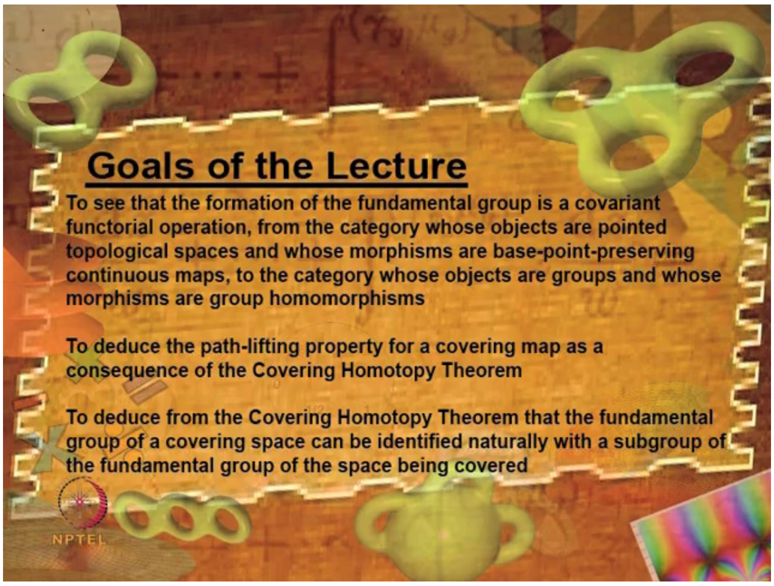
An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

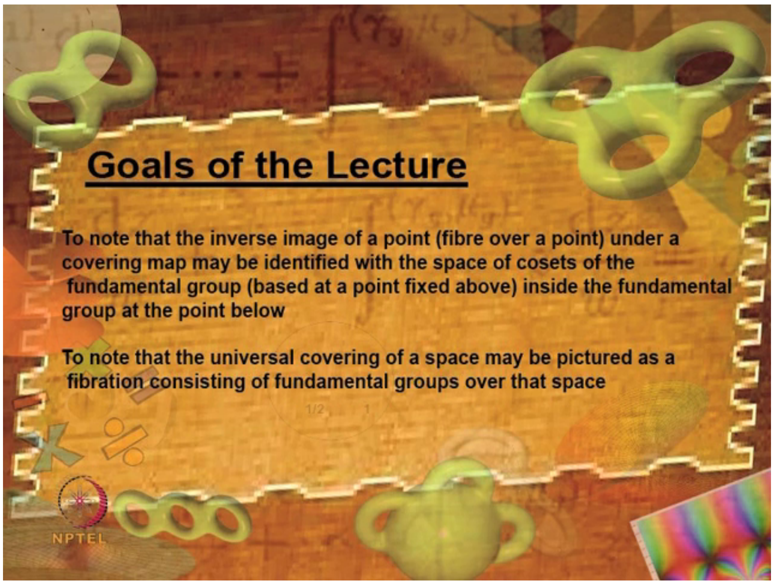
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 11
Fundamental Groups as Fibres of the Universal Covering Space

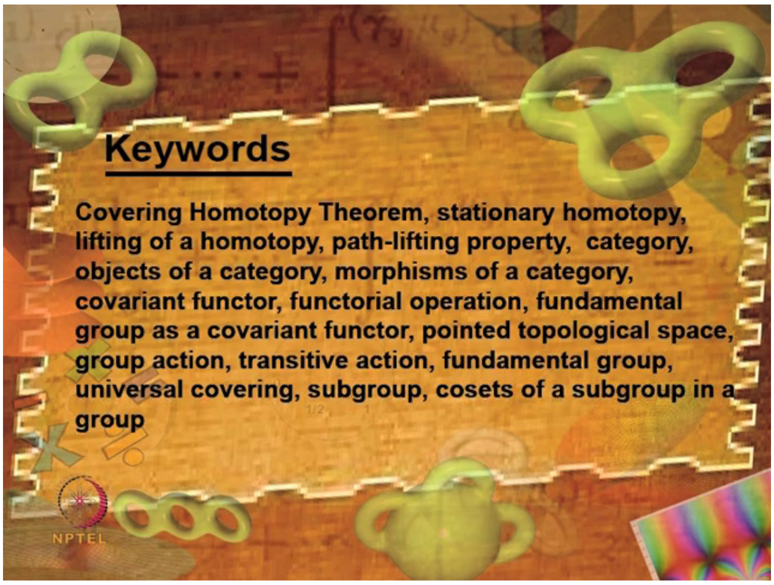
(Refer Slide Time: 00:10)



(Refer Slide Time: 00:21)

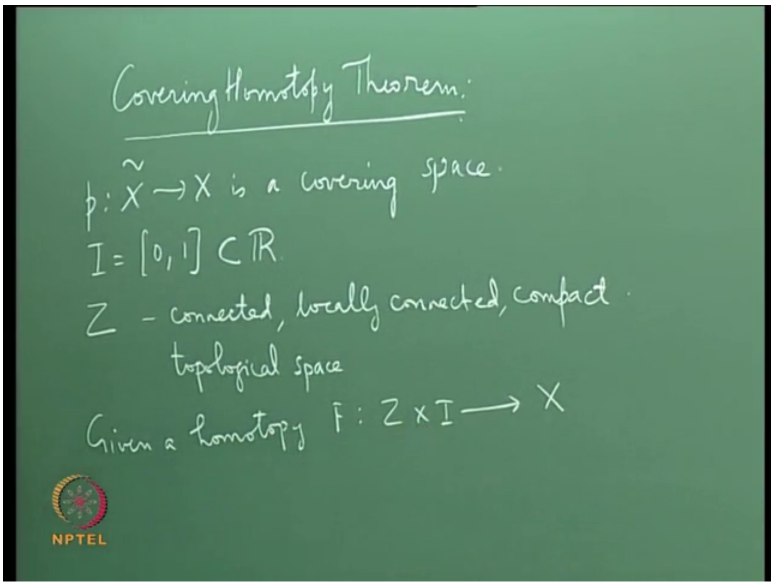


(Refer Slide Time: 00:21)



We will continue with this with the ongoing discussion; that is, to try to explain why it is that the inverse image of a point under covering map is set theoretically bijective to the fundamental group of the space being covered the base space. So, in this connection yesterday I was talking about the covering homotopic theorem. So, that is the fundamental theorem that we need to understand in this situation. So, let me recall it. So, let me write that down here covering homotopy theorem.

(Refer Slide Time: 01:10)



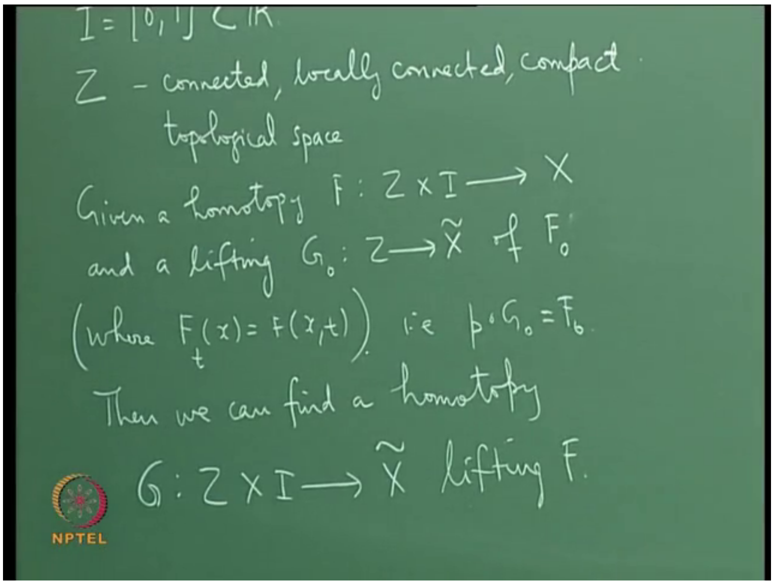
So, in the case of the covering Homotopy theorem what you have is of course so, is it? So, p from X tilde to X is; it is a covering space. And I is the open unit interval the closed unit interval in in the on the real line.

And Z is a connected a locally connected topological space which I will further assume to be compact right. So, of course, please remember that all the topological spaces that, we are going to consider we are going to assume that they are all hausdorff we are going to assume they are you know arc wise connected locally arc wise connected. And also, even locally simply connected, but of course with Z you can make a little bit of relaxation namely instead of arc wise connected, you can just assume Z is connected.

And instead of assuming locally arc wise connected, you can assume there is locally connected which of course, means that every neighborhood of a point, contains an a sub neighborhood which is connected. And of course, I am assuming it is compact and the situation is as follows. So, you are given a homotopy F from Z cross I into the base space of the covering space.

And so, this homotopy at time t equal to 0 is going to be a certain map from Z to X time being thought of as the variable in I. And at time t equal to 1, it is going to give me another map from Z to X and the homotopy is just the expression of the fact that the map at time t equal to 0 is being deformed continuously into the map at time t equal to 1. So, this map at time t equal to 0, which is a map from Z to X. Suppose there is a lifting of that map that is already prescribed.

(Refer Slide Time: 04:22)

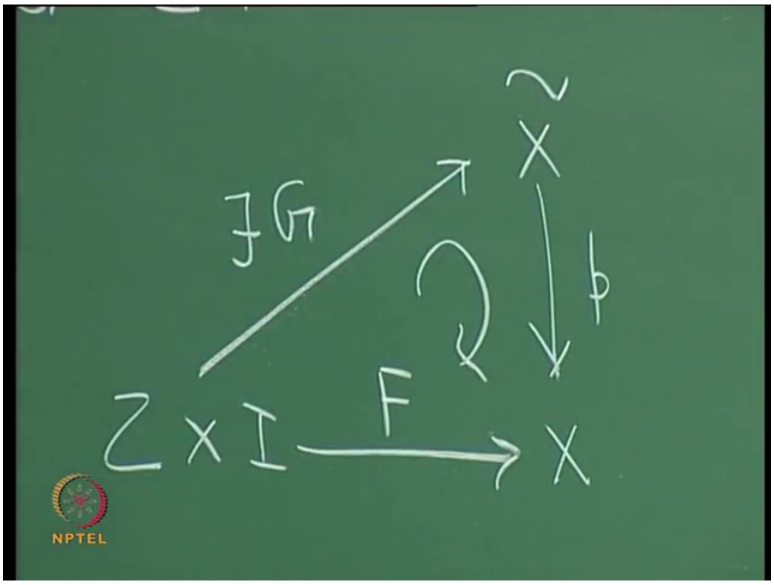


So, and a lifting G 0 from Z to the X tilde of F 0 of where you know of course, F sub t of X is F of X comma t. So, this is the standard notation. This is a notation that we have we are using.

So, then so, basically F tells you that F is a homotopy from them between the map F sub 0 and F sub 1 which are 2 maps from Z to X and the initial map F sub 0 has a lift g sub 0 which means lifting means of course this map followed by p is going to give me F 0. So, let me write that, that is G 0 followed by p is F 0. And what the covering homotopy theorem tells you is that the whole homotopy capital F can be lifted to a homotopy capital G into the covering space X tilde

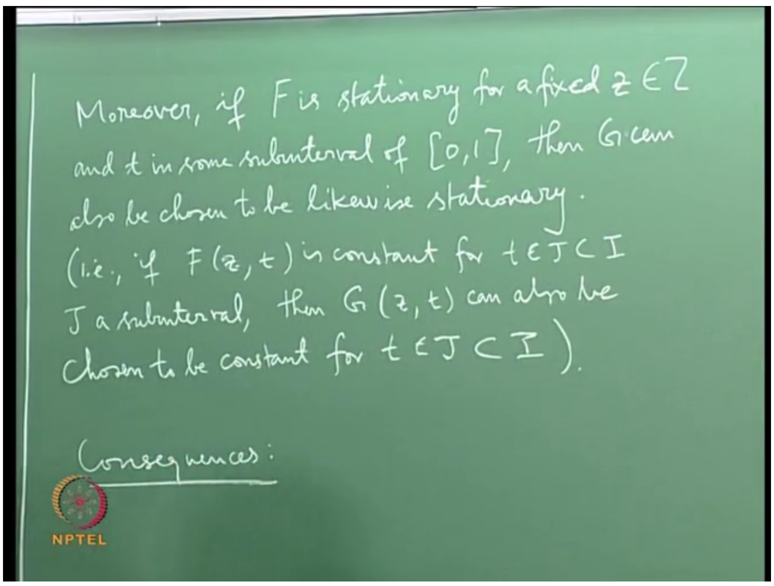
So, what is happening is that each of these F sub t's are being lifted to g sub t's and all the g sub t's with of course, g sub 0 being the one that is already prescribed, all the g sub t's they fit into a g, which is which turns out to be a homotopy on X tilde. So, that is a covering homotopy theorem. Then we can find a homotopy g which will be from Z cross I into X tilde lifting F. So, this is the covering homotopy theorem. So, if I want to put all of this in in a single diagram.

(Refer Slide Time: 06:45)



So, one can write Z cross I. So, we have so, this is X, this is F, and then this is there the covering map this is the covering space, this is the base space, which is being covered by this top space, and then the covering homotopy theorem says that you can find there exists a g which is a homotopy into X tilde such that g followed by p is just F. So, I put this circular arrow to tell you that this map followed by this map is equal to this map. Now we would like to and the, and then there is one more statement there is one more part of the a covering homotopy theorem which says that you know if so, let me write that down.

(Refer Slide Time: 07:42)



Moreover, if F is so, let me say. So, let me write this on F is stationery for a fixed Z belonging to Z, and t in some sub interval of 0 1. Then g can also be then G of then G can also be chosen be chosen to be likewise stationary. What does this mean? This means that is if you know F of X comma t F of z comma t is constant for a t in j which is an I J sub interval J a sub interval. Then then g of Z comma t can also be chosen to be constant for t belonging to J in I. So, this is the additional statement. So, now what I am going to do is, I try to first of all derive certain consequences of this covering homotopy theorem.

And then these cons this this consequences will tell you certain things first of all, they will tell you that if you have a covering map, then it has the unique path lifting property. That is one important consequence. Then it will also explain, why the fiber that is inverse image of any point is going to look like at least set theoretically like the fundamental group of the base, in the case the covering is a universal covering. So, these are the 2 consequences that I would like to immediately explain. So, what I will do is I first with so let us see some consequences. So, before I continue I am going to go over to the right side of the board.

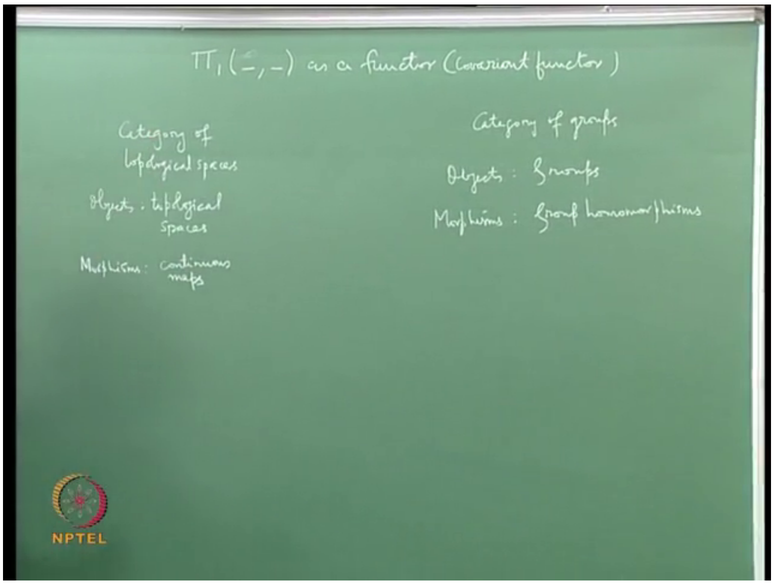
And try to explain that the fundamental group the operation of forming the fundamental group of a topological space is what is called you know it is called a Covariant Functor, from the category of topological spaces to the category of groups. So, I am I will not go into a serious definition on what a category is basically a category in mathematics consists of objects and morphemes. So, the usual categories that we are we are familiar with for example, category of sets which contains out which contains objects as sets and the orphisms are just maps of sets. Then you can also look at categories of for example, groups category of groups which is going to be the category which has objects which have groups and the morphisms are going to be group homeomorphisms.

Similarly, I can define the category of brings you know category of abelian groups if you want then you can also define category of topological spaces, the objects are going to be topological spaces, and the maps are going to be continuous maps. So, basically the idea of a category is like it consists of 2 data 2 sets of data one is the objects of the category and the other is the maps. Between these objects and the reason why this generalization is so important is because you can have general categories in which neither the neither or the objects varies sets.

And therefore, neither are the maps really maps of sets. So, you can form more complicated objects than just sets, and you can have for a map something is not a map of sets, but more data are completely the data which is not which does not run is as our intuition goes. So, it is a very important technical tool the concept of a category. And what a functor is something that maps one category to the to another category. So, since the category consists both of objects and morphisms. So, what the functor does is that it up it maps each object of the source category to an object of the target category and it maps each morphism of us of the source category to a morphisms of the target category and I just want to say that the fundamental group is forming the fundamental group of a topological space gives you F functor which is called coherent a coherent functor from the category of topological spaces to the category of groups. So, let me explain that very, very quickly.

So, you know suppose let me write this as pi 1 of X or let me say pi 1 comma dash comma dash as a functor, and I will put it in brackets as covariant functor.

(Refer Slide Time: 13:55)

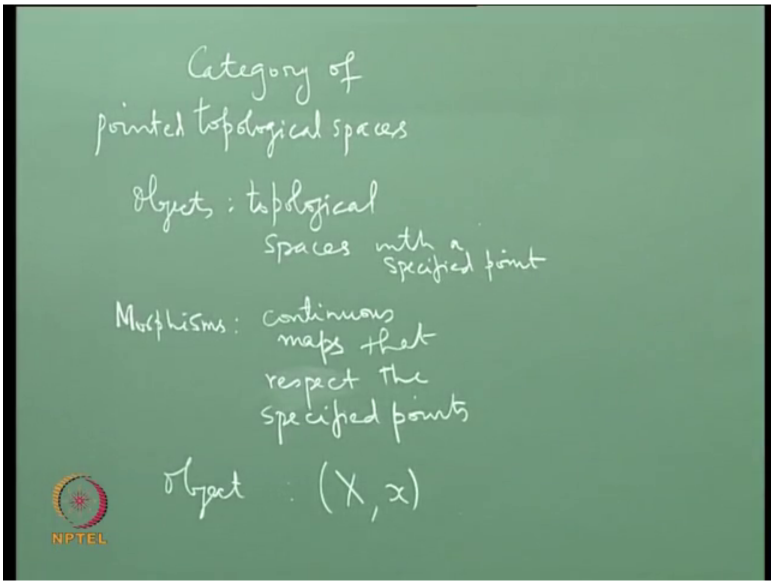


So, what does this mean? So, I have on this side the category of topological spaces. And here therefore, the objects are topological spaces, and the morphisms sometimes referred to as the arrows of the category. The morphisms are going to be continuous maps. So, the morphisms are supposed to be arrows going from one object to the category to another object of the category, in this case these are continuous maps from one topological space to another topological space. And on this side, I have the category of groups where you know the objects or groups are groups, and of course, the morphisms are group homeomorphisms.

And what does one mean? There is a functor well in fact, I have you know I have to make a further restriction, you know when I define the fundamental group this is the homotopy classes of loops that is path starting end at end starting from and ending at a given point. So, I had to fix a bit base point. And of course, I told you if your topological spaces arc wise connected, then all these various fundamental groups are based at various points they are all isomorphic.

So, let us keep track of the base point. So, what I will do is I will make this a little bit more restrictive, I will say topological spaces with a specified point. So, these are called pointed topological spaces. So, the data is a topological space be the point on it. So, let me put; so I will just modify this to point, at what does this mean this means topological spaces with the specified point.

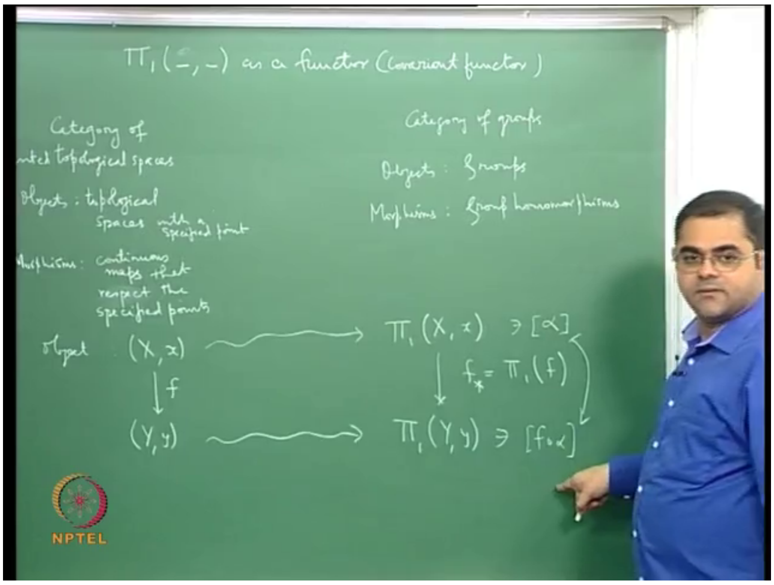
(Refer Slide Time: 16:33)



And then the morphisms so, if I take 2 topological spaces, they are going to have each one is going to have a point specified, and then I do not just consider all continuous maps from this space to that space.

I only consider those maps which map this point to that point. So, continuous maps that that preserve or that respect the respect the specified points. So, what happens see in this case an object an object is going to look like a pair capital X comma small x, all right. And what is what does my functor do?

(Refer Slide Time: 17:45)



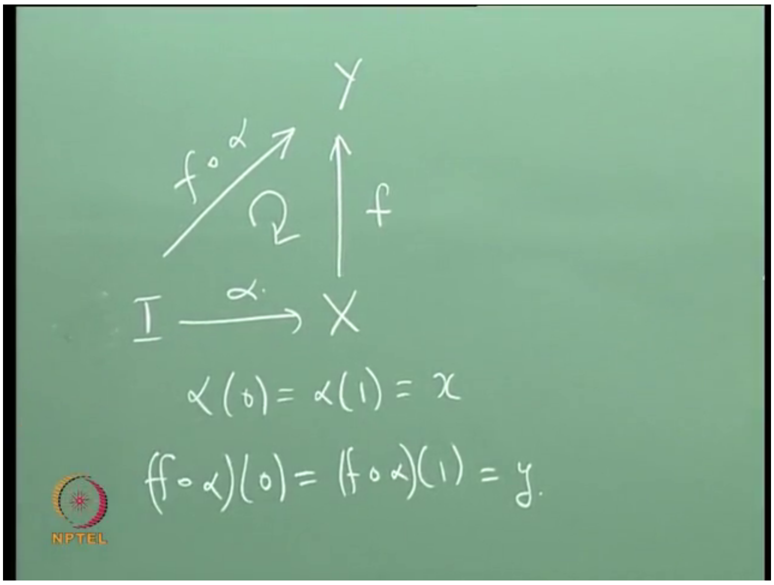
What it does is given such an object I will get a group. Namely, the first fundamental groups at this at the pole based at the point small x for the topological space capital X. So, for an object here I am getting an object there which is a group, all right. And what the functor is supposed to do a functor is a is a generalization of function the notion of function, but of course, a function has to go from one side to another side, but here you know the functor goes from something that is much more complicated than a set.

So, what does it do to? So, it takes objects to objects, and it has to take maps to maps. So, what is a map in this category? So, I take another topological pointed topological space, namely a topological space capital y and a point small y in that topological space. And of course, I will get the first fundamental group of that topological space based at that given point. And well, I also if I take a if I take a continuous map F, then this map is going to take small x to small y. Because we are going to take we are only looking at maps, the morphisms are only maps which respect this specified points. So, it has to map small x small y.

And what I am going to say is that, here I am going to get a you know what I should get there it should be a group homeomorphism. Because on this side the morphisms a group homeomorphisms. The objects are groups the morphisms are group homeomorphism. From this side the objects are pointed topological spaces maps are continuous maps which preserve the specified point. So, given me a map here I should get a map that namely which means that on that side I should get a group homeomorphism. What is the group homeomorphism s people. Usually write this as F lower star? And this is supposed to also written as pi 1 of sometimes you can also write it as pi 1 F if you want. So, we write it as pi 1 of F just to show that you know this is what is carried 2 by pi 1.

And the more common notation F flows are, but what is this group homeomorphism. So, it is very, very simple give me a loop base at X. This is the homotopy classes of loops base at X. And this is a homotopy classes of loops based at y small y. And what I will have to do is give me a loop at X, I have to produce for you a loop at y. And that is a there is a very easy way of doing it. So, you know let me start with. So, let me define this. So, I start with alpha, alpha is an element here. So, it is an equivalence class of loop at X and a loop at X is a path starting at X and at and ending at X. So, it is going to be like this. So, you are going to so, you know alpha is just a map from I to X.

(Refer Slide Time: 20:54)

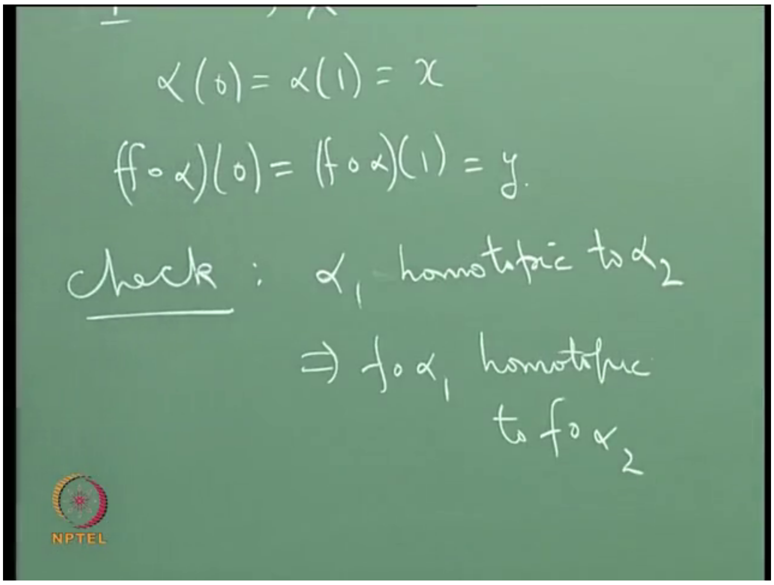


It is a continuous map from I to X with of course, alpha of 0 the same as alpha 1 which is the point small x. So, this is the loop at X, and what am I supposed to do? From this I am

I am supposed to cook up a loop at y, all right. And this is an obvious way of doing it what I do is well you know I just follow this up by this map F to Y and if I take the composition namely. So, let me rub this and write the alpha here, and I take the composition which is just alpha followed by F. So, that this diagram commutes; that means, this followed by this is this by definition. So, then you know these are continuous maps. So, alpha F circle alpha is also continues map therefore, this is also here, a path in y, and you can check that what is F circle alpha of 0, it is going to be F of X which is Y.

And F circle alpha of 1 will also be y. So, you can see that F circle alpha is at 0, it is same as F circle alpha at 1 and it will be equal to y. So, this is the obvious way in which given a loop on X based at the point small x, I am able to get by composition with F a loop on capital y base at small y. And with a little bit of work you can prove that, if alpha is changed up to homotopy then F circle alpha will also change up to homotopy.

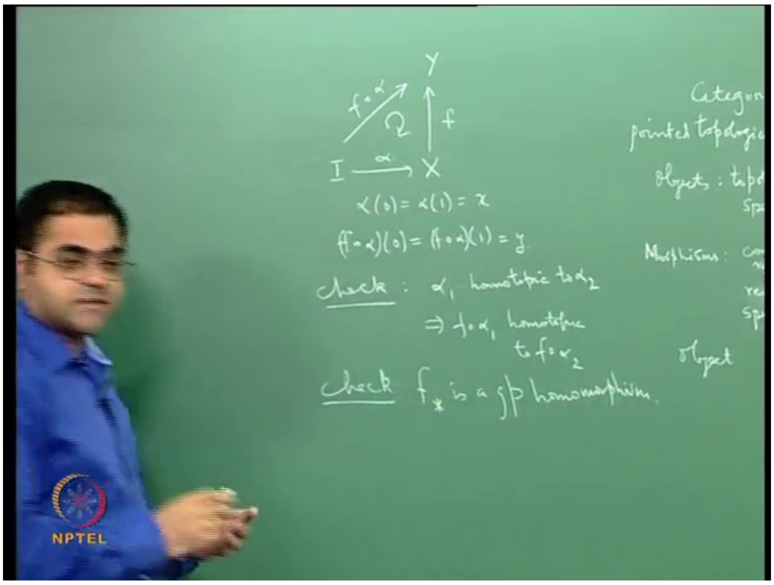
(Refer Slide Time: 22:42)



So, you can check that a alpha 1 homotopic to alpha 2 implies F circle alpha 1 homotopic.

To F circle alpha 2, and this will ensure that the map gotten by sending alpha to you know to the element F circle alpha is a well-defined map.

(Refer Slide Time: 23:46)

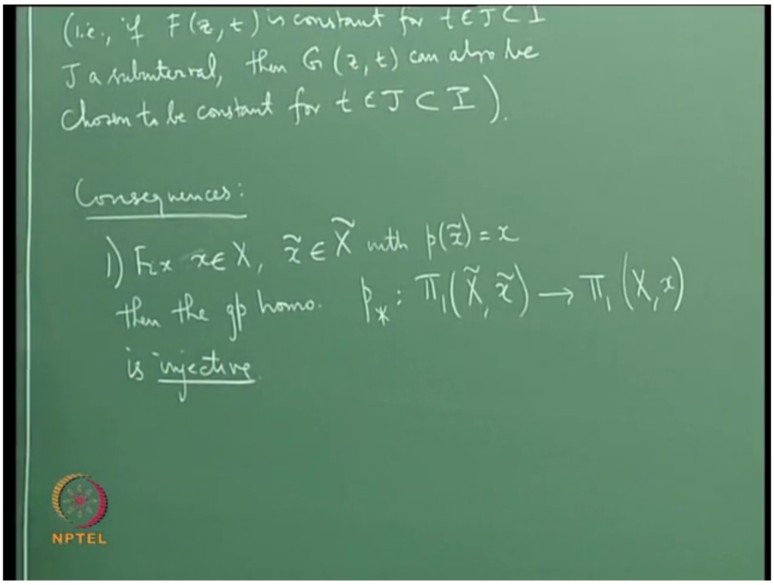


And you can check that this in fact a group homeomorphism. Namely, you can check the identity element goes to the identity element. You can check that it preserves the product that. So, the product here is given by the concatenation of 2 paths and you can check that you take the concatenation of 2 paths, and then you take the path here. That is the same as taking the images of the 2 paths and then concatenating them. So, it will be a group homeomorphism. So, you can check that you can also check that F above star is a group homeomorphism this can be checked and in this way, you get a functor.

So, this is what and we say it is a functor because, we say it is a covariant functor is because the arrows from this side or in the same direction as the arrows. On this side you can have functors in which an arrow on this side is transformed to an arrow in the reverse direction. And those kind of functors will be called contra variant functors. And you know, usually these are the functors that arise when you try to pull back maps if you have if you have a function a function of the certain property on the target space. And you give me a map in the target space if I take the pull back of that map I will get a function on the source space. So, those are the kind of operation that you give rise to contra variant functors, but these are coherent functor.

So, given this background the first consequence I want to make is that if you have a covering space you take a covering space like this. And let us let me fix this obey a point here in in capital X and a point above it so, fix.

(Refer Slide Time: 25:16)

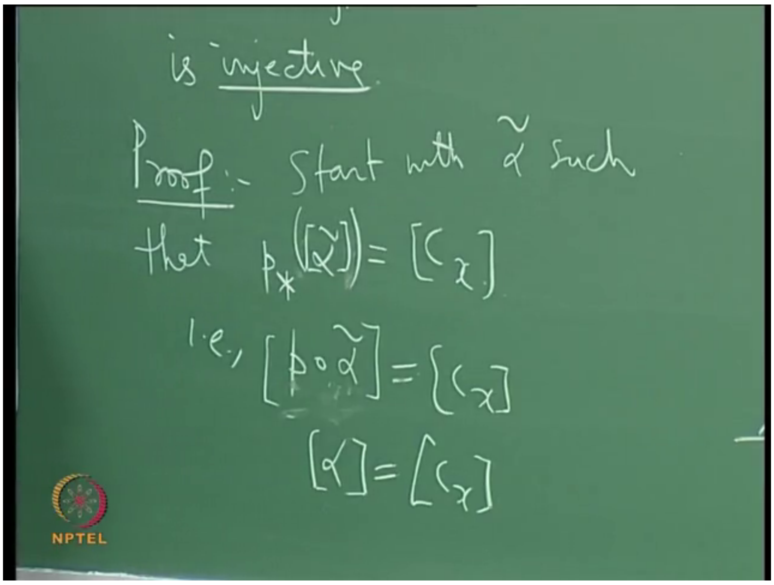


So, number 1 fix X belonging to X small x belonging to capital X, capital X tilde belonging to capital X tilde small x tilde belonging to capital X tilde with p of capital X tilde is equal to x; that means, you choose a point below and which is of the point above which goes to this point. Then you know then the group homeomorphism, the group homeomorphism p lower star which is going to be from the first fundamental group of the top space based at the point X tilde to the first fundamental group of the base space X base at x is injective.

So, the in the first consequence is that the if you take the map, that is going to be induced by the first fundamental group as a functor when you apply it to this this covering map, what you will get is a group homeomorphism, which is injective in other words the fundamental group of the top space will be identified with a subgroup or the fundamental group of the space below that is what this means. Because once you have an injective group homeomorphism then the source group can be identified with it is image which is a subgroup of the target group. So, this is the first consequence. And it is not very difficult to prove using this covering homotopy theorem.

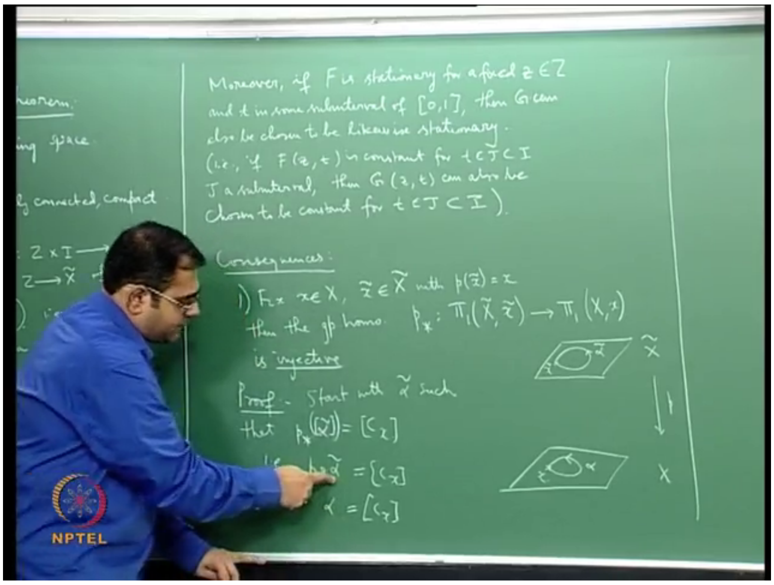
So, let us try to prove it.

(Refer Slide Time: 27:07)



So, proof is well suppose so I will draw a diagram here.

(Refer Slide Time: 27:18)



So, here is my, it is my covering space and here is my covering map or covering projection, and here is my base space. And I have fix this point small x here and a point above it capital X tilde which is mapped to small x under the projection. And what do I have to show I will have to show that if there is a point here if there is an element here that goes to identity namely an element in the kernel in that element has to be nothing else, but the identity.

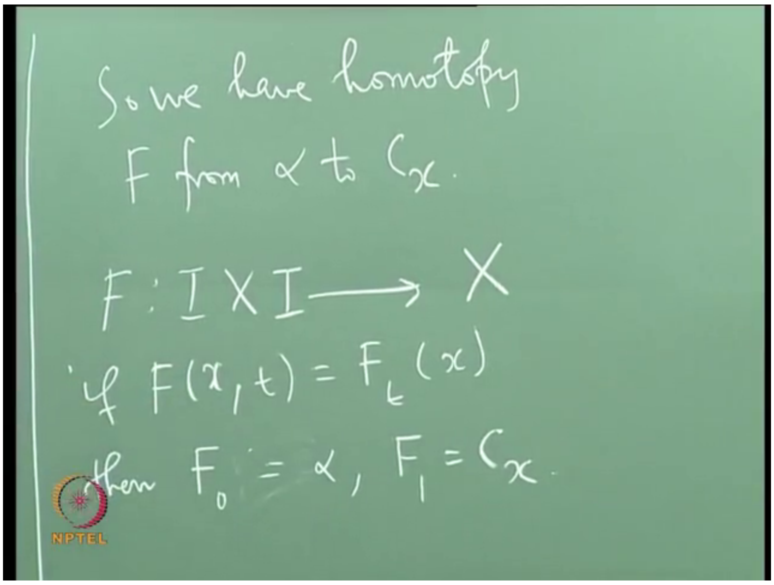
So, I have to just say the kernel is identity they identity sub group. So, I will start with I start with an alpha tilde such that that p lower star alpha tilde is a p lower star of the element alpha tilde. And I say element alpha tilde I should I should take the equivalence class. So, alpha tilde is a path here alpha tilde is a path here above. And I am assuming that the element of the fundamental group defined by this path namely it is homotopic class which I denote by putting square brackets is the trivial element here. And what is the trivial element the trivial element is the constant path at the given point the path which is which just which is just the point all the time so in the stationary path.

So, let p lower start of alpha b c x. So, then I put c x, what I mean by c x is this is the constant path at x; namely the map from the unit from the unit interval closed unit interval to the whole of the interval being mapped just to the point X. That is also by definition of path, the image is just a point. So, it is called the constant path, and that is the you see that is going to be the unit element the identity element for the group below. So, suppose this is true, but what is the meaning of p lower star of alpha tilde. If you go by that definition p lower start of alpha tilde is just alpha tilde followed by p. So, what I am saying what this. So, this is just same as saying first apply alpha tilde, then apply p. Then this is the constant path, but alpha tilde followed by p let me call this is alpha.

And what is alpha you just simply project this path below. So, I am going to get a path here. So, this path is going to be alpha. So, alpha is just you just project alpha tilde below right. And I will have to say what I will have to do is that I will have to tell this that this alpha tilde is the trivial element, this this equalance class is a trivial element. And how do I do it I make use of the covering homotopy theorem. So, what I do is the following thing see. First of all, again here I should write p circle alpha tilde alpha tilde is a path above p circle alpha tilde is a path below and then I should a strictly speaking take the homotopic class. So, here also I should and I am calling this p circle alpha tilde is alpha. So, I should take homotopy class of alpha.

So, what does this last inequality equality tell you? It tells you that alpha and c x are homotopic.

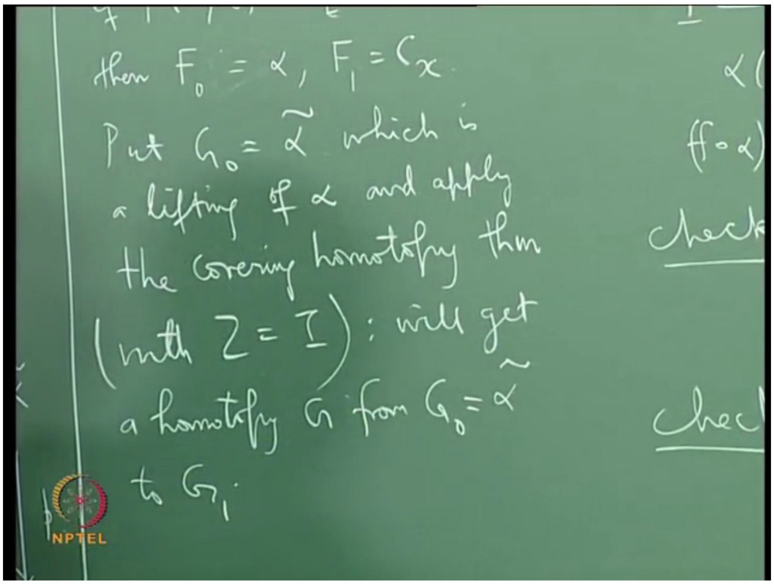
(Refer Slide Time: 31:09)



So, we have a homotopy F from alpha to c x. So, namely you have. So, how was it written I write this yes F is a map from I cross I to X, such that F of I, if I call F of X comma t as F t of X, then F 0 of X F 0 is just alpha and F 1 is cx.

This is just the a homotopy from alpha to c x which x is because of this. So, mind you if you go back to the covering homotopy theorem to apply the covering homotopy theorem I have to replace Z by I, and well I can do that because the condition on Z is satisfied by I, I certainly connected locally connected and compact. So, I can apply the covering homotopy theorem. And of course, to apply the covering homotopy theorem, I also need to have a lifting of F 0. And you see F 0 is alpha and there is a lifting of alpha, we namely alpha tilde because that is what we started with.

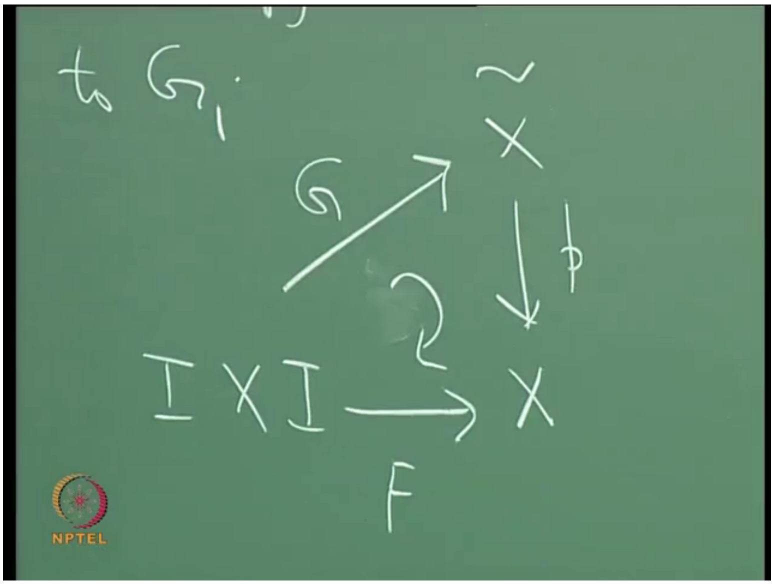
(Refer Slide Time: 32:57)



So, put G 0 is equal to alpha tilde which is lifting of alpha, which is a lifting of alpha, and apply the covering homotopy theorem, homotopy theorem with of course, capital Z equal to I. So, what you what will tell you is that you will get we will get a homotopy G from G 0 which is alpha tilde to G 1. So, basically, I am going to have a map.

I am going to have a diagram like this of course, with Z replaced by I. So, I am going to have. So, what this is going to tell me is that of course, g is lift of F so, each g sub t is a lift of F sub t, and in particular G 1, it is going to be a lift of the constant path let me write this one for a moment.

(Refer Slide Time: 34:35)

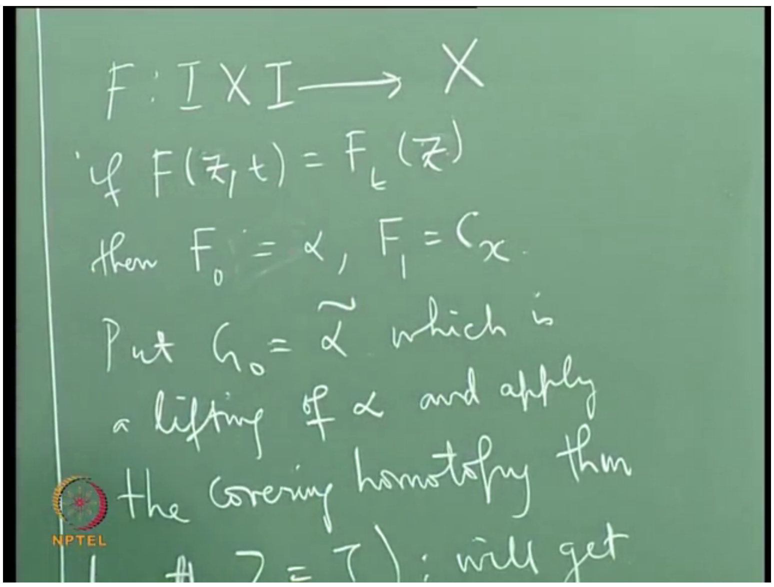


So, p so, think for a minute, I think there is a problem with notation somehow. I fix this as X and it is better not to call.

Student: (Refer Time: 35:04).

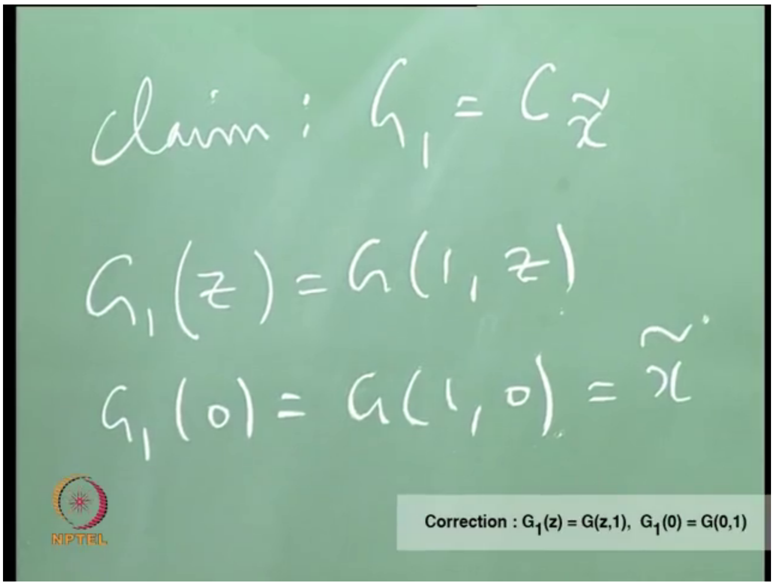
You know see this X is a fixed point here. This point X is a fixed point below. And then I am also using the X as the first variable here. Which is really bad notation? So, what I should do is let me call let me put Z, now it is better to put Z let me put Z and assume that Z is in I so this is very just to avoid notation of complications.

(Refer Slide Time: 35:42)



So, notice that F sub t, I think the easiest way to do it is first show that given a path below there is unique lift. And G 1 is lift of cx it has to be c x tilde, because that is c that is one lift and that has to be the unique lift. So, essentially that is what I have to say.

(Refer Slide Time: 36:35)

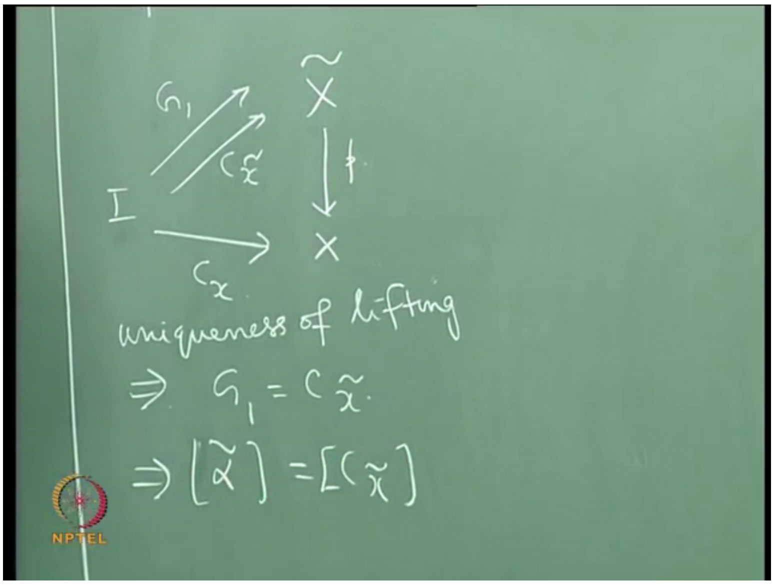


So, it appears that so, the claim is that G 1 G 1 is just a constant path at X tilde claim is G 1 is a constant path at X tilde, once you fix once you prove this claim what you are going to prove is that g is going to give you a homotopy between alpha tilde.

And constant path X tilde, and that that is the same as saying that this homotopic class alpha tilde is the trivial element that the identity element of this group which is what you want to say that the kernel is trivial. So, the claim is G 1 is the constant path of X tilde, and how does one verify this one verifies this by using this this property namely that you have uniqueness of lifting if you have a local homeomorphisms. So, you see G 1 the paths the path G 1 of Z is just G of 1 comma Z.

And you know, if I put Z is equal to 0, G 1 of 0 is going to be G of 1 comma 0, and you can see that this is going to G of 1 comma 0 is going to be a simply X tilde. And you also know so, in other words what happens is from I to X tilde there are 2 maps.

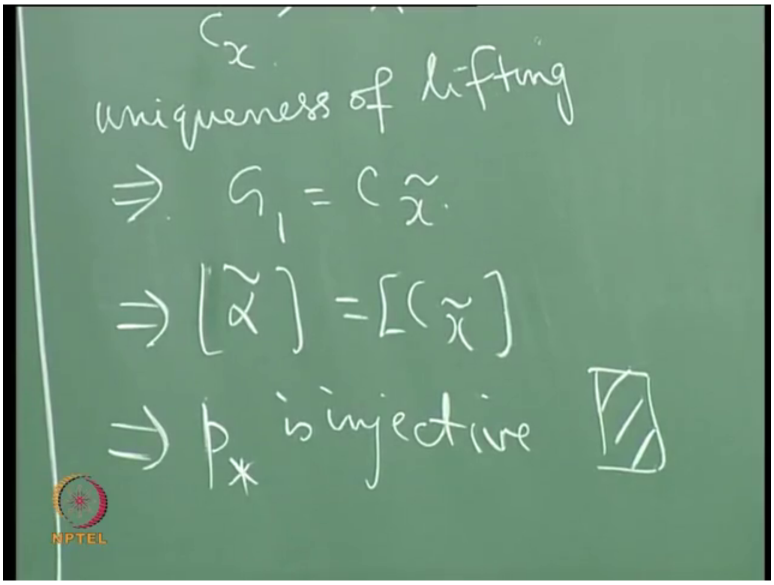
(Refer Slide Time: 38:23)



One is the path G 1, Z going to G 1 Z. Then there is the other map which is the constant path which is c sub X tilde. And the if you follow it by p, what I get in both cases I get the constant path below. So, it means that both G 1 and c x tilde are liftings of cx. So, there are 2 maps which lift cx. And they agree at Z equal to 0. And of course, this source space is connected. So, by the uniqueness of lifting you will get G 1 is equal to cx tilde and that completes the proof. So, uniqueness of lifting implies G 1 is equal to c x tilde.

And once you have this you are just saying that g is oh g is of course, a homotopy from G 0 to G 1. So, you have just saying that the alpha tilde you started with is homotopic to c x tilde. And which means that the homotopic classes are the same. And this is enough to tell you that P lower star is injective.

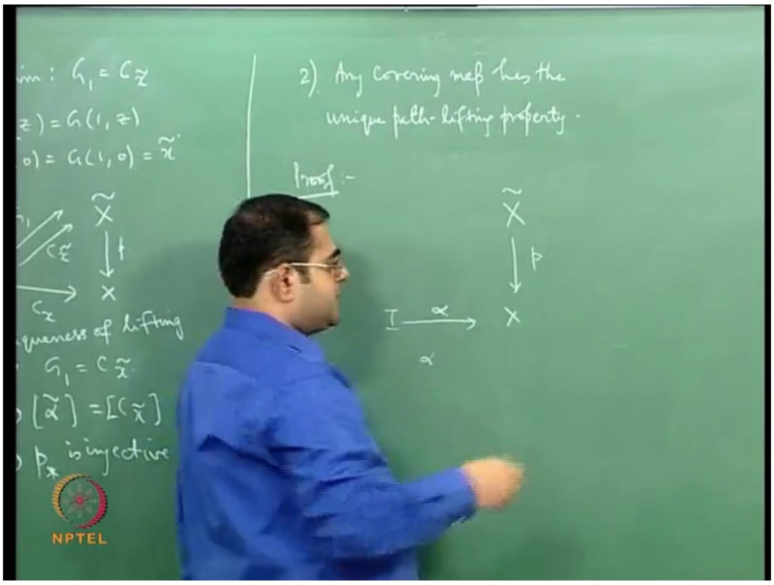
(Refer Slide Time: 40:03)



So, that is supposed to indicate end of a proof following various books. Normally you would have a solid block there, but I do not want to fill it chuck. So, well so, this is the first important consequence; when you have a covering space with the fundamental group above. It is image in the fundamental group below, the image it can be identified with this in a with it is image because the induced map is injective it is an injective group homeomorphism. And in particular notice that if X tilde to X is a universal covering. Then X tilde is simply connected. Therefore, the fundamental group above is trivial and therefore, the image of this in the fundamental group below is going to be just the trivial subgroup. So, this is the first consequence.

Then let me write down the second consequence, which we have more or less seen here, but anyway let me write it out. The second consequence is that a covering space has the unique path lifting property.

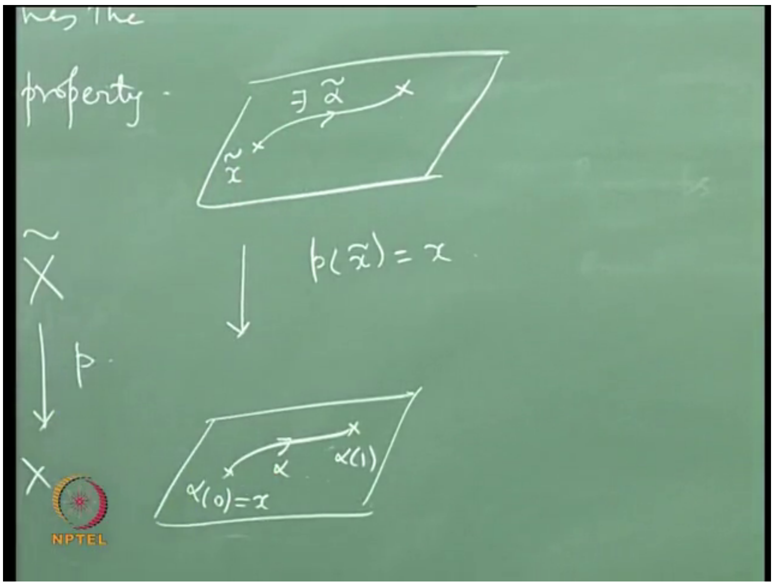
(Refer Slide Time: 41:29)



Any covering space any covering map if you want, has the unique path lifting property. So, what does this mean? It means proof given. So, you are given this covering space.

And I am given a path alpha, then alpha is say a path starting at a particular point of it X and ending at a certain particular point of X. So, alpha starts at alpha of 0 and ends at alpha of 1. So, if I draw a diagram, it is it is something like something like this.

(Refer Slide Time: 42:28)



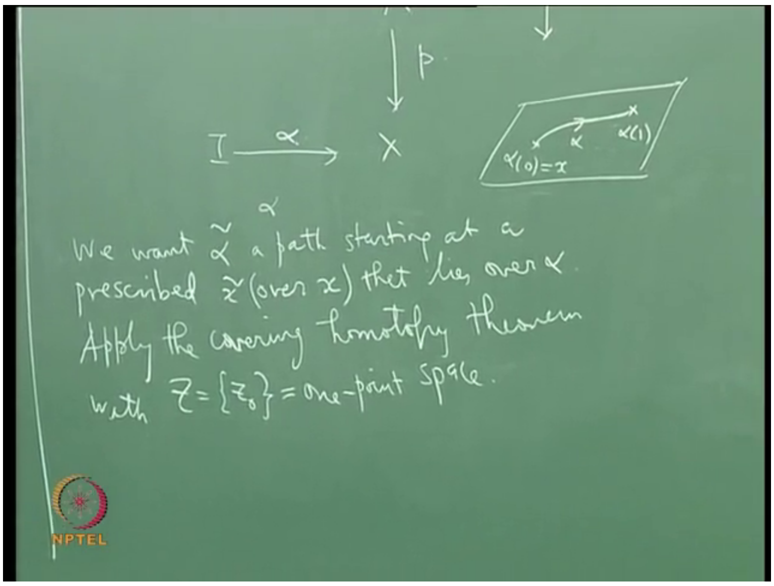
I have alpha here, this is alpha this point is alpha of 0, this point is alpha of 1. And so, this is an X, and in the covering space, suppose you fix for me a point above alpha power

0; that means, you fix a pre-image of alpha over 0. So, if you want let me call this alpha 0 is X, and I will call this point above X tilde with p of X tilde going to X. Being equal to x. So, X tilde the point that we have chosen above.

And what is the unique path lifting property it says that given a path here, you can find a path there, there exists a path alpha tilde, above which lies above this this path namely that path followed by p is going to give you this path. That path projects down to this path. And further that this path above is unique of course, uniqueness of that path would follow because of the uniqueness of lifting property that is always true of a local homeomorphism as I told you yesterday. So, because if you have 2 paths, both starting at the same point and going down to the same path below. The uniqueness of lifting will tell you that these 2 paths are the same.

So, I will just have to show you the existence of a lifting given an initial point above. And how do you get that? You get that in the following way. Namely, what you do is that you apply again the covering homotopy theorem, but this time think of Z as a one-point space. So, take Z we so, let me write this here we want.

(Refer Slide Time: 44:24)

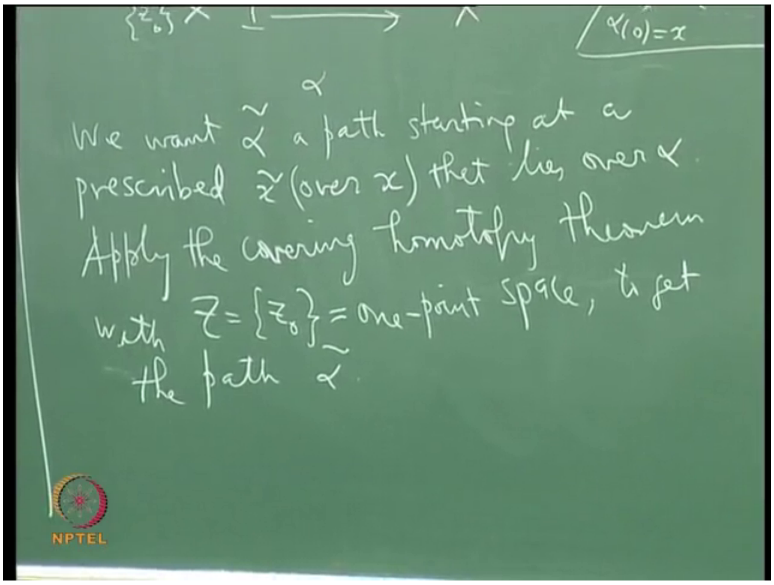


We want alpha tilde a path starting at a prescribed X tilde over X. That lies over alpha. So, you want an alpha tilde, that lies over alpha namely alpha tilde projects down to alpha, and this alpha tilde should start at X tilde.

Now you apply the covering homotopy theorem with e Z is equal to say Z naught one-point space. You take for Z one-point space which is a in a you know in every way compact connected and locally connected, because there is only one point there is nothing to check. And so, you can apply it to apply this situation apply the covering homotopic theorem to this situation. So, I get this is Z naught cross I so, this this map alpha which is a map from I to x.

Can also be thought of an map from Z naught cross I to X just that you just make alpha do not care about this Z naught. So, it is supposed to a function of several variables you need not depend on a certain variable, it is still a function. So, this still makes sense as a map like this and covering homotopy theorem will tell you that that is going to be an alpha tilde, which is a lift of alpha. And with alpha tilde of 0, already prescribed as X tilde and lying over alpha of 0 which is x.

(Refer Slide Time: 46:47)

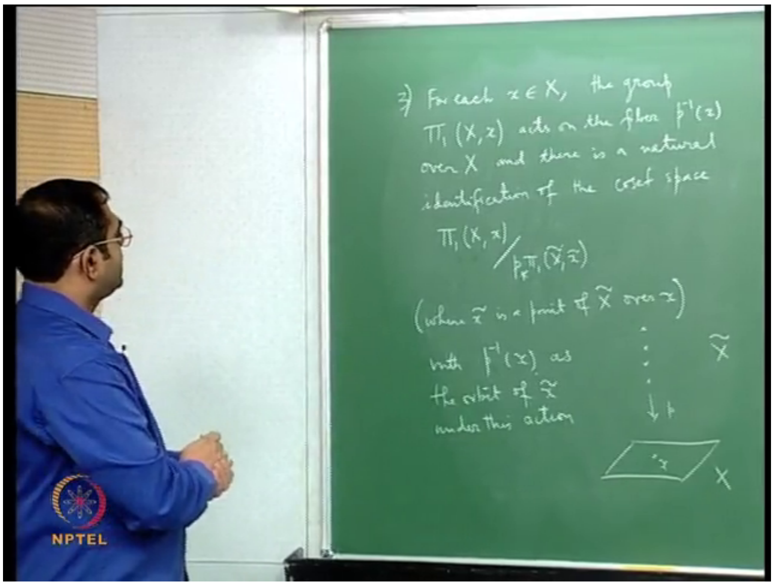


So, apply the covering homotopy theorem with this to get to get here to get the path alpha tilde.

So, the morel of the story is the covering homotopy theorem also tells you that you can lift paths. And the covering being a local homeomorphism ensures that the paths that you lift are unique, because the initial point has already been prescribed so that is the end of who proof of this statement. The third consequence is finally, the consequence that one of the things that we were trying to understand; namely why the fiber over a certain point is at least bijective to the fundamental group below in the case of a universal covering.

So, there is a more general statement which is well for any covering, and that is that is going to be consequence 3 show. So, let me write that down. So, I can write that here; so consequence 3.

(Refer Slide Time: 48:24)



For each x in X the group pi 1 of capital X based at small x acts on the fiber p inverse of X over x and there is a natural identification of the coset space phi 1 of X comma X comma capital X comma small x divided by p lower star of pi 1 of X tilde comma small x tilde where of course, X tilde is a point of X tilde over X. There is a identification of this cosets space, with the fiber.

So, what is really happening is that when you take a when you take a covering space situation, you have X here. And you give me a point X, then in X tilde there are going to be several points, lying above there that are all going to go to this point until the art of the covering projection. And these points are going to constitute the fiber the inverse image of this point here in X tilde. And what this what this statement says is that the fundamental group below is going to permute the set of points in the fiber it is going to permute the fiber; that means, it is going to act as a set of it is going to act as permutations on this on this set which is a fiber.

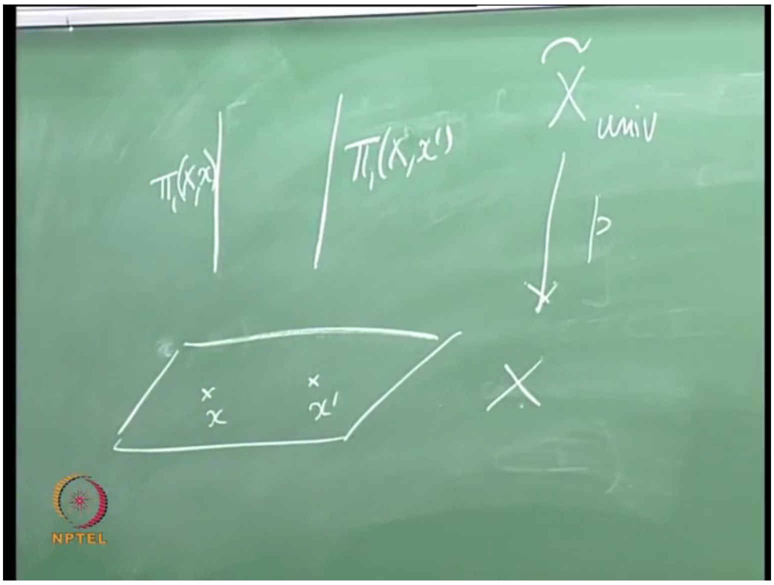
Which means each element of an fundamental group below is going to give rise to a bijective map of the fiber onto itself. And further if you take. So, the consequence one told you that the image of fundamental group above is a subgroup of the fundamental group below. So, I can take this quotient, but of course, the subgroup may not be normal. So, it is not really a quotient group, but if it still makes sense as a cosets at space we can either take it a space of right cosets or if you want a space of left cosets it does not matter.

So, you take this cosets space. The fact is that this coset space is the same as it can be by bijectively identified with the fiber and in fact, if this identification actually comes by the natural so-called orbit map. So, what is happening is that you take the orbit of X you take the orbit of X tilde and the orbit of X tilde will be the whole fiber. And the stabilizer of X tilde mainly the subgroup which of elements which will fix X tilde is going to be those which are in the subgroup. Therefore, g the group modulus stabilizer will be economically isomorphic to the orbit. So, we are we are going to think of p inverse X as the orbit of X tilde as the orbit of X tilde under this action. So, what does this mean? Let me read it for a minute.

Under the action induced this is induced by the so-called orbit map. So, let us stop for a moment and try to understand what it means for the universal covering. If you take a universal covering, then you know the covering space is simply connected. So, the fundamental group above is real trivial. Therefore, it is image below is also going to be a trivial subgroup. So, this is just going to reduce to the fundamental group below. And what this will tell you is that the fiber over a point is canonically identified with the fundamental group based at that point. And this is so, you get now a picture of the universal covering space which looks like this. So, let me rub this for a, but I can control a little picture.

So, you get a picture like this you haveX and if you have the X tilde universal I put sub I put the subscript universal to say that it is universal covering space, then it looks like this you have a point X and what you have above that is in the space X tilde is just phi 1 of capital X base that X.

(Refer Slide Time: 54:16)



Like this for various points if you take another point X prime, you are going to have a copy of the fundamental group of X base of that X prime. And in this way this the universal covering is fiber over the base space.

And the fibers are exactly set theoretically in a very natural way not in some hippos order arbitrary way, in a very natural way each fiber is precisely a copy of the fundamental group base at the point over which the fiber lies. So, you get this kind of a very beautiful picture which makes which portrays X tilde or X as a vibration, containing the fibers all being fundamental groups at various points. And of course, if you if we are going to assume X is arc wise connected which we have, then all these are all going to be isomorphic. You know, the fundamental groups based at different points are all going to be isomorphic.

So, essentially what you get is this this this fiber kind of situation. So, that is so, you can see that this is what we saw when we looked at the case of you know the complex structure on a cylinder or complex structure on a torus we found that you take any point below, the inverse image somehow turned out to be bijective to the fundamental group below. And the reason for that is because of this consequence. And this consequence can be also derived from these the earlier consequences and the covering homotopy theorem.

So, I will try to do that in the next lecture.