An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 12
The Monodromy Action

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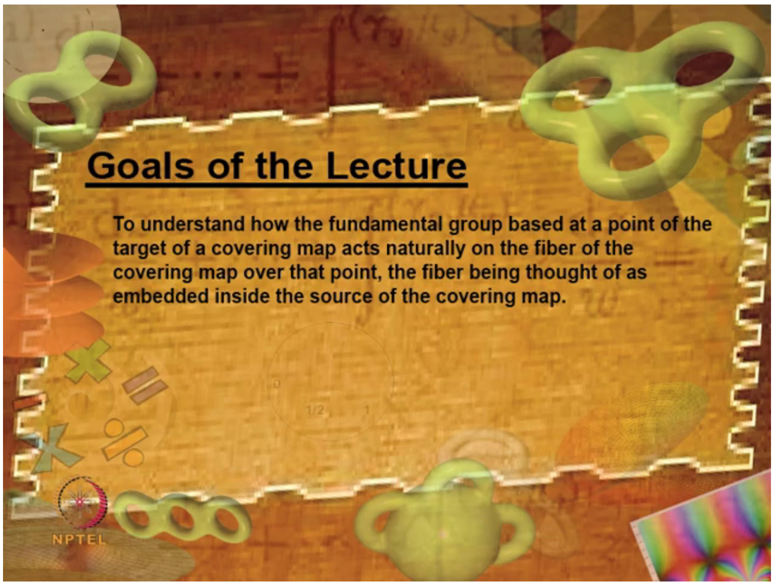

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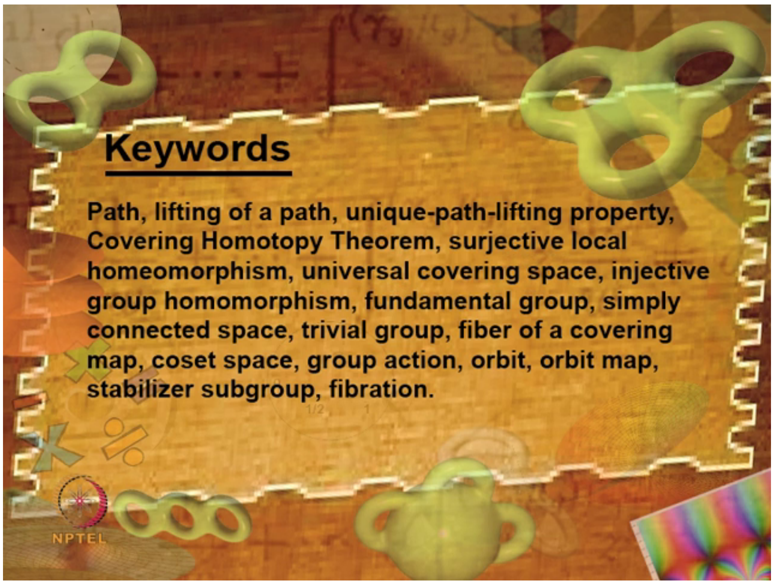

So, let us continue with our discussion. We were trying to derive the important let us say, few important consequences of the covering multiple theorem. So, all this was a it is part of our effort to understand why the fiber over each point in the base space of a universal covering is set theoretically bijective to the fundamental group of the base space. And the other question was why is it that the fundamental group of the base space can be identity can be identified with the subgroup of automorphisms of the universal covering space. So, the key to understanding, this is trying to understand the covering homotopy theorem. So, I said it to the covering homotopy theorem and I was deriving consequences.

So, let me quickly recall consequences of the covering homotopy theorem.

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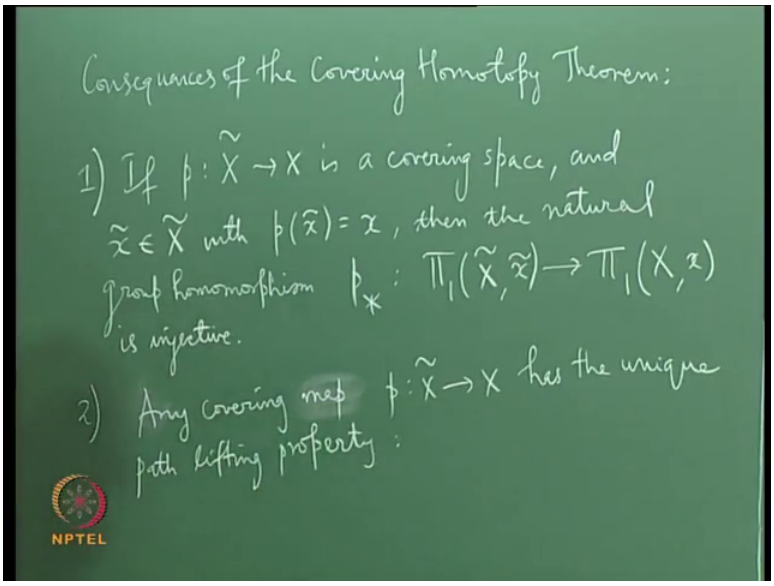

Number one, the first consequence was the fact that if you take a covering space then the image of the fundamental group of the covering space can be a in the fundamental group of the base space is isomorphic to the fundamental group of the covering space. Namely, the natural map induced by the functor of forming fundamental groups is injective. So, let me state that, if p from X tilde to x is a covering space and X tilde is a point of capital X tilde with p of X tilde is equal to x. Then the natural map the natural group homomorphism, p lower star from the first fundamental group of the space above the covering space based at the point X tilde to the first fundamental group of the base space base at the point x is injective.

Therefore, you can identify the first fundamental group of the covering space as a subgroup of the first fundamental group of the base space. And if you recall I told you that the formation of the fundamental group first fundamental group is what is called a functorial operation; which transforms also a map into group homomorphism that is how you got this.

So, the injectivity was of course, derived using the covering homotopy theorem right. So, this and of course, in the case when this is the universal covering, the space X tilde is simply connected. So, the fundamental group above is trivial. So, what you get as a subgroup of a fundamental group below is just a trivial subgroup. So, the second consequence was the existence of unique path liftings.

So, let me write that down existence. So, rather I should say the any covering space has unique path lifting property. Or let me say even covering map has the unique path lifting property, what does this mean? So, if I draw a diagram. So, you have you have X tilde here.

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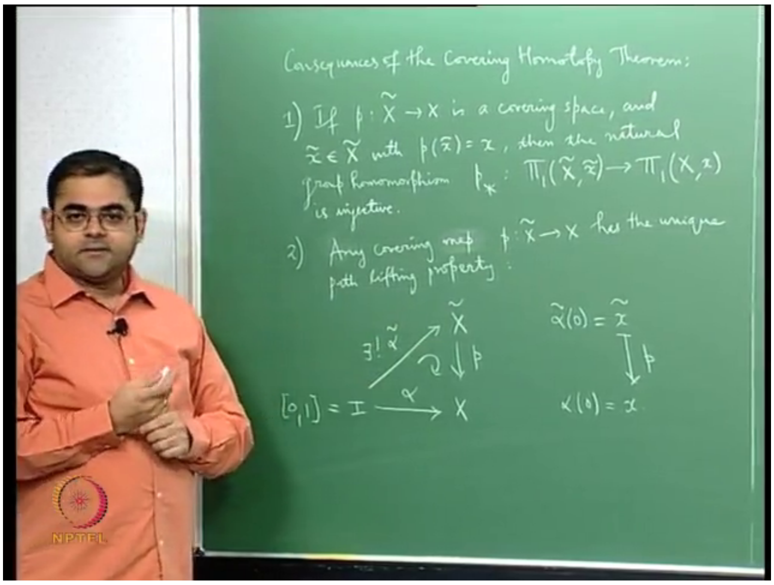

You have x here, this is the covering map the covering the so called covering projection. And suppose I have a path in x. So, I is the closed unit interval in the real line, and suppose I have a path. And suppose the path starts at the point x in x. And suppose I am given a point X tilde which lies above the point x under the map p.

So, p of X tilde is x. Then there exists a unique path alpha tilde in X tilde which lies over alpha. Namely, the there is a path in a X tilde with alpha tilde of 0 equal to X tilde namely starting at x at the point X tilde we have chosen. And such that if you project that path down, you will get the path alpha projecting the path alpha tilde, down gives you the path alpha.

So, in other words the path alpha can be lifted to a path alpha tilde. Provided of course, you fix a starting point an initial point which has to of course, lie over the starting point of alpha. And that recovering homotopy theorem tells you, that you can certainly get this alpha tilde. And I have told you earlier that the property of this map being a local homomorphism ensures that the lifting is unique.

So, you get the unique path operating property. And this is very, very important because, as I told you the difference between a subjective local homomorphism and a covering map is precisely this unique path lifting property this path lifting property. So, there is a there is a statement that you take you take a surjective local homomorphism. And if you are sure that it has the unique path lifting property, the path lifting property, then you can deduce from this that it is actually a covering map.

So, of course, I should again remind you that in general we are always assuming that all our spaces are houses of spaces. They are all you know arc wise connected locally arc wise connected and locally simply connected. So, then we come to the third consequence, which is exactly the one that is going to explain why the fiber over each point the inverse image of each point and recover under a universal covering map is canonically bijective to the fundamental group below.

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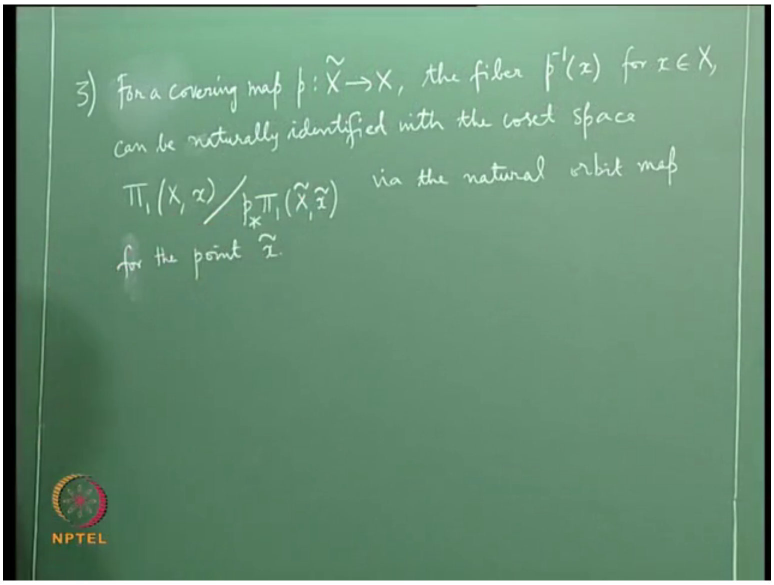

So, let me write that down the fixing (Refer Time: 08:28) let me not say fixing for a covering map p from X tilde to x the fiber p inverse of x for x in x. Small x in capital X, can be canonically or let me say naturally identify with the coset space by one the fundamental group below mod the image of the fundamental group above, where of course, small X tilde is a point lying over small x. That is a point that is mapped by p to small x.

So, there is the fiber can be identified with this via the natural orbit map of the point orbit map for the point X tilde. So, in particular you know if this was the universal covering, then pi 1 x, then this this is going to be a trivial subgroup. And what it will tell me is that every fiber the fiber over each point can be naturally identified with the fundamental group of the base space based at that point.

So, you get a very nice picture namely the picture of a universal covering in this form.

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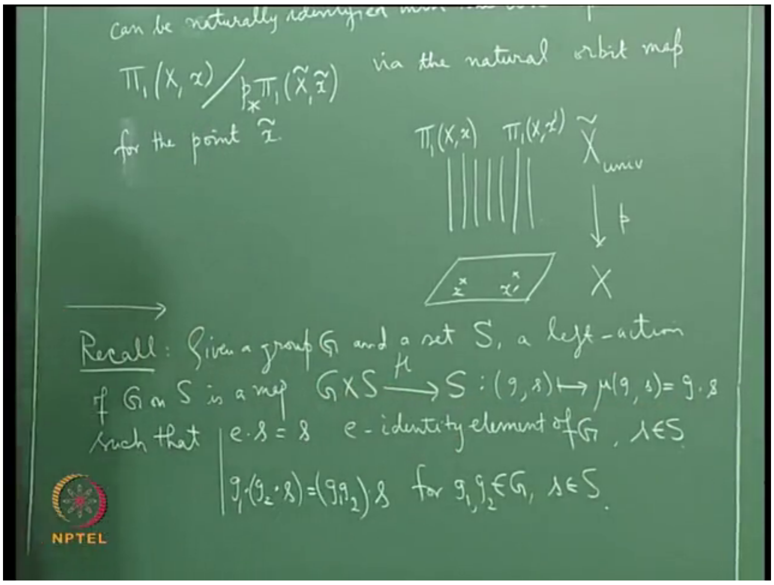

So, let me again draw this picture, this is a universal covering to x. So, it looks it looks like this, over each point x. You put pi 1 capital X comma small x. If you take some other point x prime, you put over that pi the fundamental group pi 1 capital X comma x prime. And of course, I am assuming capital X is so, in this way I get all these fundamental groups. And what you get is that the space X tilde is fibration. The fibers are all isomorphic, because you know I am I am assuming the topological spaces are of my topological spaces are all arc wise connected.

And then you know that the fundamental groups at different points they are all isomorphic. So, you get this nice picture of the universal covering space as some kind of, as a kind of what may be called a pi 1 bundle over the base space x. So, it is a all the fibers are by ones. And pi 1s of the base below of the space below. So, this is what I am we should try to now understand.

So, for that I will first begin by trying to explain the idea of group actions of a set, where I will quickly recall. So, recall given a group a group capital G and a set capital S, your left action of a g on s is a map g cross s to s, let me call this as mu. So, it is written as g small g comma small s going to mu of g comma s, which is usually written as G got s, there is a notation. Such that, e dot s is equal to s e the identity element of G and of course, s is an element of S. So, this is one condition.

The second condition is g 1 dot g 2 dot s is g 1 g 2 dot s for g 1 comma g 2 belonging to g, and of course, the s remains. So, this is the notion of a group acting on a set. So, what the first one tells is that; so, basically this is this is a way, I mean this is a method of trying to take an elements small s of the set capital S. Take an element of g e and let the small g operate on the small s to produce a new element of s which is denoted as G got s.

And in this way if you give me an element of g a given any element of s I am going to get another element of s. So, each element of g is going to induce a map from s to s. And the definitions will tell you that the map that is induced is a bijective map. Namely, it is a permutation of s. So, In fact, you can see that you know, so let me write that down, but before that let me explain these 2 points. This, this, this condition tells you that the identity element of the group simply does nothing. It does not produce a new element. And this one tells you that the way you form the new element is compatible with the group operation. So, this is this g 1 g 2 is multiplication of the 2 elements g 1 g 2 in the group.

So, you multiply these 2 elements and then operate on s to get a new element of s. Or you do it a one at a time, but you keep this order of one followed by 2 in both cases, then if you get the same result. So, the point is that each element of g will give rise to a permutation of s. So, each element of g, g belonging to g gives rise to a permutation of s; namely, you see in other words I have map from g to permutations of s; which is just g going to let me call this as mu sub g.

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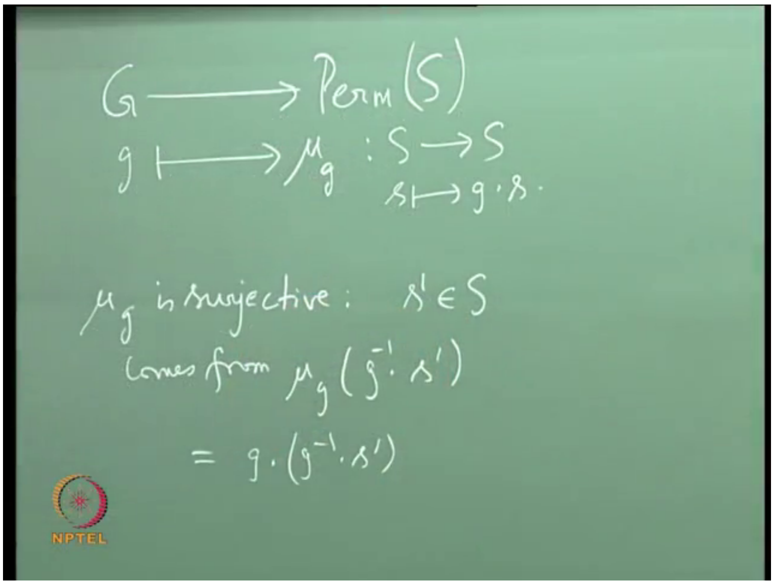

And what is this mu sub g? Mu sub g is the map from s to s which simply takes s to G got s. And so, every element of g gives rise to a permutation of s. And why is it a permutation? It is because of these. It is because of these conditions.

You can see that ug is surjective because if I start with an element of s, if I start with s prime belonging to s comes from mu g of g inverse dot s prime. So, mu g of g inverse s prime is going to be just g dot g inverse dot s prime. And that is going to be by this by this condition, I can put the g and g inverse together I will get e I get e dot s prime. And but e dot s prime is just as prime. So, this will turn out to be s prime.

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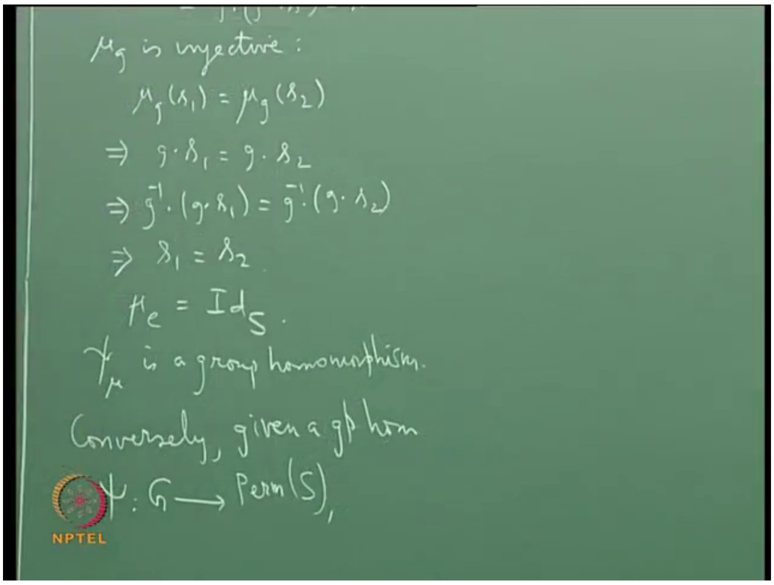

So, it is mu g it is certainly surjective, and mu g is also injective, because you know mu g of s 1 is equal to mu sub g of s 2 will mean G got s 1 is equal to G got s 2. And then I operate by g inverse on both sides.

So, I will get g inverse dot G got s 1 should be equal to g inverse of G got s 2. And then again, I combine this g inverse and g I will get e by using this and this, and that will tell you that s 1 equal to s 2. So, it is very clear that each mu g action of an element is going to produce a permutation of s. And in this way, we are going to get a map from the group g to set of permutations. And you this this condition also tells you that the permutation that you get corresponding to the identity element of g is just the identity map on s u sub e is just the identity map on s, because it simply does not do anything.

Operating by e we when we write G got s we say g operates on this element s to produce a new element G got s. So, when e operates on s it does not do anything. It keeps us it returns s. So, mu is identity of s is identity map of s, and now notice that this map what do you have on the left side is a group, we which we started with the right side is also group. The set of permutations of a group of a set at the set of permutations of on any set is a group under composition of mappings.

The composition of 2 permutations is again a permutation, and you know the inverse of permutation is also a permutation. So, the beautiful thing is that this map is a group homomorphism. So, so if I maybe I will call this as psi, it is a mu, psi is a mu is a group homomorphism. So, is sub mu is a group homomorphism. So, which means so this this is something that you can easily check. So in fact, if you give me an action of a group on a set; mu I am getting a group homomorphism of g into a set of permutations. It is a group of permutations of that set, and the converse is also true if you give me a group homomorphism from g to the group of permutations of s; that is actually defining an action.

So, conversely given a group homomorphism psi from g to permutations of s so, again I will draw small line here.

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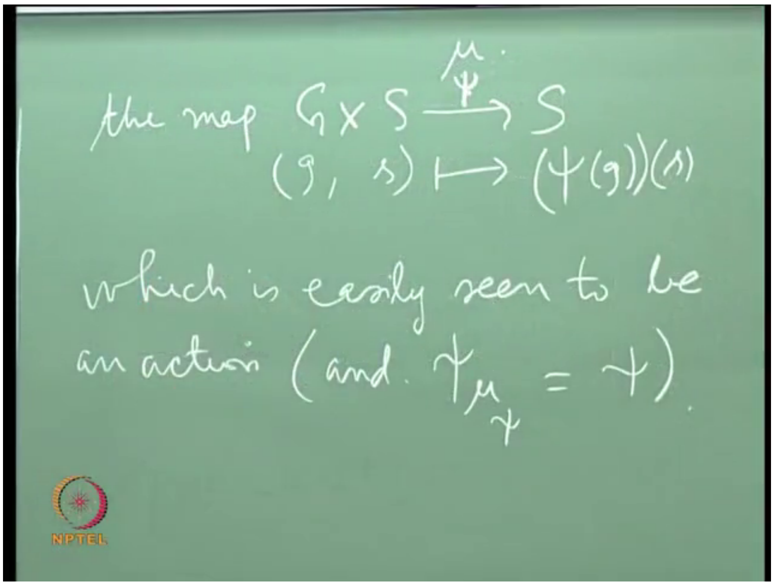

The map g cross s to s mu sub psi defined by g comma s going to, so this is sub mu sub psi, and if you give me an element of g I get psi g it is a permutation, psi of g is a permutation on s, and then I let it act on the elements small s. That will write an element of capital s; so conversely given a group homomorphism. You get a map, and which is easily seen to be an action; which is easily seen to be a to be an action.

And in fact, if I if I write down the psi sub mu for that; you will get back your psi; so what I am trying to say is that I am just trying to tell you that giving a group action as a map in this way with these property is the same as giving a group homomorphism into the permutations of that set fine. So now the next thing that I wanted to tell you is about orbits.

So, let me say the following thing.

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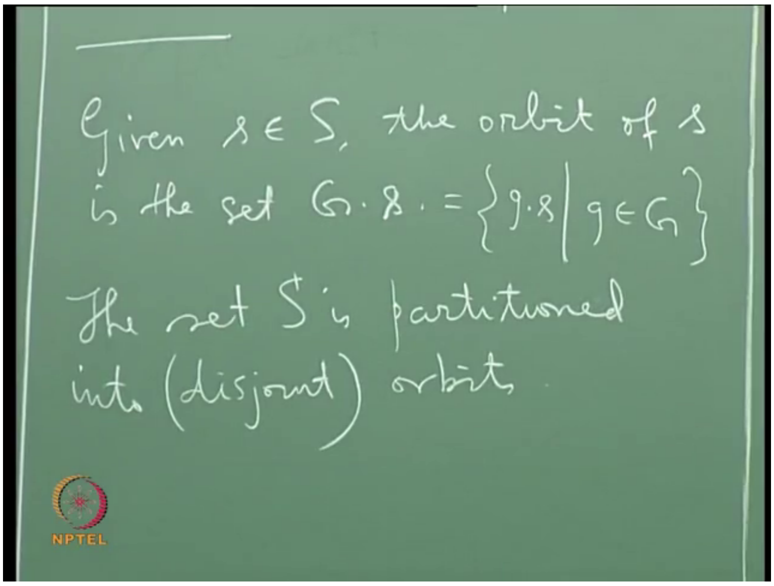

Given s in S the orbit of s is the set G got s. What is this G got s? It is just it is just the set of all small G got s where g varies in G. So, you if you if you give me a single element if you let g act on it a various elements of g act on it to produce new elements and that is the orbit of that of that element.

And what happens is that the set s is broken down is partitioned into orbits. And it that means, either 2 orbits are the same or they are completely disjoint. So, the set so, you can check this, the set s is partitioned into disjoint, of course, partition means disjoint orbits. The set s is part partitioned into various orbits under the group action correct.

And so, the question is how does each orbit look like. So, for that I will have to explain what is meant by the stabilizer of an element. So, you see the idea is to set is broken down into various orbits. And we would like to think of each orbit completely in a group theoretic way. The way of doing that is to calculate what is known as a stabilizer of an element.

So, again let me write that down.

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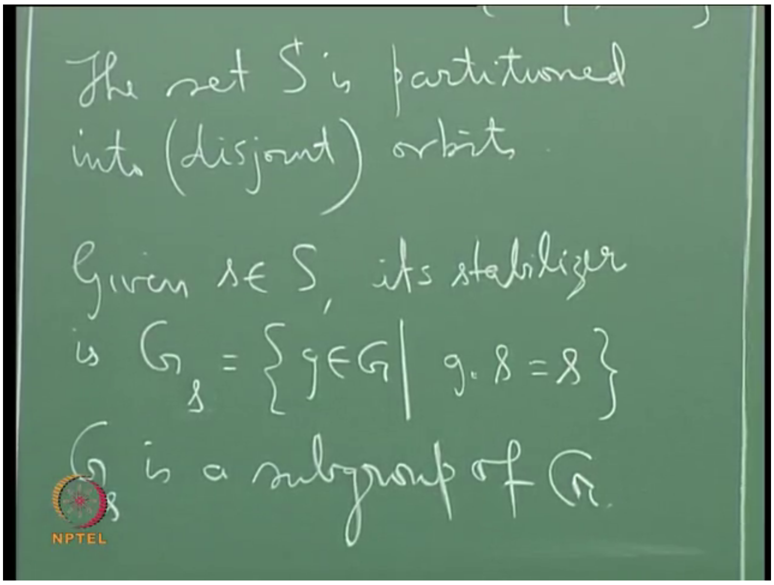

Given an element small s and capital S it is stabilizer is; let me write it as g sub z is the set of all g in G, small g in capital G such that g sub g sub s sorry, G got s equal to s. So, these are exactly the elements of g which do not do anything to that given element which fix that element.

And you can check that this is a sub group. G s is a subgroup of g. This is a subgroup of the group. And once we have the notion of the stabilizer, which is the sub group. Then, you can describe each orbit the orbit is just the coset space g mod G s mind you G s is only a sub group it is not a normal subgroup. So, g mod G s does not make sense as a quotient group, but it of course, makes sense as a coset space you can either take it as left cosets or a right cosets which way you want.

So, for that I will I will have to explain what the orbit map is; so, this is exactly what I have written down here.

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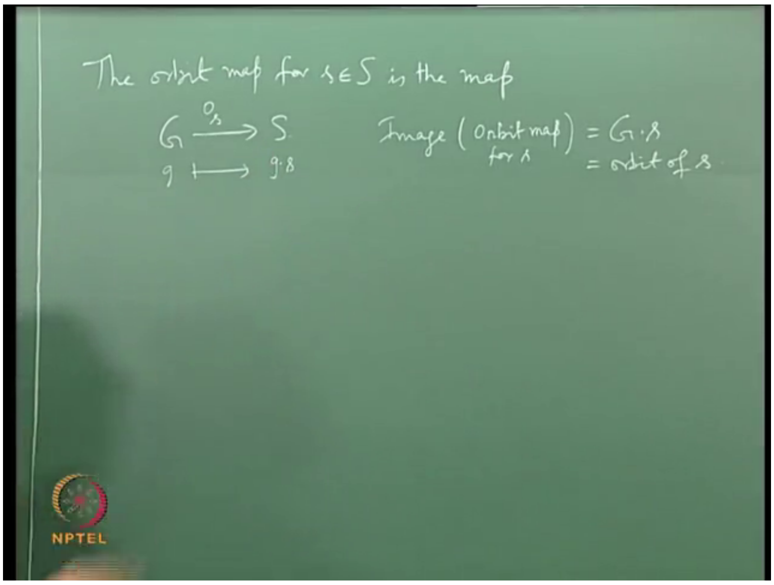

So, the orbit map for an element of s is the map from G to S given by g going to so, let me write this o sub s G got s. So, this is the orbit map right. And you can see the (Refer Time: 25:42) the orbit map is precisely the orbit. The image of the orbit map for s is exactly the image is exactly capital G got S which is exactly the orbit of s the image of the orbit map is just the orbit.

In other words this is a surjection onto that subset G got s which is the orbit. And in fact, so we get a surjection G to G dot s orbit map, I still write this map as the orbit map.

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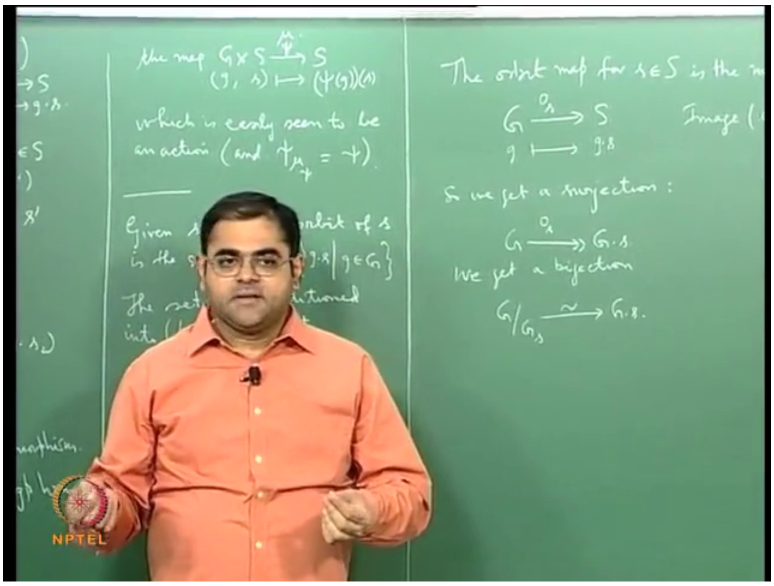

Because essentially, I have just taken the image, in in principle I should denote this by another letter, but it is essentially the same map.

I do not want to do that. So, you get a surjective map like this. And the question is the fact is that if you now go modulo the stabilizer g sub small s then this induces a bijection of the coset space g mod g sub s with the orbit. If we get we get a bijection G mod G s with G got s. So, G sub s is the is the subgroup stabilizer subgroup, consisting of elements of the group which fix the element s and G got s is the orbit. So, this is a picture that you get when a group acts on a set the set is broken down into orbits. And each orbit looks like the coset space g modulo stabilizer of an element in that orbit.

Of course, you know I fixed an element s, if I had taken any other element the stabilizer would only change up to a conjugate, and still this bijection would hold. And there is another thing I want to say, you can also interpret this as a as coming out from an equivalence relation. So, the relation of 2 elements of the set lying in the same orbit is an equivalence relation. And under this equivalence relation, the orbits are precisely the equivalence classes.

And you know every equivalence class; I mean every equivalence relation will partition your set into a disjoint you know into a disjoint collection of subsets, which correspond to the equivalence classes. And these are precisely the orbits; this is exactly the partitioning into orbits. So, I just want to again emphasize that what is happening here is that you are giving the equivalence relation, that 2 elements are equivalent if one can be moved to the other by a group element.

So now with this background let me go back and try to explain this this third consequence. First of all, you can imagine by comparing this with that that, I lead to think of an action of the fundamental group on the fiber. And show that for a fixed point in the fiber the fiber is exactly the orbit. And the stabilizer is exactly this. Then this statement would follow from this generality. So, that is what I am going to do.

So, this is the whole point the point is the fundamental group below at a point acts on the fiber above, that is the whole point. So, let me explain this is the action of pi 1 capital X comma small x on p inverse x.

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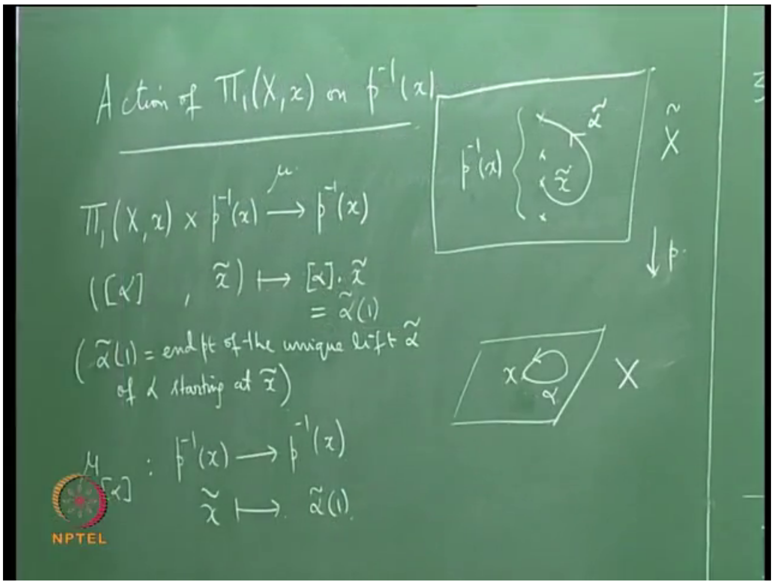

So, you see the picture is like this. So, you so, I have here, I have x I have the point small x. And I have various points above which are all in the fiber over this point. And all this is in X tilde. So, this is the projection p.

So, this is v. So, it looks like this roughly now. So, I will have to explain first of all I have to show that there is an action. So, what do I do? So, I will have to give you a map from pi 1 I have to give you an action map pi 1 capital X comma small x cross fiber into the fiber. I have to define an action map just like this. And what is his action? So, I start with an element here, what is an element here it is homotopy equivalence class of a loop alpha at the point loop at the point x.

So, this is alpha and. In fact, it is a homotopy last class right. Of course, fixed endpoint homotopy equivalence class with both endpoints being alpha being xx. So, this is the point x. And I take a point in the fiber above. So, I take say X tilde a point in the fiber above, and I will tell you what is a new point I am going to get. So, I am going to tell you what happens when I take this alpha and act on X tilde, I had to produce a new point on the fiber.

So, how am I going to do it? I am going to do it by using consequence 2, I am going to use the unique path lifting property. So, what I am going to do. So, you see I have this path alpha it is a loop. I have a point above which is being prescribed to me. So, there is a lifting alpha tilde of this loop to your path starting at X tilde. So, and mind you so, if I if I get a lifting and call that as alpha tilde this is lifting is unique given this alpha this lifting is unique, because I have fixed starting point.

And but notice that the end point of alpha tilde has to be again lying over x. That is because alpha tilde is a lift. So, if I project it down I should get alpha. Which means that both the (Refer Time: 33:08) also that the end point of alpha tilde should be a point on x; namely, the end point of alpha tilde is a point in the fiber. So, we define the new point. So, we think of X tilde as going to this new point. And what is that new point? That is alpha tilde of 1. That is the end point of the unique lift alpha tilde which starts at X tilde.

So, let me write that down alpha tilde 1 is equal to endpoint of the unique lift alpha tilde of alpha starting at X tilde. So, in this way give me a loop below and a point above I am cooking up for you another point. So, the claim is that this is an action. First of all, that this is well defined is clear, because if I replace alpha by another loop that is homotopic to this, then if I replace alpha by say alpha prime which is homotopic to alpha, then alpha tilde and alpha prime tilde will also be homotopic because the covering homotopy theorem tells you that you can also lift homotopies. And that will force that if I replace alpha by alpha prime, then alpha tilde and alpha prime tilde will still have the same endpoint.

So, the covering homotopic theorem will tell you that if you change alpha up to homotopic; this point is not going to change. So, that is that is crucial tutor to say that this map is well defined. Right now, I have to tell you that this map is a to essentially check that this is an action as I told you, I will have to tell you that each element each element of the group should act as a permutation. So, I will tell you that each element of the fundamental group below has to act as a permutation of the fiber p inverse x.

So, let me explain that. So, I let me take mu sub alpha. So, mu sub alpha in that notation is the map from p inverse of x p inverse of x. And it is defined by sending X tilde to alpha tilde of one. Now I need to say that to verify that this is an action I need to say that the each mu sub alpha is a bijective map. I have to say it is a permutation of p inverse of x.

So, how do I say is surjective?

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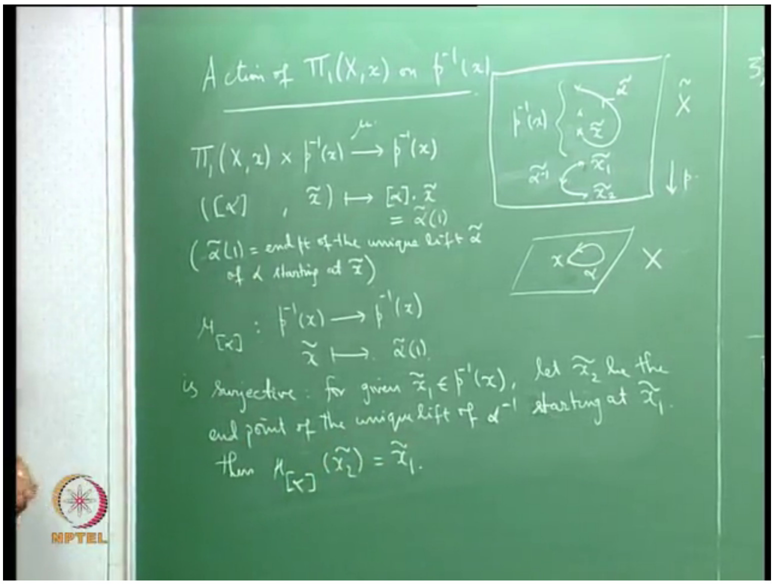

It is surjective. So, this map is surjective for given a point say x 1 tilde in the fiber. We proceed as follows. So, you have you have this point x 1 tilde, and what you want is you want to find another point on the fiber which goes to this under this map. So, how do we do it? What we do is that we take the loop alpha inverse, you take the loop alpha inverse; that is also a loop based at x, and you take it is lift it is unique lift which starts at x 1 tilde.

So, what I am going to get is I am going to get a path above which is alpha inverse tilde, and this is the unique path which is the lifting of alpha inverse to a path starting at x 1 tilde. And take it is end point. The end point let me call it as say x 2 tilde. Then you can see x 2 tilde will go under this map 2 x 1 tilde, because why is that so? That is because how do I, what will x 2 tilde go to? It will go to the end point of the unique lift of alpha to a path starting at x 2 tilde which turns out to be the inverse of this path and therefore, has to end at x 1 tilde.

So, that that tells you that given an x 1 tilde, I am able to find an x 2 tilde which maps to it. So, let me write that down. Let x 2 tilde be the endpoint of the unique lift of alpha inverse to starting at starting at x 1 tilde. Then mu sub alpha of x 2 tilde turns out to be x 1 tilde. So, this gives you this surjectivity. Then I will tell you about the injectivity. So, that is also very easily understood.

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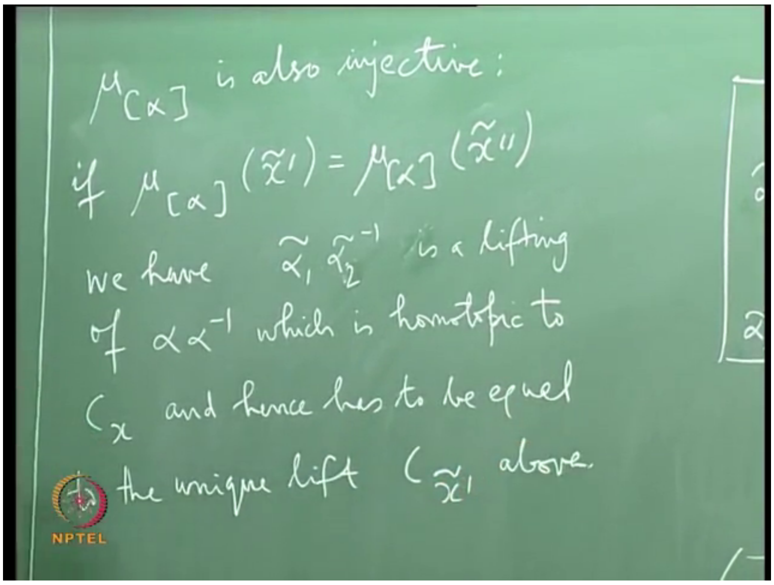

Mu sub alpha is also injective, and how do I prove that? So, let me draw another diagram.

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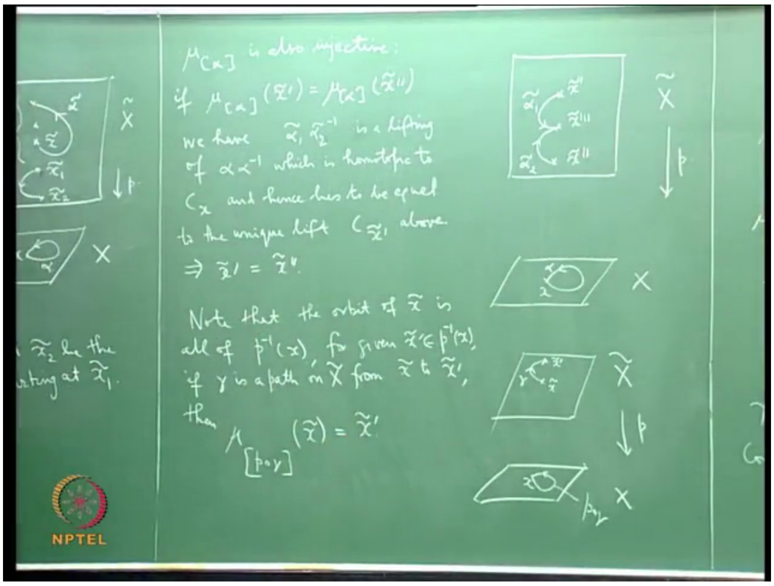

So, I have so here is my x, this is my X tilde this is p. And I have this point x, I have this loop alpha. And I have to assume that if 2 points x prime tilde and x prime double tilde go to the same point, then the points are equal.

So, I assume that I have a point x prime tilde, and another point x double prime tilde. And suppose they go to the same point under this map that is mu sub alpha of x prime tilde is equal to mu sub alpha of x double prime tilde. So, if you assume this, let us see what it means. So, it means you. So, what is the way this map goes, given a point in the fiber you take a lift of alpha starting at that point, and the end point is what is the point which it goes; so given x prime tilde I take a lift of alpha to alpha tilde.

And it goes to a certain point; say, let me call this as x triple prime tilde. And in the same way I take another lift of alpha, but now starting at x double prime tilde. And this assumption is that that also ends at x triple prime tilde. Mind you I am writing alpha tilde the same alpha tilde, but mind you these are both these are actually different paths, because the starting points a different.

So, if I want to be very strict, I should actually label this alpha tilde by if you want x prime tilde and this alpha tilde by x double prime tilde, but I am not going to do that. It will look very it look very complicated. Now I will have to show that x prime tilde and x double prime tilde are the same. And I think that is that is quite obvious that is because, you see if I if I for go by alpha tilde, this alpha tilde, and then I go by this the inverse of this. Then I am going to get a path from x prime tilde to x double prime tilde. That goes to a path which is homotopic to the trivial path, because the composition of these 2 paths is just going to give me alpha followed by alpha inverse.

Because alpha tilde goes to alpha, and this alpha tilde also goes to alpha. So, this followed by this in the reverse direction is going to give me alpha followed by alpha inverse, which is homotopic to the constant pathetics; that means, the constant path that x has the lifting up to homotopy this followed by this, but there is only one lifting; namely, the constant path above. So, this followed by this must be homotopic to the constant path at x prime tilde. And that forces that x double prime tilde has to be x prime tilde. So, that is the argument it is again uniqueness of part lifting. And the fact that if you have a constant path there is one there is, but one lift namely the constant path above nothing else.

So, let me write that down. So, let me if you want let me call this as alpha 1, let me call this as alpha 2, just for clarity, and then say we have alpha 1 tilde followed by alpha 2. Alpha 2 tilde inverse lie is a lifting of alpha inverse, which is which is homotopic to the constant path that x which is the identity element for the fundamental group. It is it is homotopic classes identity element of the fundamental group below. And hence has to be equal to the unique lift, constant path that x prime tilde above. So, that is going to tell you that so, this will imply that x prime tilde is the same as x double prime tilde. So, that that finishes the injectivity.

So, what I have established is that this is indeed an action. The fundamental group below acts on the fiber. It permutes the elements of the fiber. And now to tell you that the fiber is precisely that the so, I will have to tell you 2 more things I will I will have to tell you that they take any point on the fiber; say, X tilde the my first claim is that the whole orbit I mean the orbit of that point is a whole fiber. So, in other words I am saying that there is only one orbit. And whenever a group acts on a set which has only one orbit it means that any element of the set can be moved to any other element. And this is this is what we call as a transitive action.

So, what I am trying to say is that the action of the fundamental group is transitive. So, you get all the points in the fiber are equivalent. So, you get only one orbit, and that is the whole fiber. And why is that true that is also because of you know arc wise connectedness of X tilde. So, note that the orbit of X tilde is all of the fiber. Why is that true, because if you want let me draw another diagram. So, here is here is X tilde. And suppose you give me some other point let us say, x prime tilde this is this is capital X tilde, and this is the projection p onto the space below; which is x and these are all points lying over small x.

Now, what I can do is that since capital X tilde is arc wise connected. I can find a gamma a path gamma, that connects X tilde and x prime tilde. Now I push that path down. I will get a loop let me. So, so I will get a loop here let me call this. So, this loop will be this loop will be gamma followed by p, p circle gamma. That be a loop base at x. And the lifting of this p circle gamma is precisely this gamma, with initial point X tilde. And what is the final point? It is x prime tilde.

So, what does it mean? For given x prime tilde in the fiber, if gamma is a path on X tilde from X tilde to small x prime tilde, then you see let me write this down mu sub p circle gamma the equivalence class of the point X tilde is exactly x prime tilde. So, the mod of the story is that the whole fiber is a single orbit, all right. And I told you that the group model of the stabilizer is bijective with respect to the orbits. So, I will have to just tell you that the stabilizer is precisely the subgroup of the fundamental group below, which is the image of the fundamental group above. So, I will do that, and then we are done.

So, let me write that down, what is the stabilizer of the point X tilde?

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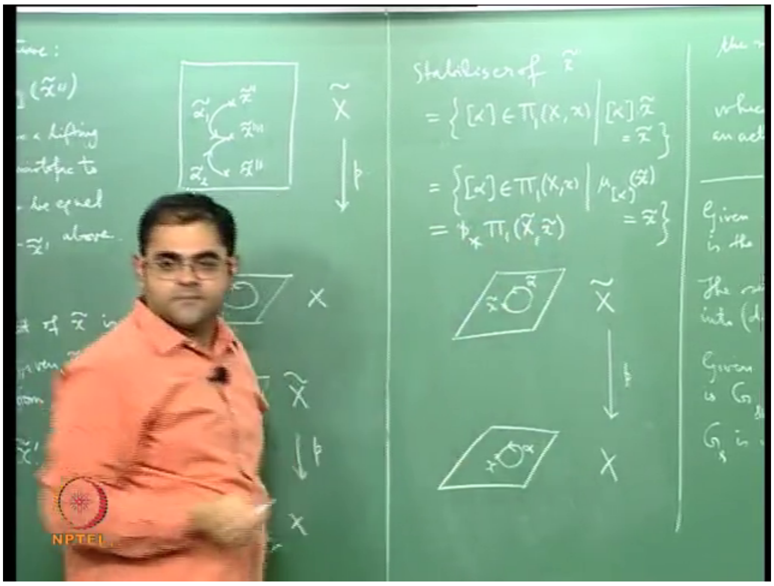

What is the stabilizer of the point X tilde? It is all those alpha in the fundamental group, such that alpha acting on X tilde is equal to X tilde. Which is the set of all elements alpha in the fundamental group below, such that mu sub alpha of X tilde is equal to X tilde.

Now, try to understand what this means. So, again let me draw another diagram. So, here is here is X tilde. So, here is p here is x. And so, I have point small x here, I have point X tilde above it. And I am trying to calculate the stabilizer of this point X tilde. So, what is in the stabilizer? Suppose an element alpha is in the stabilizer, it means that, when I take the unique lift of this alpha namely alpha tilde starting at X tilde it will also end at X tilde. So, it means that alpha tilde is going to look like this it also has to end at X tilde. Because where it ends is what is where X tilde has to go to under action.

I am saying that it does not X tilde X tilde stays put means that, this is going to also be a loop above. But then what is the image of this loop? The image of this loop is just alpha and where is this; this is in the fundamental group above. So, what you have proved is that anything in the stabilizer comes from a loop above which means you have shown that the stabilizer is precisely the image the fundamental group above. Of course, conversely if I take any element in any loop above, it is going to be in the stabilizer obviously.

So, you can see that this is just p lower star of I 1 of capital X tilde small X tilde. So, therefore, we get.

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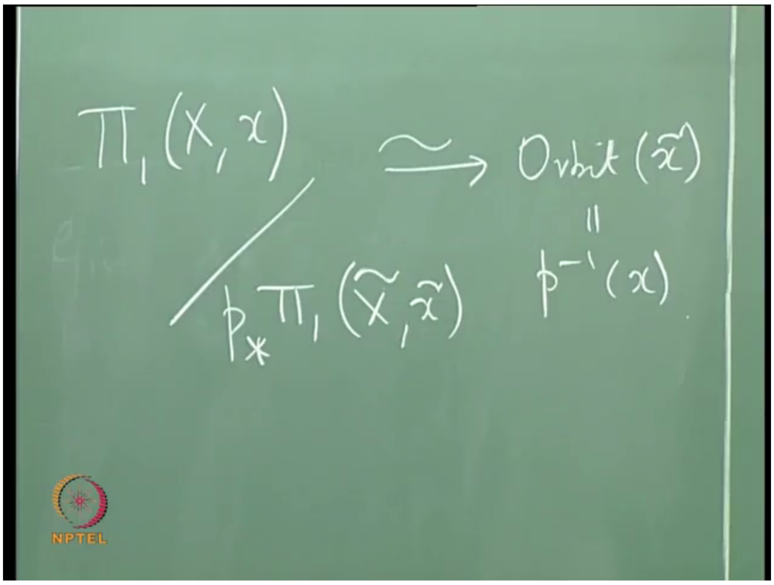

We get the following we get the following identification which is what we set out to understand. Namely, the group which is a fundamental group placed at the point small x below. Modulo the stabilizer, which is the image of the fundamental group above this treated thought of as a coset space is bijective to the orbit of X tilde; which is nothing but p inverse of x. So, each fiber is just identified with the coset space of the fundamental group below by the image of the fundamental group above.

And again, let me repeat that this implies that if x capital X tilde is universal covering then this is trivial. So, it tells you that every fiber is just a copy or the fundamental group below. The fiber over each point is a copy of the fundamental group based at that point so that that explains. So, that that explains why in the case of you know, the complex structure on the cylinder on the or on the complex structure on the torus. The inverse image of a point was bijective to the fundamental group of the respectively the cylinder of the torus. So, it is a very general thing that happens in any covering space in the sense.

So, (Refer Time: 53:07) we will stop here.