood D on the surface. So, you see so that is the purpose of a coordinate chart. A coordinate chart provides you with a coordinate with which you can do complex analysis. So, the idea is that if you call a varying point here as x then you can call a

varying point here which is in its image under phi as z. So, you have z is equal to phi of x and this z is actually a coordinate on this complex plane.

So, what you have done is that you have somehow tried to give up a position at a unique position to every point on this disc D, disc like neighborhood D. You are provided it with a different symbol in a continuously isomorphic way, which is a complex variable. So, that the resulting function becomes a function of one complex variable a function which is defined concretely on an open subset of the complex plane for which you can do complex analysis.

So, more generally what is a complex coordinate chart more generally I need not have taken here disc like neighborhood I could have taken just any open set containing the point and then I would have to choose again a topological isomorphism of that open set with an open set in the complex plane. So, more generally what is a complex coordinate chart more generally a complex coordinate chart is a pair u comma phi, where u is an open subset of x, phi from u to v is a homeomorphism of u onto an open subset v of the complex plane. So, this is what a coordinate chart is. And you can now see that somehow, so the more general diagram will look like this. So, I have this point x say x naught and then. I have some open say q on the surface and I have an identification of this by a homeomorphism a topological isomorphism of this with a subset of the complex plane.

So, here is my subset of the complex plane v. And this is chart this pair consisting of u and phi is a chart. And why is it useful, it is useful because whenever I have a function f defined on new taking values in C, I can call f to be holomorphic if the composite function which is given by phi inverse followed by f is holomorphic. So, I chose to begin with a disc because it is intuitive, but then instead of disc I could have had an open set.

So, at this moment it would appear that we could take for our definition of the Riemann surface just a surface that we can imagine in three space, and which is equipped with a set of charts like this such that these charts cover all of x that means, basically I want to do complex analysis on the Riemann surface. So, given any point of the Riemann surface, it should be contained in a chart it should be contained in the u member of a chart, so that I can use that chart to do complex analysis in that neighborhood of that point. So, this could be taken as the as a working definition or the first definition.
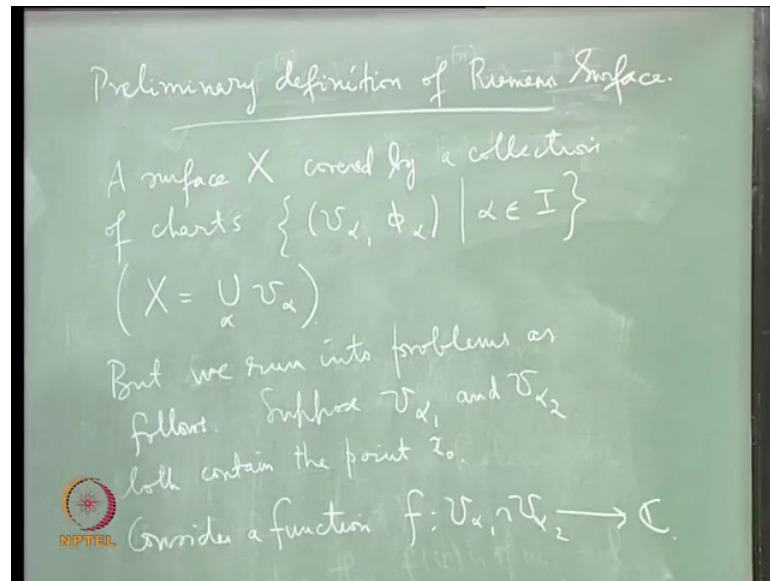
So, let me write that down, but let me also caution you that we will run into problems very soon and that will tell you how the definition has to be modified. So, let me write this down. So, preliminary definition of a Riemann surface is a surface x covered by a collection of charts collection of charts u alpha phi alpha where alpha trans over some indexing set. So, this capital I is an indexing set and for each element in I, I have a chart u alpha comma phi of alpha and these u alpha should cover x.

So, let me write that down X is equal to union for alpha U alpha this ensures that at every point of X, I can really do complex analysis using the chart that is available at that point, but we immediately run into problems. What is the problem that we will run into it is a kind of it is the following kind of obvious problem namely given a point it might occur in more than one chart. So, let us look at this situation, but we run into problems as follows. Suppose u alpha 1 and u alpha 2, both contain the point x naught. So, let us look at this situation. So, let me draw diagram I would need a larger diagram.
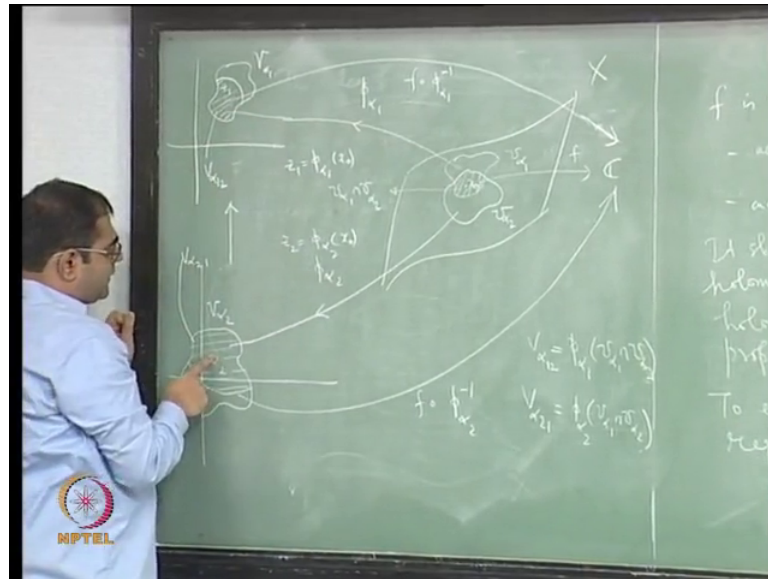
So, let me go and here. So, here is my surface. And I have one open set u alpha 1 and it is equipped with a chart namely I have a phi alpha 1 which is homeomorphism of some u alpha 1 with some subset of the complex plane which I will call it which I will call as v alpha 1. And well on the other hand, my point x naught also lies in another open set which is u alpha 2 which is the first member of the chart. So, there is again another homeomorphism phi alpha 2 which identifies this u alpha 2 with another open set let me call that as v alpha two in the complex plane. So, this is my situation.

And suppose that I have a function f that is defined on this intersection. So, let me write that there consider a function f from u alpha one intersection u alpha two to c suppose you have a function is defined on the intersection. So, in particular you could have considered a function is defined in a neighborhood of the point x naught and you could consider a small enough neighborhood, so that it is in the intersection. And the difficulty we will run into is that is the following it is in trying to decide whether the function is holomorphic at x naught or not. The reason is because we have two ways of defining f to be holomorphic at x naught.

So, there is one way one way of saying that f is holomorphic at x naught is to say that this composition this composition namely which is this f circle phi alpha 1 inverse, this composition is holomorphic at the image of x naught onto this map. So, if you want I will call that as z 1, so z 1. So, z 1 is phi alpha 1 x naught. And well the other way of

deciding that f is holomorphic at x naught is to require that this composition which is f now followed by which is now phi alpha 2 inverse followed by f is holomorphic at the point z 2, where z 2 is the image of x naught under phi alpha 2.

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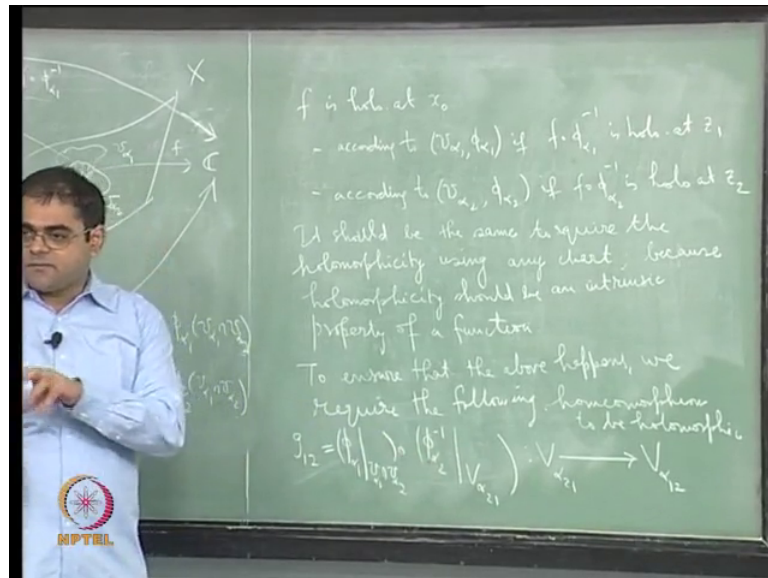
So, let me write that down clearly f is holomorphic at x naught according to the chart u alpha 1 phi alpha 1 if f circle phi alpha 1 inverse is holomorphic at z 1, and according to u alpha 2 comma p alpha 2 if f circle phi alpha 2 inverse is holomorphic at z 2. So, you get two definitions of f being holomorphic at the point x naught that is because you have chosen two charts which are available around the point. And in fact, you could have an infinite collection of charts and then that would give you a infinite set of definitions for f being holomorphic at the point x naught.

Clearly this is not something that we want because we you know that the idea of holomorphicity for that matter any property of a function has to be intrinsic to the function it should not be a property that should be ambiguously defined. All good properties of functions like continuity, differentiability, analyticity or holomorphicity these should be intrinsic properties of functions they should be properties which do not depend on let me say reparametrization, because after all what these charts are doing are just reparametrzing that neighborhood around that point in terms of complex variables.

So, what we really do not want to happen is that for example, that f circle phi alpha 1 inverse is actually holomorphic at z naught, but f circle phi alpha 2 inverse is not

holomorphic at z 2. What should not happen is that one of these is true and the others are false the other is false such a thing should not happen. So, how do you remedy the situation, you remedy the situation in the following way. You see I have this kind of a situation where I do not want the function to be holomorphic with respect to one chart at a point and it is not holomorphic with respect to some other chart.

So, how do I avoid this? So, you see it should be same to require the holomorphicity using any chart. So, this is what the ideal situation is that no matter what chart you use. The idea of a function being holomorphic at a point should be unambiguous because holomorphicity is an intrinsic property it should be an intrinsic property of a function, because holomorphicity should be an intrinsic property of a function.

So, let me repeat it should not be that because I have different charts my definition of holomorphicity depends on a chart if I change the chart the holomorphicity is false, and for some other chart the holomorphicity is true, I do not want such a thing to happen. So, this tells you that the charts have to be compatible in a certain sense. And how do you get this compatibility? So, you get this compatibility in the following way. You see to ensure that the above the above happens we require we require the following.

So, let me first explain it using the diagram here. So, you see I have this, this shaded region here this shaded region here is u alpha 1 intersection u alpha 2 is the shaded region here and that is of course, it is an intersection of two open sets. So, it is open. So, in open subset of u alpha 1 and since phi alpha 1 is a homeomorphism. The image of this here is going to be an open set in v alpha 1. So, I will get an open set here which I would call as v alpha 1 2; and v alpha 1 2 is nothing but is just the image under phi alpha 1 of u alpha 1 intersection u alpha 2. And similarly so this shaded region goes to this shaded region here. And similarly this open set u alpha 1 intersection u alpha 2 goes to another shaded region that I draw here which is again an open subset of v alpha 2 which I would like to call as v alpha 2 1. So, v alpha 2 1 is phi alpha 2 the image under phi alpha 2 of u alpha 1 intersection u alpha 2.

And now what I want you to understand is to look at this map. So, I look at this map from this shaded region to this shaded region from this open set phi alpha 1 2 to this open set v alpha 2 1. And how do I how do I get this map I first take phi alpha 1 I take yeah. So, let me just for convention let me change the direction of the map. So, I first

take phi alpha 2 inverse, phi alpha 2 inverse will take this shaded region which is v alpha 2 1 into u alpha 1 intersection u alpha 2 and then I apply phi alpha 1.

So, actually I have a map from this shaded region to that shaded region. So, I call that map as g 1 2. So, what is this g 1 2. So, g 1 2 is first apply phi alpha 2 inverse restricted to v alpha 2 1 that will start that will take v alpha 2 1 to homeomorphically on to u alpha 1 intersection u alpha 2 and then apply phi alpha 1 restricted to u alpha 1 intersection u alpha 2. So, I will compose this with phi alpha 1 restricted to u alpha 1 intersection u alpha 2. So, it will go from v alpha 2 1 to v alpha 1 2. And you can see that this is a composition of homeomorphisms. So, this is a homeomorphism, it is a homomorphism. This is a homeomorphism because the restriction of a homeomorphism to an open set is also a homeomorphism and a composition of homeomorphism is again a homeomorphism. So, this is homeomorphism.

And what is it homeomorphism it is a homeomorphism of two open subsets of the complex plane. So, I can require that to be holomorphic. So, require the following homeomorphism g 1 2 to be holomorphic; after all its a homeomorphism basically it is a mapping from an open subset of the complex plane to another open subset of the complex plane I just wanted to be homeomorphism, I just wanted to be holomorphic. But the point is it is already homeomorphism which means it is already injective. So, you see by remark that I told you earlier it is an injective holomorphic map. So, it is a holomorphic isomorphism.

So, what it will tell you is that it will tell you that g one two is not only not only is g 1 2 holomorphic, but g 1 2 is an open map and its inverse is also holomorphic. So, g 1 2 inverse is also holomorphic. So, putting this condition helps us it makes sure that you do not get a conflict in these two definitions and why is that so it is because of the following a simple observation.
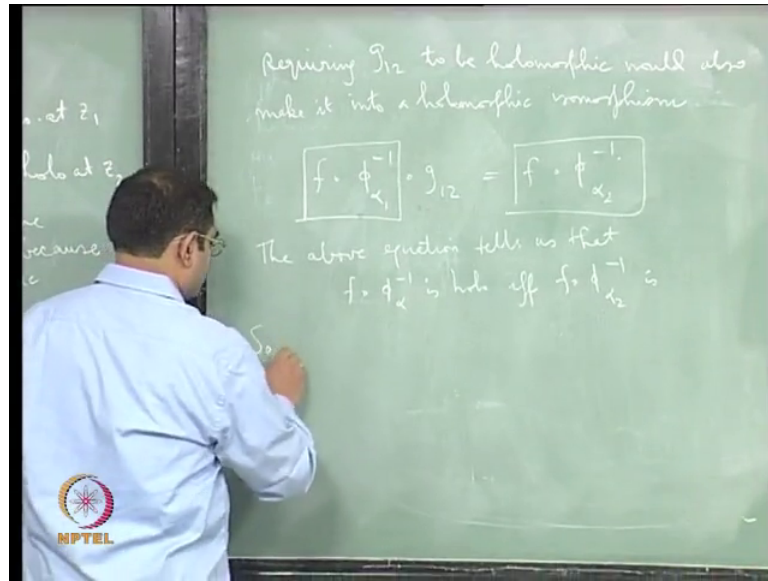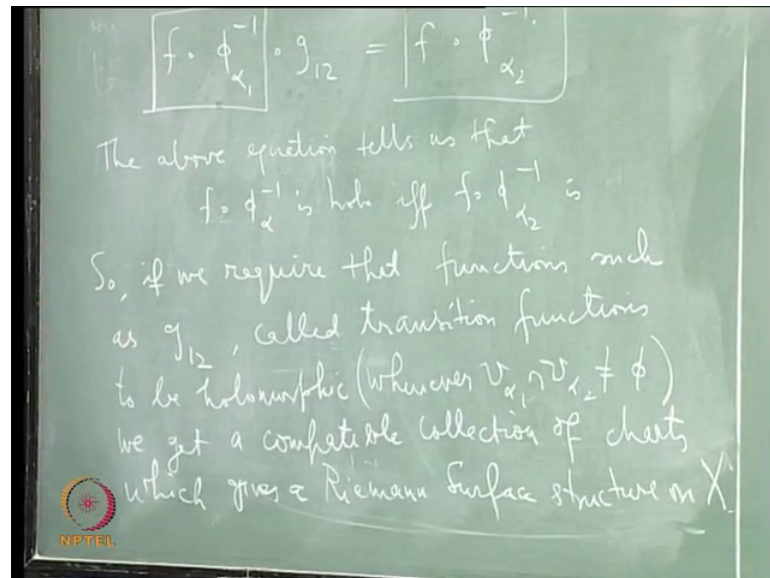
So, let me write that down. So, requiring g 1 2 to be holomorphic would also make it into a holomorphic isomorphism by a remark that I recalled some time ago in the beginning. And now why does this help, why does this condition help it helps because the following reason. Because you see f circle phi alpha 1 inverse if I compose it with g 1 2, I get f circle phi alpha 2 inverse. Because if I do not worry about writing these restrictions which is a little cumbersome I just write this as phi alpha one circle phi alpha 2 inverse with the meanings as to where these maps are being taken understood then I just write g 1 to as phi alpha 1. So, to phi alpha 2 inverse and then you know so if I in g 1 2 if I plug in a phi alpha 1 circle phi alpha 2 inverse then you see I get f circle phi alpha 2 inverse.

And you see that therefore, this map and this map they differ by holomorphic isomorphism and therefore, this is holomorphic if and only if that is holomorphic because g one two has an inverse if this is holomorphic g 1 2 is already holomorphic and there is a composition of holomorphic maps. So, that is holomorphic and conversely if that is holomorphic I can multiply on the right by g 1 2 inverse to get this is holomorphic. So, the above equation tells us that f circle phi alpha 1 inverse is holomorphic if and only if f circle phi alpha 2 inverse is, so there is really no conflict in using these two charts to define holomorphicity.

And now if you require this condition to happen whenever you have two intersecting charts in which case we say that those two charts are pair wise compatible then you are

in a good situation. So, we make this requirement that not only is the Riemann surface just a bunch of is a surface which is covered by a collection of charts, but these charts whenever they intersect on the intersection, these functions g 1 2, which are called the transition functions these are called transition functions. And we want these transition functions to actually be holomorphic.

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So, let me write that down and that gives us a very ah concrete first definition of a Riemann surface. So, let me write that down. So, if we require that functions such as g 1 2 called transition functions to be holomorphic whenever u alpha 1 intersection u alpha 2 is non empty, we get a compatible collection of charts which gives a Riemann surface structure on x.

So, let me again repeat that. So, we started with trying to do complex analysis on a surface and we realize that we could do that if we had these complex coordinate charts. And we want to be able to do that every points, so these charts should cover the whole surface, but then we run into problems deciding whether a function is holomorphic at a point because there may be more than one chart available at that point. And in order that such ambiguity does not arise, we put this extra condition that for any two intersecting charts, the transition function is holomorphic. And once you do that everything is fine. So, you just take a collection of charts which are compatible with each other, this is the

compatibility condition and that gives you a Riemann surface structure on x. So, this is the beginning definition of what Riemann surfaces. So, let me stop here

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1 Holomorphic (or) analytic functions. Let $f : U \longrightarrow \mathbb{C}$ be a complex valued function defined on an open subset U of the complex plane $\mathbb{C}$. Consider the following conditions on f:

(a) f is continuous at every point of U;

(b) the first partial derivatives $\frac{\partial f}{\partial x}$ of $f = f(z)$ with respect to x the real part of z, and $\frac{\partial f}{\partial y}$ with respect to y the imaginary part of z, both exist at every point of U;

(c) the first partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous at every point of U;

(d) the first partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and satisfy the Cauchy-Riemann equations $\frac{\partial f}{\partial x} = -\sqrt{-1}\frac{\partial f}{\partial y}$ at every point of U.

In a first course in Complex Analysis, all the above hypotheses together are shown to imply that f is a holomorphic (or) analytic function of z on U (of course (c) or (d) implies (b)).

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It is a deeper fact, called the Looman-Menchoff Theorem, that the continuity hypotheses on the first partial derivatives, namely (c), may be omitted and the analyticity of f still remains true. For a proof of this, refer to the book by Raghavan Narasimhan and Yves Nievergelt titled *Complex analysis in one variable*, Second edition, Birkhäuser, 2001. In fact, we may also replace (a) with

(a'): f is locally bounded on U i.e., for each $z \in U$, there is a small disc centered at z inside U where the modulus $|f|$ is bounded by a finite (non-negative real) quantity.

For more details on this and on other possible hypotheses that would imply analyticity, refer to the article of J. D. Gray and S. A. Morris titled *When is a function that satisfies the Cauchy-Riemann equations analytic?* in the American Mathematical Monthly Issue 85 (1978), no. 4, pages 246–256.

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2 **Biholomorphic maps (or) Holomorphic Isomorphisms.**
Recall the following results that are usually proved in a first course in Complex Analysis. They would help in understanding why an injective holomorphic map is a holomorphic isomorphism. For example, you may consult the book by S. Ponnusamy and Herb Silverman titled *Complex Variables with Applications*, Birkhäuser, 2006.

(a) (Open Mapping Theorem) A nonconstant analytic function is an open mapping, i.e., the image of an open set under such a function is again an open set.

(b) If an analytic function has nonvanishing derivative at a point, then that function is one-to-one i.e., injective as a mapping in some neighborhood of that point.

(c) An injective analytic function on an open connected set has nonzero derivative everywhere.