An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves
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Lecture - 10
The Importance of the Path-Lifting Property

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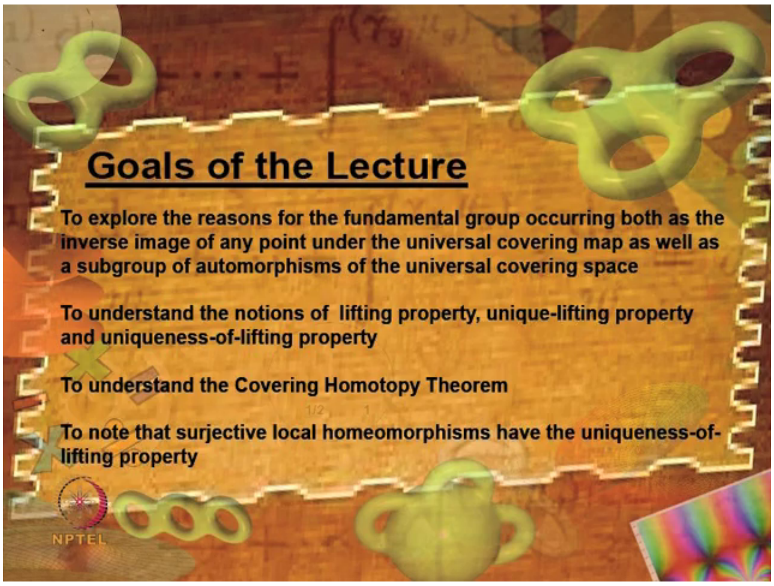

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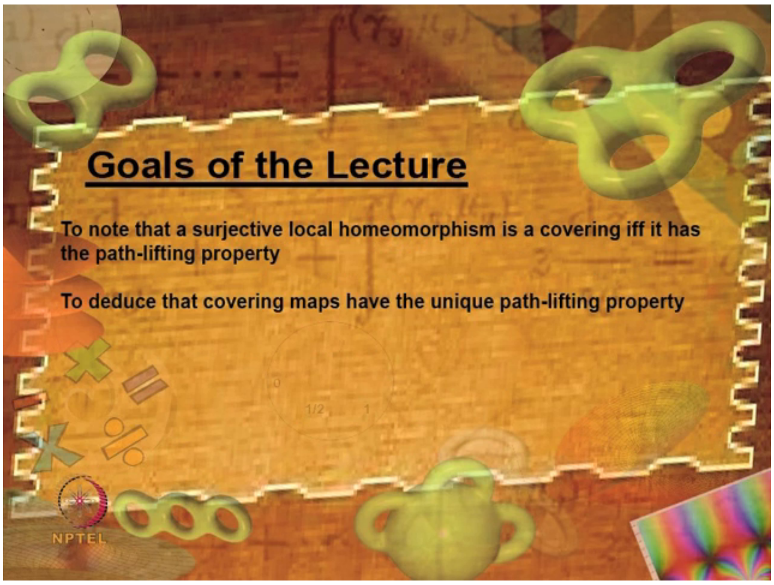

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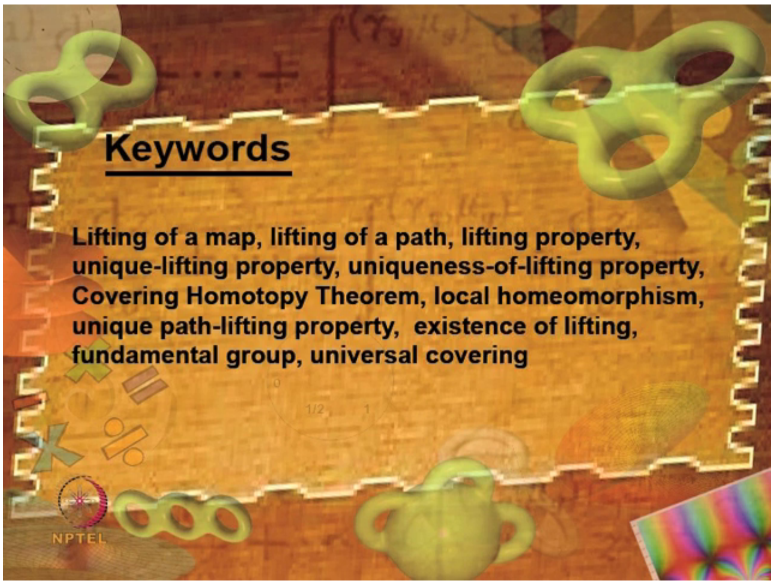

What I am going to try to explain in this lecture is, how the notion of homotopies and the notion of lifting of paths comes into the picture when we look at covering spaces. So, let me recall that, there were 2 examples of covering spaces of Riemann surfaces that I gave you one was.

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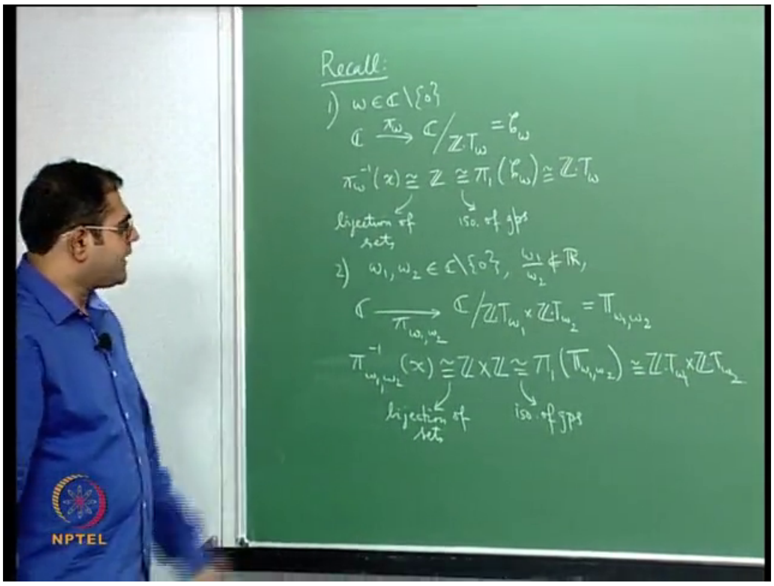

So, let me write that here, one was we had omega a complex number 1 0 and then we had the covering map C to C modulo the group which is the translation by integer multiples of omega, and I told you that I if you call this map as phi sub omega, I told you that this is a covering map and.

This quotient is topologically a cylinder and it acquires Riemann surface structure such that this map becomes holomorphic. And then I told you that if you take a point in the cylinder and take the inverse image, then the inverse image turns out to be set theoretically the same as this group of translations by integer multiples of omega, which also is isomorphic to z which turns out to be the fundamental group of the cylinder . So, what I wanted to say was that when you take this covering space situation, the fiber over a point there is the inverse image of a point is set theoretically the same as the fundamental group of the base space.

And similarly, pi omega inverse let me write that down pi omega inverse the inverse image of a point x is isomorphic to Z and which is also isomorphic to the first fundamental group of if I call this as script C sub omega this is the cylinder with Riemann surface structure given in this way. So, this is I 1 C sub omega. So, of course, this isomorphism is an isomorphism as a bijection of sets and. In fact, this is an isomorphism of groups because the first fundamental group is z and this also turns out to be. So, so let me write that down this is this is bijection of sets and this is isomorphism of groups and in fact, what I wanted to say is that this this is of course, isomorphic also to this the z dot T sub omega.

So, this was the group of automorphisms of C which you have to go modulo to get the Riemann surface structure on the cylinder. So, you see the fundamental group of the cylinder is occurring in 3 ways or you can see it occur in the covering space this is the universal covering space because you see the top space is simply connected. So, you see the fundamental group appearing in 3 ways, one thing is it is bijective to the fundamental group of the base is bijective to every fiber that is this statement. The fundamental group of the base is also isomorphic to a subgroup of automorphisms of the covering space.

The universal covering space and that sub group is precisely the subgroup modulo which you have to go to get the base below the base space below you see. So, you see this is how the fundamental. So, this involves this tells you 2 things, first thing is the fiber being identified with the fundamental group is one point, the other point is the fundamental group below being realized as a subgroup of holomorphic automorphisms of the space above. These are 2 aspects of covering space theory which are very very important that we would like to understand. So, I try to explain in this lecture how this happens.

So, let me also recall the other example that the other example is that of the holomorphic structure on a cylinder on a torus. So, what did we do? We took we took 2 complex numbers omega 1 omega 2 nonzero complex numbers and of course, we assume that they are linearly independent over r so; that means, omega 1 by omega 2 is not a real number; that means, these 2 the vectors represented by these 2 complex numbers are 2 linearly independent vectors. They form a parallelogram of nonzero area and then I told you that you get a holomorphic structure on the torus.

In the following way namely you simply go modulo the group of translations by integral multiples integer multiples of these two. So, it was Z dot translation by omega 1 cross Z translations by omega 2. And in this case also the same thing happened what namely you take a point x here the inverse image by omega 1 comma omega 2 universe of a point x, C gave you a grid of points in the complex plane and the grid was just isomorphic to you know Z cross v which also turns out to be the fundamental group of this complex torus mind you.

This is a Riemann surface structure on the real torus yes this is homeomorphic to s 1 cross s 1. So, I think I call this 1 can call this T sub omega 1 comma omega 2. So, the fundamental group of the torus is also Z cross Z. So, this is an isomorphism. So, again here we have a similar situation namely this is isomorphic to the first fundamental group of the torus of course, I can whenever I say first fundamental group, it is just first fundamental group as a topological space I can forget the Riemann surface structure mind you; because the first fundamental group is defined only on the underlying topological space and well.

Again in as in this case you see that this is a bijection of sets and well this is an isomorphism of groups and in fact, this is isomorphic to Z dot this this this subgroup. So, again it is a same picture, this is again a holomorphic universal covering for every point here the inverse image is isomorphic to the fundamental group below as a set, and the fundamental group of the base can be realized as exactly a certain subgroup namely this 1 subgroup of automophism holomorphic automorphisms of C of the coverings universal covering space, modulo which you have to go to get this holomorphic structure on the torus.

So, these are 2 nice examples and they tell you what happens in general. So, we one has to one would like to understand the following questions. First of all why is it that the fundamental group of the base shows sup set theoretically as the fiber over each point that is the first question to answer or to understand? The second question is how is it that the fundamental group of the base a how is it that it can be realized as a subgroup of automorphisms of the covering space. So, the key to this understanding is what is called the covering homotopic theorem which is a fundamental tool in the study of covering spaces. So, in which is what I am trying I am going to try to explain. So, to begin with, let me make some definitions.

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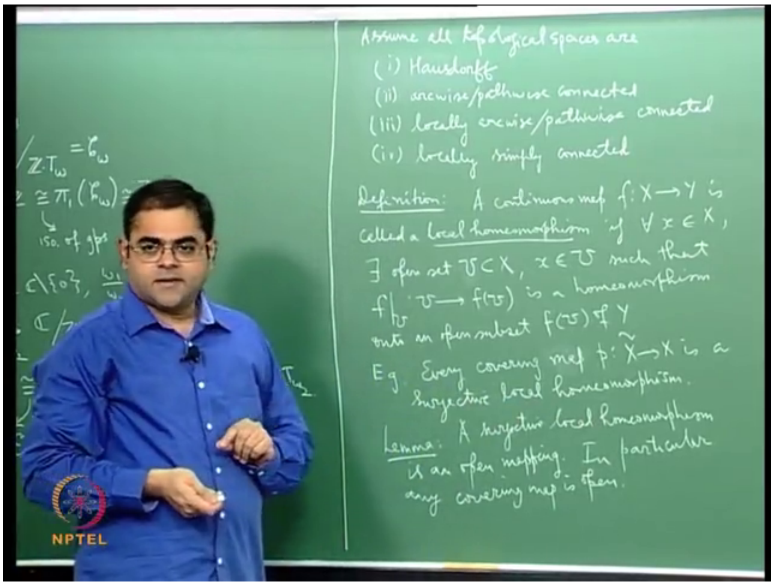

So, the first definition is the following. So, let me make a few blanket assumptions, in the course of our discussion we will need several hypotheses and so, I am going to make some blanket hypotheses about the kind of spaces that we are going to work with we may not be using all the hypotheses or maybe we may use some weaker hypothesis in certain cases, but just to make the exposition simple I will make some blanket assumptions assume all topological spaces.

Are the I think I should erase this assume all topological spaces are by that I mean all topological spaces that we are when you talk about. Number 1 Hausdonff which is essentially that any 2 distinct points can be separated by disjoint open neighbourhoods, then number 2 arcwise or pathwise connected.

Number 3 of course, if you recall arcwise or pathwise current it means that any 2 points any 2 distinct points can be collect connected by a continuous image of an interval a closed interval and then locally arcwice or pathwise connected. A weaker condition than 2 is just assuming connectedness which is weaker. So, if you assume arcwise or pathwise connected, it implies that it is connected. So, weaker condition for 2 would be connected and whenever we can assume that I will make a mention of it. Then a fourth one is locally simply connected locally simply connected. So, this is the condition that every point has an open neighbourhood, which has the property that it is simply connected namely that any closed path in that neighbourhood can be continuously shrunk to a point. So, closed path means a continuous image if an interval it starts at one point and ends back at the same point 1.

Sometimes refers to this as a loop based at a point and the condition of for simply connectedness is that this loop can be continuously shrunk to 0, and that can happen only if there are no holes in that neighbourhood. So, we put all these conditions in what follows. So, I will make the first definition the first definition, I want to tell you about this that of a local homeomorphism. So, a continuous map f from let me say y to let me just put x toy is called a local homeomorphism.

If for every point x in x there exists an open set; v let me call it as u exist an open set u in x, x belonging to u such that f restricted to u from u to f of u is a homeomorphism on to an open subset f of u of y. So, the definition of a local homeomorphism is a continuous map, such that at every point I am able to find in a neighbourhood which this map maps homeomorphically on to an open neighbourhood in the target topological space.

So, what is the connection? The you know that every covering map in fact, uh these covering maps if you remember I explained that they are all open. Because you take any set here then its inverse image is just translates of a fixed copy of this set above and all these translates are disjoint if you choose a set below small enough they all be disjoint, if you choose a set if you do not choose them small enough they will still be a union of open sets. And the quotient topology will tell you that you know a set here is open if and only the inverse image is open.

So, if I start with an open set here I take its image there, then you can check that this is an open then this is an open set here and how why is it an open set? That is precisely because of the quotient topology because its inverse image will be the all translates of the original open set I started with here by this group and all these translates are all open. So, that is the reason by the image of an open set is open same thing happens here. So, let me repeat that why are these open maps, because I started an open set here, I take its image there I want to say that that is open.

But by the quotient topology this is open if and only if the inverse image of that is open, but what is the inverse image? The inverse image is just translate for the original open set I started with translates by these maps. The union of all such translates and each such translate is a is an open set and therefore, the inverse image is the union of all such open sets which are translates and therefore, it is open and therefore, this is an open map and this happens in general for a covering space.

So, what I want to say is that and of course, you know a covering space has this property, the property of being a local homeomorphism. Because the covering space definition is you give me a point, every point below as a neighbourhood such that the inverse image is a disjoint union of neighbourhoods each of which is homeomorphic to the point to the neighbourhood below, that miscible neighbourhood below. So, if you look at it, that it will tell you that every covering space is in fact, a surjective local homeomorphism. So, example every covering space I should say every covering map is surjective local homeomorphism.

So, this is an example and, but of course, the covering map is more just it is not just the surjective local homeomorphism homeomorphism, but its more than that, but why did I single this out? Because of the following fact namely you take a surjective local homeomorphism then it will be an open map ok.

So, lemma a sujejective local homeomorphism is an open map. So, this is very very easy to prove you can you can convince yourself of this, and as a result what it tells you is that every covering map is also an open map and by of course, by open mapping you need a you mean a map, which snaps open sets to open sets and that is what is happening in these cases. So, what is happening in these cases it is also true gently. Covering map is always an open mapping it takes open sets to open sets and the reason is because it is a surjective local homeomorphism.

So, in particular any covering map is open. So, this is a very simple exercise in topology we can take we can check it. Now having said this, the next thing that I would like to worry about is I would like to talk about the converse condition. So, suppose I know that I have a surjective local homeomorphism which is of course, true for a covering map what more conditions do I have to put to a surjective local homeomorphism so that it becomes a covering map ok.

So, the surjective local homeomorphism is a weaker condition than a covering map. So, my question is can you put some more conditions to this so that you can get back the covering property the property for covering map. So, this is where the lifting property, the unique lifting the lifting property of maps comes into the picture. So, I will explain what that is.

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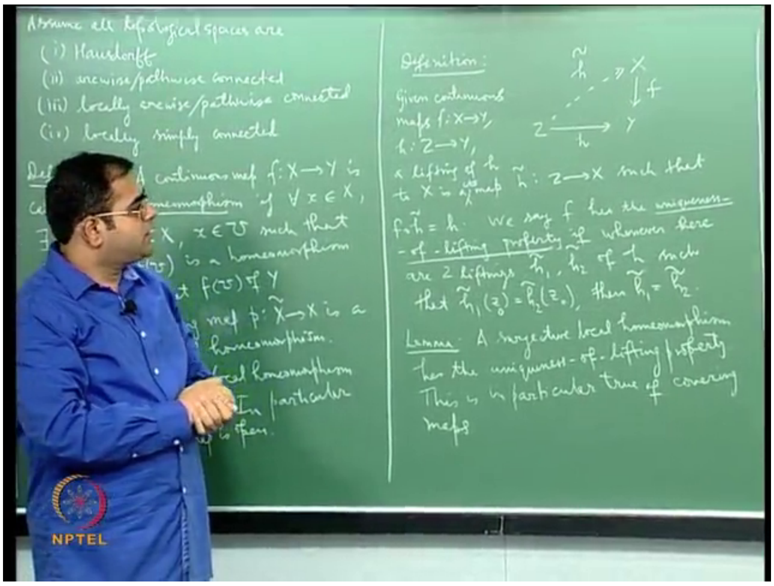

So, maybe I can use and draw this line here I remove this. So, let me talk about lifting of maps. So, the situation is the following you see I have a map f from x to y, which I am purposely writing it vertically and suppose I have a map from z to y let me call this as h. So given, let me say that given continuous maps h f from x to y, h from z to y. So, these are continuous maps; a lifting of h to x to x is a map, h tilde from z to x such that h tilde followed by f s h. So, this is a lifting of a map. So, there is a h tilde of course, when I say map I mean of course, continuous map we are in the category of topological spaces So, if you want I should include here continuous I just put cts.

So, what has happened is that this map h has been lifted to a map h tilde and why do we call this is a lift? Because this followed by this is this that is this condition. So, we say that h tilde is a lift of the map h. Now there is a very nice lemma. The lemma says that you know if you have a surjective local homeomorphism, suppose this map f is a surjective local homeomorphism and suppose this set z is say connected and locally connected which is slightly weaker than you know arcwise connected is locally arcwise connected.

So, in that case there is something very nice that is going to happen, it says that if you have 2 liftings suppose you have 2 liftings and suppose both liftings coincide one point of z then they coincide everywhere. So, the moral of the story is, if you have a lifting with a prescribed value at a point of z, then that is unique. Namely if there are 2 liftings which have the same value at one point of z then they have to be everywhere equal. So, this property is called the uniqueness of lifting property and this uniqueness of lifting property happens whenever f is surjective local homeomorphisms.

So, let me state that we say f has T uniqueness of lifting property, if whenever there are 2 liftings let me call them h 1 tilde, h 2 tilde of h such that h 1 tilde of z naught is equal to h 2 tilde or z naught, then h 1 tilde is equal to h 2 tilde. So, what I am defining is this uniqueness of lifting property. So, what is this uniqueness of lifting properties property? It says that if you have 2 liftings h 1 tilde h 2 tilde, namely 2 maps like this which when composed by f give h.

And of course again when I say maps I always mean continuous maps, and suppose these 2 are going to coincide one point of z then they are going to coincide everywhere right. So, the nice thing is that a surjective local homomorphism is going to have this uniqueness of list lifting property all right. So, let me write the down, I will just check the let me just check the hypothesis maybe if one it spends a little bit more time, some of the hypothesis may can be weaken, but anyway let us not worry too much about that. So, let me state this lemma.

A surjective local homomorphism has the uniqueness of lifting property. So, in other words if f from x to y is a surjective local homeomorphism, then it has the uniqueness of lifting property namely you give me any map from any other topological space to y of course, when I say any other topological space I am assuming at least that for example, it is connected and I am assuming all these things.

So, I mean I may not need all of them to get the conclusion, but for safety I am assuming all of these conditions. So, in particular I am assuming that is connected and locally connected. So, then any map if you give me a map from z to y which has 2 liftings and if these liftings are same at one point of z, then they have to be same everywhere . So, this is in particular true of covering spaces because covering spaces are of course, covering maps are of course, sujective local homeomorphisms. So, this is in particular true of covering maps.

I will tell you why this is important, it is important for the following reason is.

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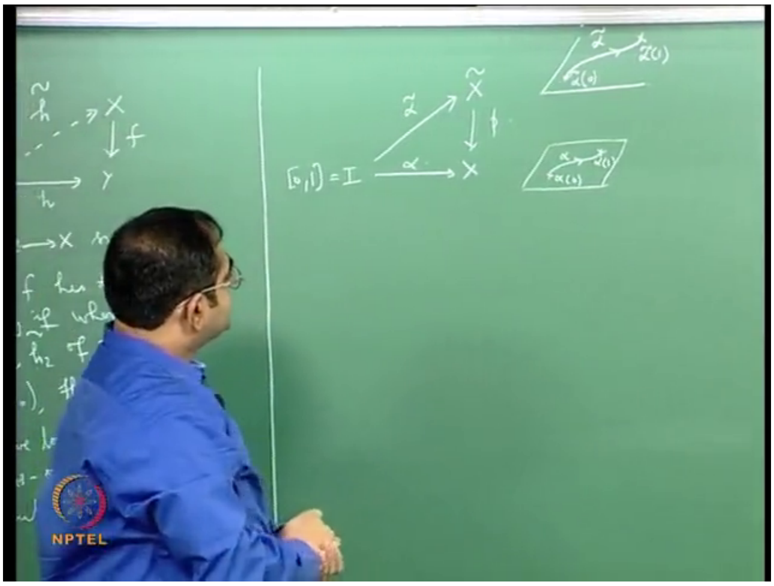

So, you see suppose from x tilde to x suppose I had a covering space and suppose I take I to be the unit interval and I take a map from I to x. You know the image of I is going to be a path in x all right. So, if I draw a diagram then this is my x and I am going to get a path this path is going to start at alpha of 0 and its going to end at alpha of 1 right now.

Suppose I am able to find. So, you see. So, this point is alpha of 0 and this point is alpha of 1 and that is this that is the covering space above and let me draw it a little bit more space so that I can write like this. So, here is my covering space above suppose for this point here. Suppose I choose a pre image above after all occurred the covering map is surjective and you know given any point here I can choose a pre image. Then if I am able to find a lift alpha tilde of this path alpha it is going to be up.

So, this is my path alpha and I am going to get another path here above alpha tilde. The only thing is that this point is alpha tilde of 0 that point is going to be you know alpha tilde of 1 and you know under the projection p alpha tilde of 0 will go to alpha 0 and alpha tilde 1 is going to go to alpha of 1. And suppose I have lifting like this, and then this lemma will tell me that the lifting is going to be unique in the sense that if I had another starting at alpha of 0 then I will get starting at alpha tilde of 0 then that is going to give me another lift here and both the lifts coincide at the point 0 alpha tilde of 0 they both coincide here they coincide at the point 0 here both maps. So, what I am trying to say is that If I draw it you can imagine such a situation namely that you know I have alpha 1 tilde.

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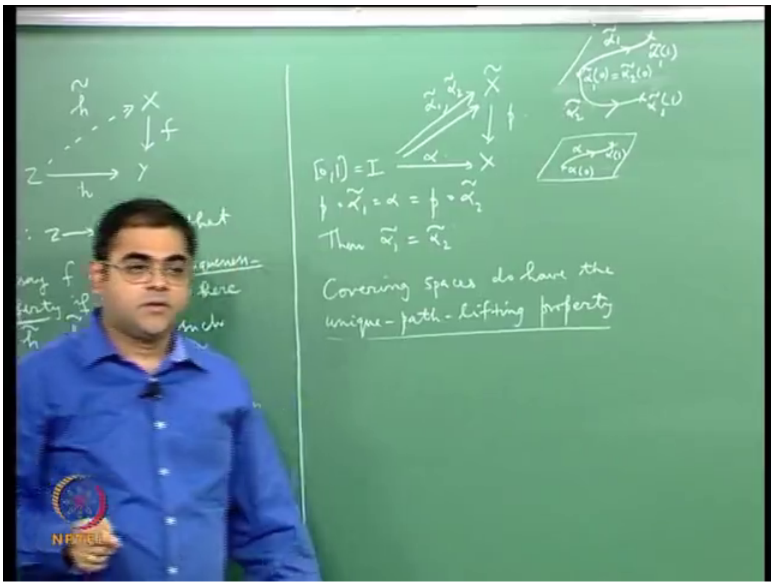

I have also alpha 2 tilde, I have 2 lifts of this map which of course, means that alpha 1 tilde followed by p is alpha and the same is true of alpha 2 tilde also 2 lifts. And suppose that alpha 1 tilde and alpha 2 tilde have the same starting point. So, this is alpha 2 tilde of 0 this is the same as. So, let me write that here and this is alpha 2 tilde of 1. So, this map is alpha 2 tilde this path is alpha 2 tilde.

So, you see the path below is alpha and it has been lifted let us say to 2 paths well alpha 1 tilde and alpha 2 tilde. So, but I have assume that you know this starting point is the same the initial point for both path is the same of course, you know alpha 1 tilde and alpha 2 tilde at 1 they should lie above alpha of 1, because of this condition these paths lie over this path. So, if I follow this path by your projection I should get the path below. So, its clear that alpha 1 tilde of 1 and alpha 2 tilde of 1 the endpoints are certainly points laying above the end point below, but that is not the point the point is that this the both the maps alpha 1 and alpha 2 they agree at the point 0, that is what it means to say that they start from the same.

Point alpha 1 tilde of 0 is equal to alpha 2 tilde of 0 and what does uniqueness of lifting property; now say it says that these 2 paths have to be the same. So, the moral of the story is if you have a covering space like situation you take a path below, and for this path you take the initial point and choose a point above. Then there is a unique if at all there are paths above that lift of this path there can be only 1. So, you see a path below is going to give you a unique path above, if you fix the initial point above. So, you see now you can guess that in some way the fundamental group is going to be involved, because after all the fundamental group is connected with closed paths and taking homotopic classes.

So, this is how it intersects the picture. So, then you know alpha 1 tilde is equal to alpha 2 tilde. So, if given a path below if at all you can lift it, you can lift it only to one path provided you fix the initial point above. At this point matters you could have fixed some other initial point and then you were to get some other path, but once you fix the initial point above lying above the initial point below then there is only one path you can get. So, we say that you know these uniqueness of path lifting property is true for covering spaces.

Now, there is another question; look at this definition on the uniqueness of lifting property, mind is the a uniqueness of lifting properties for any maps not just paths, I specialized to the case of a path, but its true it is supposed to be defined for any maps all right. You see we still do not know whether given a map you can actually lift it for example, what this condition says is if you get 2 lifts which agree at one point then they are the same, but it did not it does not guarantee you the existence of a lifting.

So, where do you get that from? So, it happens for covering spaces. So, in the case of covering spaces you have the existence of a lifting and then because it is also a local homeomorphism the lifting is unique. So, we say that the covering spaces have the covering maps they have the unique path lifting property. So, to explain that let me draw a diagram just to. So, let me say the following covering spaces do have the unique path lifting.

So, what is this unique path lifting property? It unique path lifting property is given path below you can get a unique path above provided the initial point of the initial point above has been fixed. So, it gives you existence of a lifting and of course, uniqueness of a lifting follows from this lemma. So, if I draw a diagram. So, I have a diagram like this. So, I have I have a covering map.

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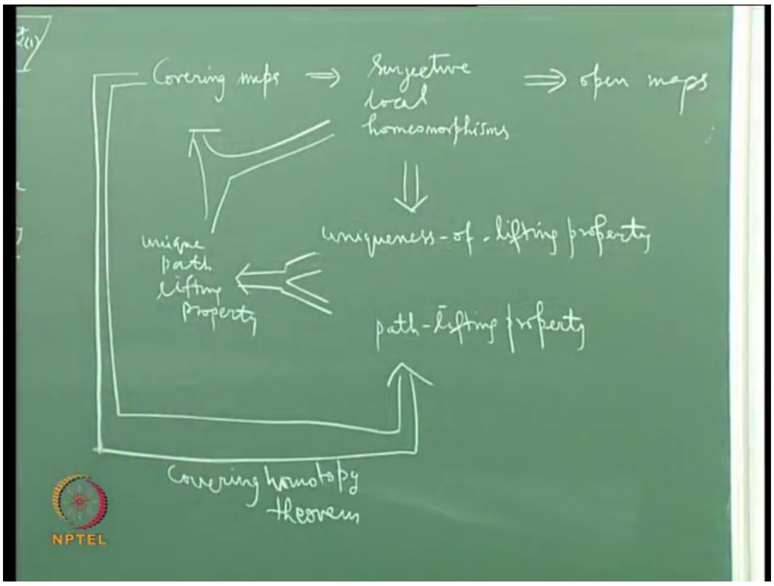

So, of course, I should actually write.

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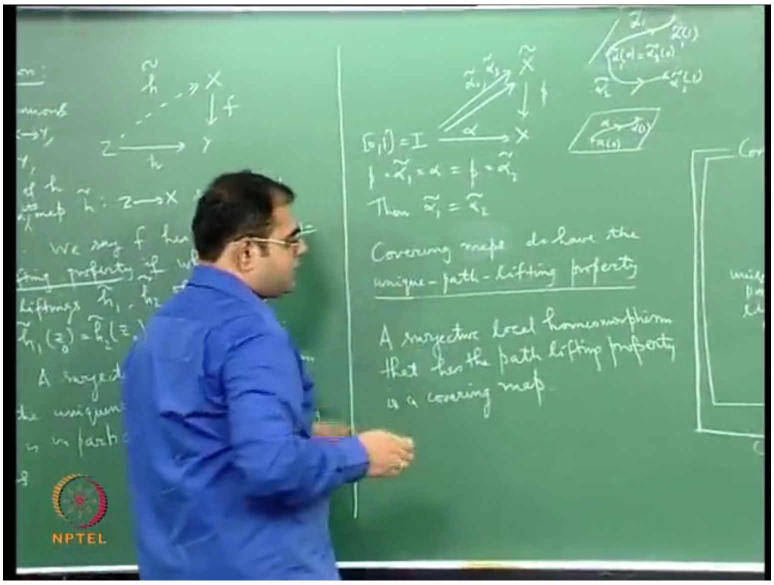

Covering maps let me do that of course, you know a when I say it is a property of a map of course, the source and the target spaces are also involved. So, if I say covering space of course, it also involves a covering map and if I say covering map also it involves the source and target spaces. So, sometimes by abuse of language one interchange is this, but anyway.

So, the point is you see you take covering maps, they are subjective local homeomorphisms which are of course, open maps and subjective local homeomorphisms have this uniqueness of path lifting property of course, uniqueness of lifting provided you know the lifting at one point is prescribed. I am not just saying that case take any 2 liftings they are the same that is not correct 2 liftings which agree at least 1 point is what I want. So, here also I am not just saying you take a path below there is a unique path above know.

A point a path below will give you a unique path above provided you fix a starting point which has to be a point you fix lying above this point below. So, you have uniqueness of lifting property and then you have the path lifting property. See a map can have any map can have a path lifting property which you can define in a very simple way a path below can be lifted to a path above. So, I can define it for any map; any map can be said to have a path lifting property if given a path below you can lift it to a path above so well you.

So, what I was trying to say is if you take a covering map you are going to get a uniqueness of lifting property because of this series self-implications and then I am go I have also told you I am also telling you here that there is a unique path lifting property which means that you also have the existence of lifting paths. And these 2 put together let me put it like this; these 2 put together give you the unique path lifting properly. So, this is the property that ensures that you know given paths below you can always lift it to a path above and that path is unique. Of course, whenever I say unique path above the starting point has to be fixed and the covering maps.

They have this path lifting property how this comes about is by what is called as the covering homotopy theorem. So, this is the. So, I will write it here covering homotopy theorem; it is the basic tool to link these ideas and the beautiful thing is I you know I started with this question you see a covering map is a surjective local homeomorphism which has a uniqueness of lifting property and I asked you what more should you add to a surjective local homeomorphism to make it a covering space the answer is as follows. You take a surjective local homeomorphism which has the path lifting property of course, if it has a path lifting property it has to be a unique path lifting property because the surjective local homeomorphism is going to imply uniqueness of lifting.

So, you add this property of path lifting to surjective local homeomorphism and what you get is a covering map. So, it is a very beautifully result. So, I will just signify that by you know putting a map like this putting an arrow like this. A surjective local homeomorphism, which has if the path lifting property has to be covering map; so, this will tell you in a certain way how covering maps behave. So, you can think of a covering map also as a surjective local homeomorphism with the path lifting property.

Of course uniqueness of the path lifting uniqueness will come because it is already a local surjective homeomorphism. So, well now what remains is somehow to explain the covering homotopy theorem. So, let me do that. So, let me state the covering homotopy theorem and indicate to you how the inverse image of a point under covering map is bijective to the fundamental group of the base. So, I need to explain that. So, I will do that next. So, let me write this down covering homotopic theorem.

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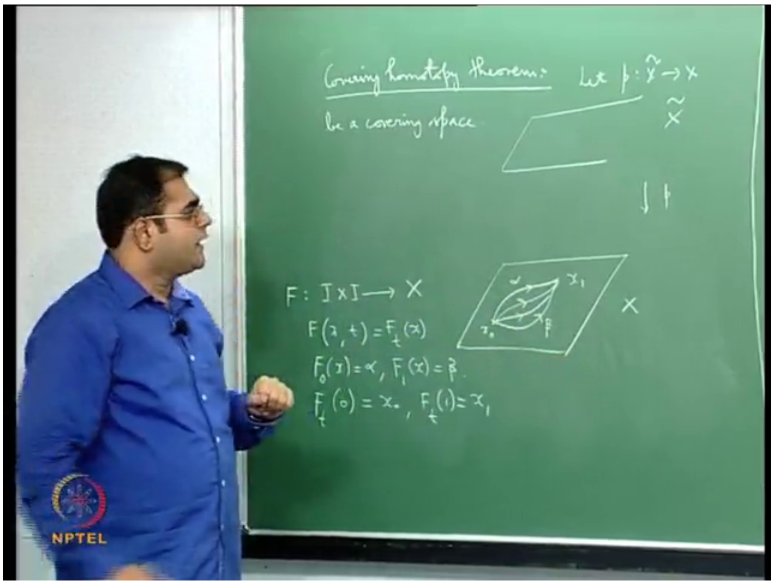

So, my situation is the following. So, this is as you can see this is this is a theorem that describes the property of covering maps and what it tells you is that you can lift homotopics. So, let me write down the statement let p from x tilde to x be a covering space, then it be a covering space let. So, I will first draw a diagram. So, that you know you can visualize what is happening. So, I have I have x here, I have p, I have I have x tilde and roughly what I have is. So, let me think, think of the following situation let us assume that you know there are there is a path here, it starts see a path alpha and suppose there is another path beta and let us assume that alpha and beta are homotopic fixed end point homotopic. So, you know. So, there is a family of paths like this. So, this means that you know.

So, I have a homotopy which is a map F from I cross I to x such that F of. So, this I is treated as the let me treat this is a time parameter, which means that you know F of x comma t is written as F sub t of x. So, and I want f 0 of x to be alpha, F 1 of x to be beta and whatever I get in in the intermediate paths they will be F sub ts all right and of course, you know F all these paths start at the same point and they end at the same point and they all ended the same terminal point. So, I will put that condition as F t of 0 is let me say x naught and F t of 1 the terminal point is always is it say x 1.

So, I have a I have 2 paths here alpha and beta on x which are homotopic. Now suppose I fix a point above x naught.

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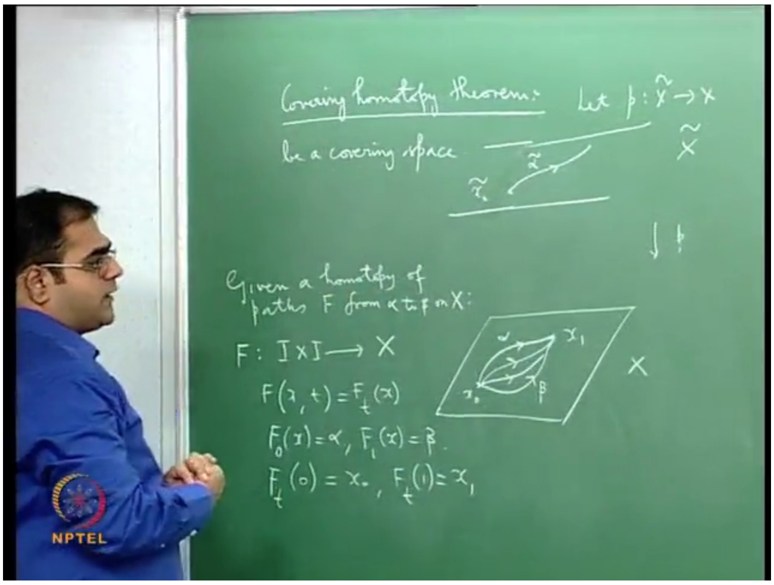

So, I fix a point let me call this as x naught tilde that is a point above and suppose I have a path here which lies above over alpha. So, I have a path alpha tilde then the conclusion of the theorem is that I can lift the whole homotopy to a homotopy of paths above. So, I will write that down and given. So, let me write this here given a homotopy of paths from alpha to beta on x which is given by this data and given a lifting of the path alpha of to alpha tilde. So, let me rub the rest on this board and given a lift a lifting alpha tilde of alpha to x tilde.

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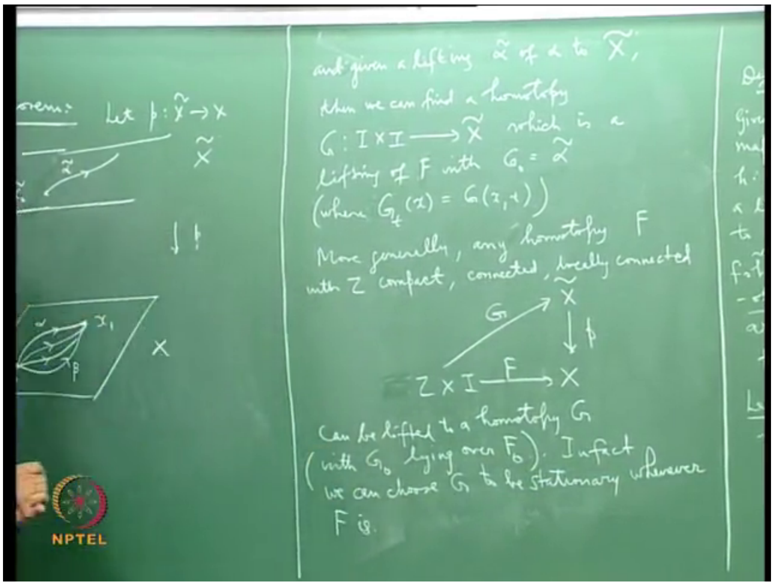

So, this is the picture this alpha tilde followed by p goes to alpha then we can find a homotopy G I cross I to x tilde which is a lifting of F which is a lifting of F with G 0 is equal to alpha tilde where of course, G sub T of x is G of x comma t.

So, when you put t equal to 0 you get G 0 of x and G 0 at of x lies above F 0 of x, which is alpha and G 0 of x is G 0 is equal to alpha tilde . So, this is. So, what I have written is actually its special of the covering homotopy theorem. So, this is not the full covering homeopathy theorem and the full covering homotopy theorem is more general, it is not just about lifting of maps, it is not just about lifting of homotopys of paths, it is about lifting of homotopys of maps and you will have to replace this I here by z which is a compact connected space. So, let me write that down. So, I wrote this particular case because this is this is a special case of the covering homotopy theorem. So, let me write this more generally any homotopy.

So, let me write this as F from Z cross I to x with Z compact and of course, you know 1 has to assume z is connected and locally connected I think in this case connected locally connected . So, any homotopy F can be lifted to a homotopy F. So, G. So, F is going to be like this, I do it able to lift it to homotopy G this is the homotopy can be any homotopic can be lifted to a homotopy, but of course, you need to you need to prescribe the value at a point. So, I should say with G 0 lying over F 0 .

So, G 0 is its just g of x comma 0 and F 0 is F of x comma 0. So, you see of course, when I say it can be lifted to homotopy G, with G 0 lying over F 0 this is this is understood because g followed by p is F. So, its G T will lie over F T for each T in I right so, but the point is of course, as I told you the this being a covering map this homotopy is a lift of F and. In fact, you can choose G in a very special way the theorem says that the G can be chosen in such a way that you know whenever F is stationary, that is F does not depend on time for a particular value of z, you can choose g also not to depend on time for that particular value of z.

So, let me write that down in particular in fact we can choose G, G to be stationary whenever F is; what this means is the following. So, let me write that down here that is.

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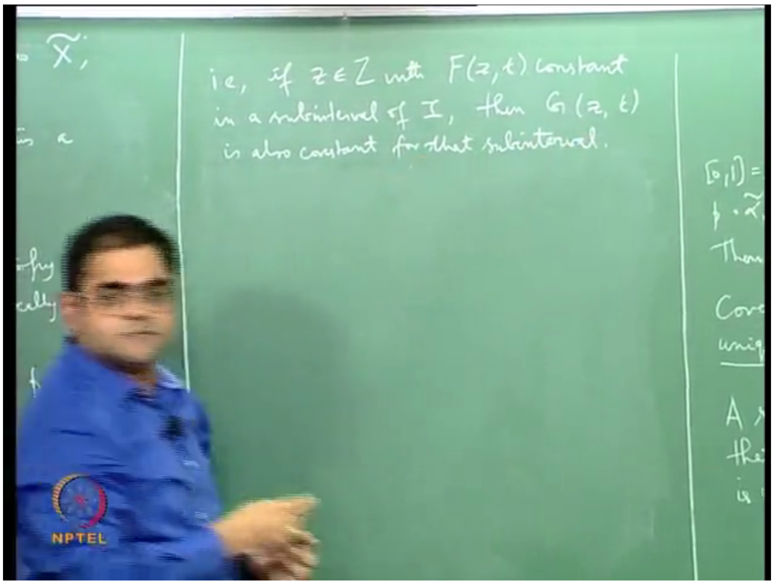

If is that is a point of z with F of Z comma t constant in a sub interval of I, then G of z comma t s also constant for that of course, when I say if Z is a small z belongs to capital z with the f of that comma t constant in a sub interval of time which means t should belong to that sub interval for all values of t for this value of z and for all values of T in that sub interval. Then the lift G we also have the same property namely g of the z comma t will also be constant for that sub interval. So, F of z comma t being stationary for a particular value of z means that for that particular value of z you vary time it does not change and the same property will be true of G also.

So, you can see that this is the general segment of the covering amount of a theorem, this is a special case that you get when you put z equal to I. And now this has beautiful consequences right which I will try to explain.