An Introduction to Riemann Surfaces and Algebraic Curves:
Complex 1-dimensional Tori and Elliptic Curves

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Lecture-15 Part A
Completion of the Construction of the Universal Covering: Universality of the Universal Covering

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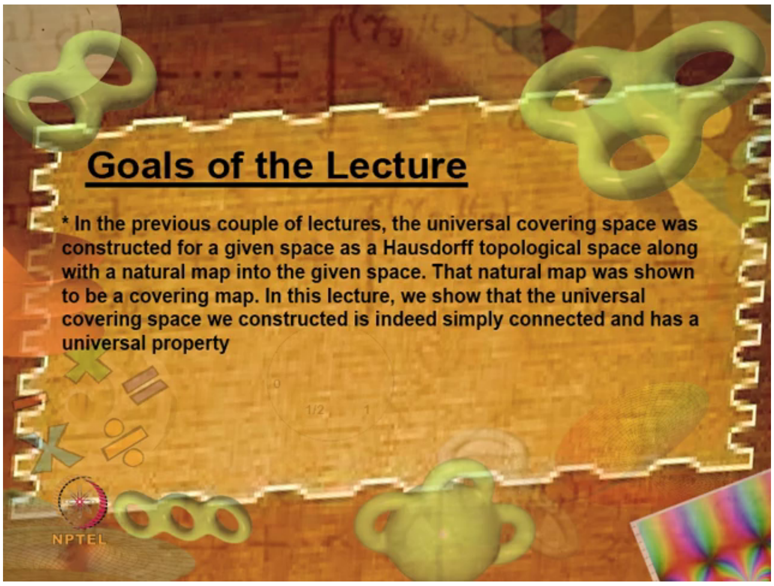

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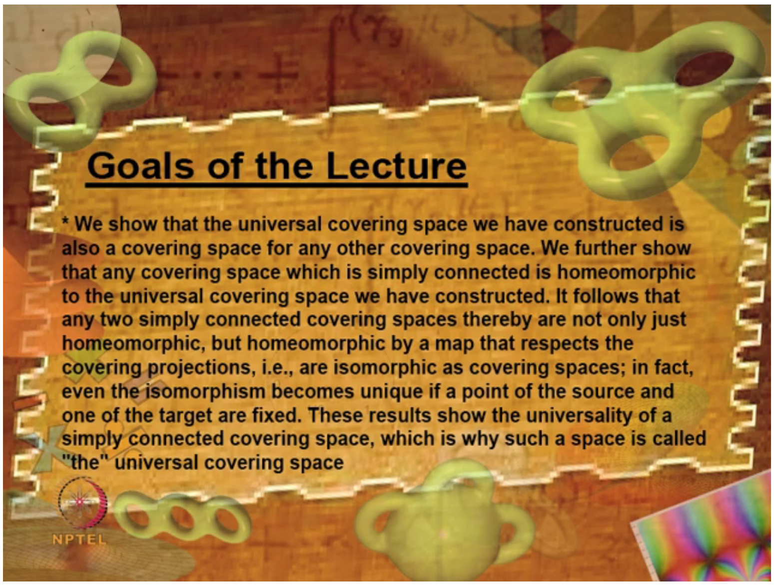

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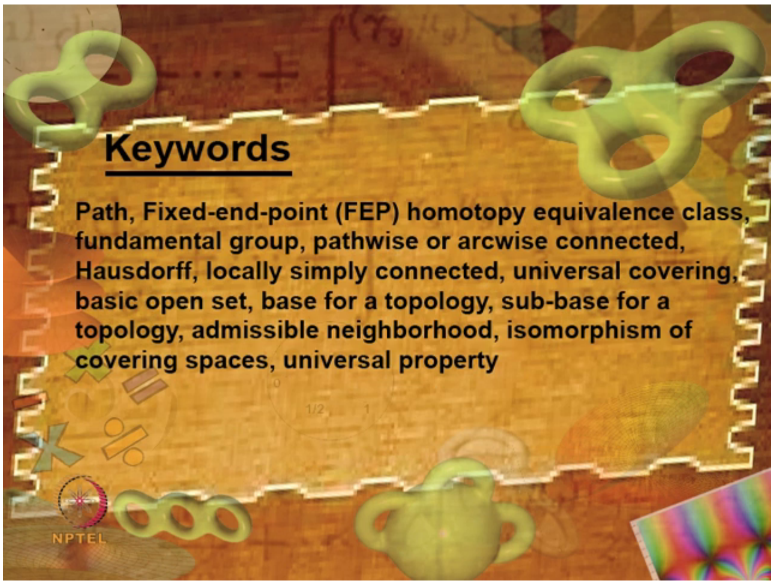

So let us continue with our discussion of the Universal Covering. So, what we have gotten so far is that we have proved.

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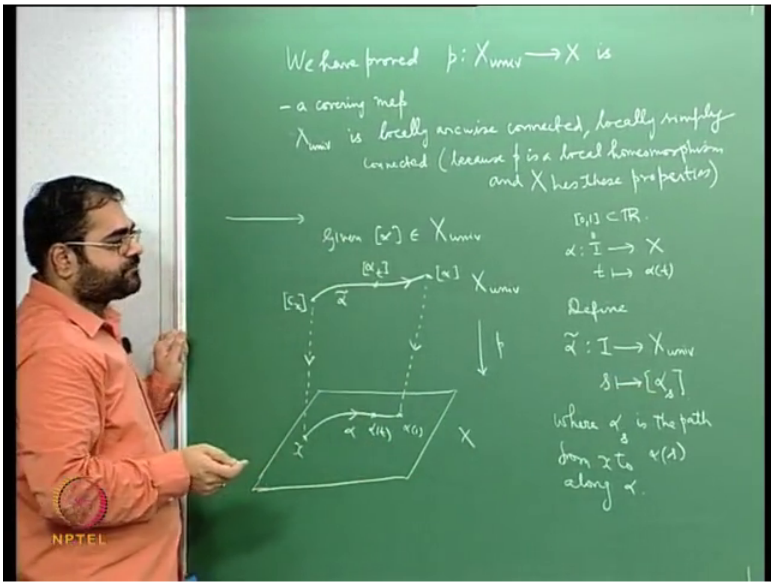

We have proved that the map p from X univ to X is covering map, we have also shown that X sub univ; is locally arc wise connected; locally simply connected basically because p is a local homeomorphism and X has these properties.

So, we have proved that these are covering map; so, X sub univ is of course, locally arc wise connected locally simply connected, the only thing that remains to prove is that X sub univ is actually arc wise connected; globally arc wise connected and we have to show that it is simply connected which is what we are going to do. And then towards the end you would like to see why the fundamental group of X can be identified as sub group of auto morphisms of this universal covering. So, first let us try to prove that X sub univ is arc wise connected.

So, the key to that is the existence of a very canonical lifting of a path starting at X; 2 X univ; so it is in the following way. So, given suppose I am given an alpha in X sub univ; so how is it going to look like? It is going to look like well; so, let me draw a diagram again. So, here is my X, here is a point small x i fixed and in X sub univ I have this point alpha and this alpha is just going to lie above the end point of the path alpha; that starts from X and ends at alpha of 1 and this point is going to lie over this namely it is going to be mapped to this, under the map p which goes from the universal covering space to X.

Now, you see and of course, over x there is this path; there is a unique point C x; which is a constant path x; that gives me a point that lies over x. Now I am going to the following thing, what I am going to do is; you see, if after all what is alpha? See alpha is just a continuous map from I to X; that since a t to alpha of t.

Now I am going to do the following thing; take any t in I of course, I is the closed interval 0, 1; the closed unit interval on the real line. Take t, then I am going to get a point here in the alpha of t and I am going to the following. I am just going to consider the path; that starts at x; at time t equal to 0 and at time t equal to 1, ends at alpha of t

So, that you can believe is going to give me a point here; which I would like to call as alpha sub t and as t varies, I will get a path above; which is exactly lying over this path below. And this path above is going to be the unique lift of the path alpha, so this path above is going to be alpha tilde; that is unique lift of the path alpha; to the universal covering, starting at this point; this point which corresponds to the constant for the things. So, how do I write this down? So, you see what I do is; I define a path alpha tilde.

So, this is from I to the universal covering and you know what this path is; it is just going to send s to; well let me put s, s is going to go to alpha sub s and I am going to put this where, alpha sub s is the path from X to alpha of s; along alpha. So, you see what is; so you know this there is a, I have put in another variable s here that is because of obvious reasons because I will have to define this alpha sub s.

So, you know basically what is happening is that you see given. So, what I am actually doing is you see; I am just shrinking alpha along itself, from alpha of 1 to alpha of 0; which is X and this shrinking provides me naturally a homotopy of alpha with the constant path of X and this homotopy shows up as a path above.

So, what is happening is see points above our paths below starting from x; and. In fact, the homotopy below is showing up as a path above; that is what is happening.

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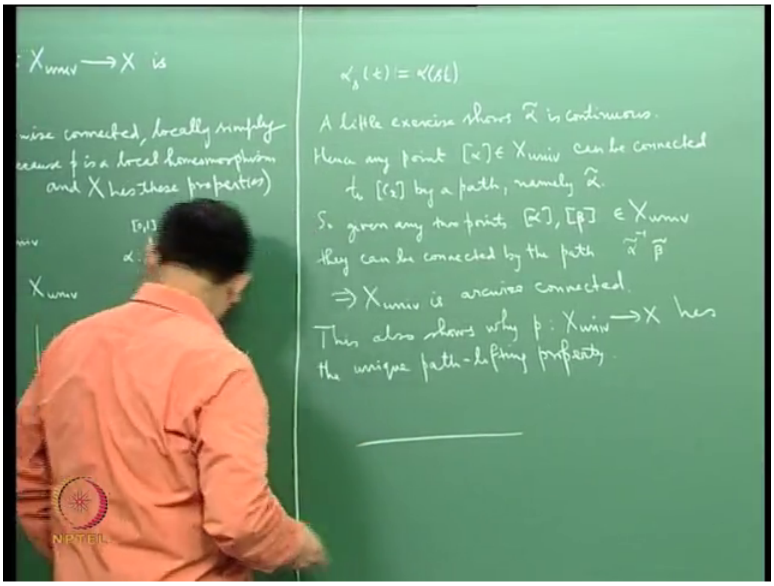

So, it is very easy to define alpha of s; it is not difficult to define alpha of s, so alpha sub s is the path which at time t is just alpha of s t. So, this is the path so you see you can see that, when t is 0; you will get alpha 0 and when you will get alpha of 0; which is x and when t is 1, you get alpha of s . So, it is a path which starts at X and ends at alpha of s.

So, it will lie above this end point alpha of s; so and it is also clear that there is a multiplication of variables here, which is continuous. So, a little bit of work has to be done to show that this map alpha tilde is actually a continuous map and once you establish that what happens is that we see that given any alpha; starting with x, I am able to get a lift alpha tilde, that lift is unique that is because you know the moment you have a local homeomorphism; the lifting is a unique, if you fix the initial point and the lift is a path that connects this point alpha to this point C x.

Now, you know if I had two points I can connect them by two paths to C x and therefore, I can connect them to each other by a path; that will tell you that this phase is arc wise connected. So, a little work; a little exercise shows alpha tilde is continuous hence any point alpha in X sub univ can be connected to C x by a path; namely alpha tilde. So, given any two points alpha and beta; here in X sub univ, they can be connected by the path. Well you know I will have to go, I have to shrink alpha to X and then I have to unshrink beta.

So, the path will be just you first apply alpha; tilde, that alpha tilde will go from so in fact, I should take alpha tilde inverse because I want it to go like this and then I will apply beta tilde. So, this path will do the job alright, so it tells you that therefore, this is arc wise connected. In fact, it tells you even more; what it tells you is that I told you that covering a space has the unique path lifting property; that was one of the consequences of the covering homotopic theory. But in this case that; this covering has the unique path lifting property, can be directly deduced from this construction.

So, this construction; this also shows why p from has; why this covering has the unique path lifting property; that also follows from this. Why? Because you see of course, I can I need not worry about the uniqueness because the uniqueness will follow, because it is a local homeomorphisms once I fixed the initial point. But why does this argument show that? Because you see, so let me draw; maybe there is not enough space here, I should go to the go to the next board; so let me do that here.

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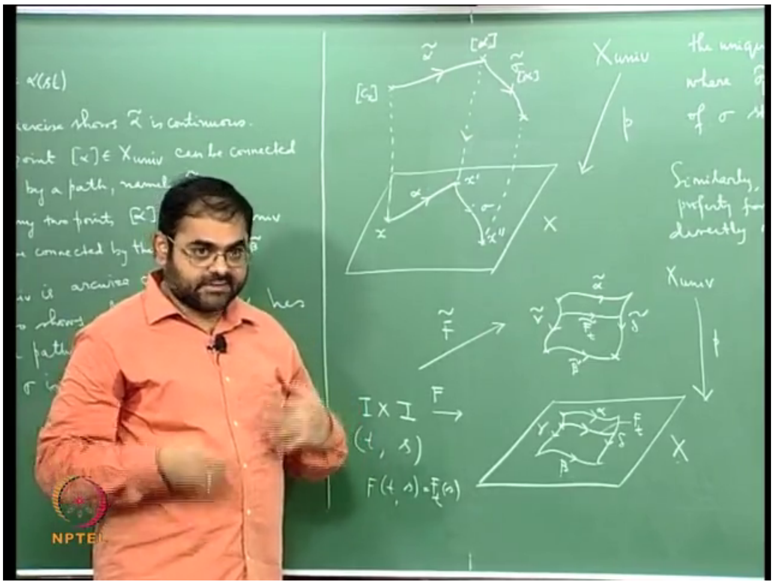

So, I have; here is my X and suppose I am given a path, from a point let us say x prime to another point x double prime.

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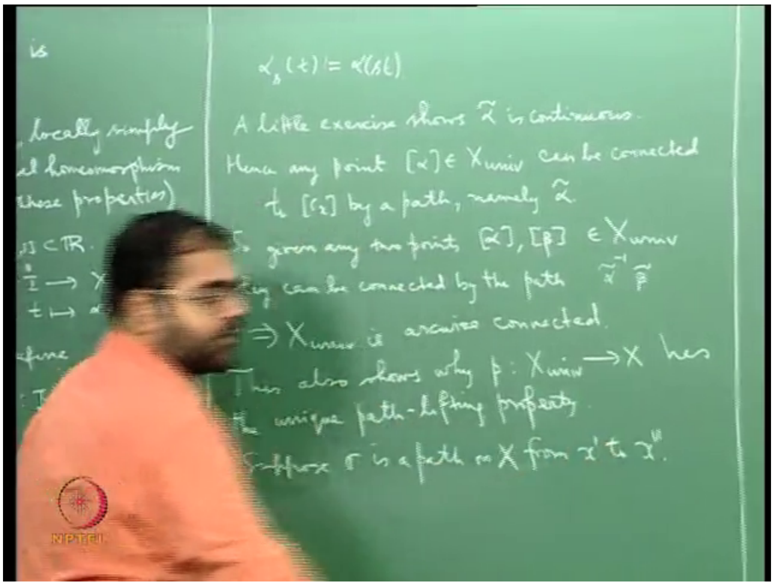

And let me call this path sigma; suppose sigma is a path on X; on capital X from say x prime to x double prime. Now, so what is a path lifting property? Give me a path below and give me a point above the initial point so; that means, I am going to be given, have to start with an alpha, which goes to this x prime and I have to show that there is a lift of this path. Now what I do is; well you see there is this path x; this is point x giving this alpha amounts to giving a path like this.

And you see now consider alpha followed by sigma by this construction alpha followed by sigma has this lift, which is literally got by shrinking alpha followed by sigma. You can see if this is the point C x that is lying over x; then you can see that the lift that I have constructed there; of alpha followed by sigma will be this lift of alpha namely alpha tilde that I have constructed there, followed by a certain other lift of sigma.

So, this will be sigma tilde, but this will be sigma tilde; the lift of sigma with starting point alpha. So, this will be sigma tilde; sub alpha and then this piece is going to just be the lift of sigma.

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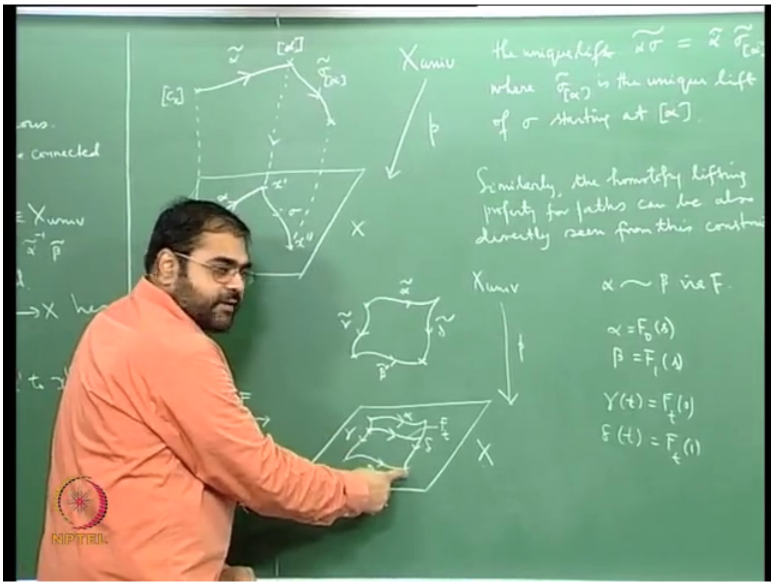

So, let me write that down; the unique lift; alpha, sigma, full tilde which you get by you know shrinking sigma and then shrinking long alpha; that is going to be equal to I mean it has to pass through this point alpha. Because at some point of time, you are going to shrink it to alpha and therefore, it has to it lie here and so, this will be alpha tilde followed by a sigma tilde; sub alpha; where the sigma tilde sub alpha is the unique lift of sigma; starting at alpha.

So, this construction also tells you that you can directly see the path lifting property, just looking at this shrinking homotopy. And a similar argument will tell you that you can also lift homotopies of paths. So, that is also something that is promised by the covering homotopy theorem, but that is something that you can directly see in this case. So, let me write that similarly the homotopy lifting property property for paths can be also directly seen from this construction. So, that is also something that I can quickly explain for you.

So, basically if I have X here and so, suppose there are two paths in a X; let us say alpha and I have another path beta and suppose alpha and beta are homotopic; it is a free homotopy which means endpoints need not be fixed. Suppose they are homotopic, then basically you will get a diagram like this; which is the image of a map capital F which is a homotopy from I cross I o X and what will happen is; if I call this as well t comma s; then and if I call F of t comma s to be F sub t of s, then saying that F is homotopy from alpha to beta is the same as saying that F sub 0 of s is alpha; beta is F sub 1 of s and so, alpha is homotopic to beta via F; that is your homotopic. So, and mind you; this is also going to be a path and this is also going to be a path.

What is this path? This is going to be there. So, at some intermediate time t your homotopy will give you a path something like this. So, this will be F sub t; intermediate F sub 0 is alpha; F sub 1 is beta. And F sub t is something in the middle and all these F sub t's, if you take the initial points; that is going to give you a path, which I can call as gamma. So, this gamma of t is F sub t of 0 and then if I take all the F sub t's of 1; the end points of all the F sub t's; that is going to be, that is another path I can call it as delta.

So, delta of t will be F sub t of 1; now these are also paths because they are again continuous images of the interval, their restrictions of this continuous map to suitable subsets here, so they are also paths. Now how do you get the covering homotopy? See how do you cover this homotopy? So, you know you have p here, you have X sub univ; so I do the following thing; you see for this point, I choose a point above. And then I lift this alpha to alpha tilde, but this is alpha tilde is starting at this chosen point.

Then I also lift this gamma, I will get a gamma tilde; I will get a unique lift. I will also get; from this point I will get a unique lift of beta namely beta tilde. And from this point I will get a unique lift of delta and that will be delta tilde and the fact is that the end point of delta tilde and the end point of beta tilde will be the same. That is because, actually if you look at this picture; it tells you that alpha followed by delta is fixed end point homotopic to gamma followed by beta.

This picture itself will tell you that this path alpha followed by delta is F e p homotopic to gamma followed by beta. And since these two paths have the same endpoint, this point above that you get has to be the same, by the very definition of X sub univ. So, the moral of story is you can lift the homotopy and in fact, similarly this F t will lift to an F t tilde. So, the whole homotopy will lift and one can actually sit down and verify that you get lifting like this; F tilde and you can check that this lifting is actually continuous.

So, using this construction you can actually see why this X sub univ has the unique path lifting property and also the homotopy lifting property. And you see further you see, if this was a fixed end point homotopy; then this gamma will collapse to a point, delta will collapse to a point and the unique lift of that will also have to be constant paths above. So, what will tell you is that fixed endpoint homotopies will lift to fixed end point homotopies.

So, these are all things that the covering homotopy theorem in general promises for any covering space. But in the case of the universal covering space, you can directly see it because of this construction.

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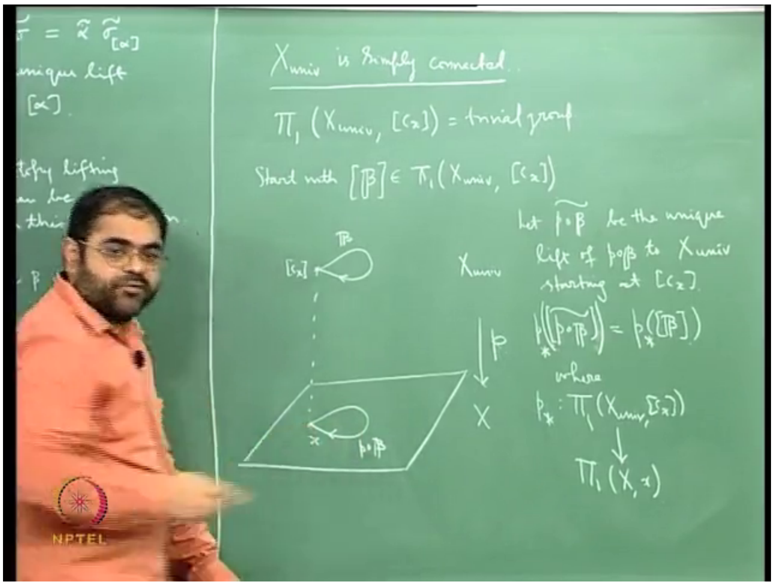

So, now, there is only one more thing that is left and that is to show that the X sub univ is simply connected. So, let us do that X sub univ is simply connected and I have to show this. So, for that I am going to do the following thing; so, what I will have to do is; so, I will have to show that the fundamental group at any point is trivial.

So, but you have a fundamental group at different points are going to be isomorphic. So, I can choose any one point and show that the fundamental group at that point is trivial. That is because I have already shown that X of univ arc wise connected; so, what I will do is, I will show that the fundamental group at C x is trivial.

So, what I will do is pi 1; X unv comma C x is a trivial group. So, what do I do? I take a loop here; I take a loop centred at, loops starting at C x and ending at C x, I will have to show that again continuously deform it to the constant path at C x; that is what I have to show.

So, what I will do? So, let me do that; so let me draw a diagram, start with a beta in pi 1 X sub univ C x. So, let me draw a diagram again, so I have this situation. So, here is x; in capital X this is ,p this is X sub univ and over the point x in the fiber; well I have C x. And what I have done is; I have taken a point, I have taken path beta above and of course, I should take in principle homotopy class because elements of fundamental group are F e p homotopy classes.

So, I will have to show that this beta is homotopic to the constant path C x. Now let us project this beta down under p, if you project this beta down under p what I will get is; I will get a path below. I will get a closed loop below and what is this close loop? It is just going to be p beta followed by p. And well now take this beta followed by p and take its unique lift to X univ; starting at the point C x; by this construction, given a path starting at X alpha, I have an alpha tilde.

Now, this alpha tilde is just the shrinking homotopy; it is gotten by the shrinking homotopy. So, you take for this path ;you just take the unique lift there, so what will happen is that; you will see p circle beta tilde. Let this be unique lift; lift of p circle beta; 2 X sub univ, starting at the point C x. Take this unique lift, now notice that by the very definition; both this as well as beta, they go to the same element below. So p circle beta; tilde; if I take p lower star of that, that will be equal to p lower star of beta.

So, I will have to put square brackets where p lower star is the group of some from the fundamental group above, to the fundamental group below. So, now I have p circle beta tilde and beta both going to the same thing, but you know that this is injective. This uses the covering homotopy theorem, but in many case we have verified the covering homotopic theorem; namely the important corollaries; namely the unique path lifting property and the homotopy lifting property. So, even from that you can show that p lower star is injective.

So, what this will tell you is that these two are the same; so, let me go back there.

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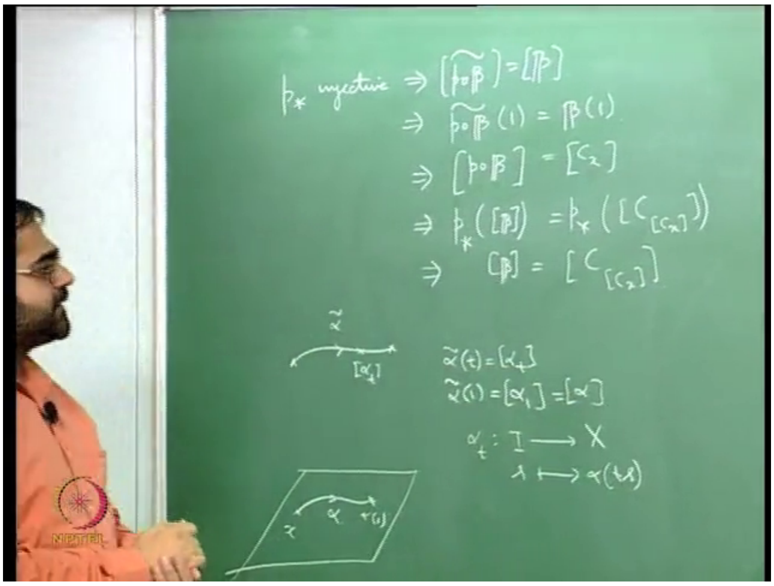

So, p lower star injective will tell you that those two elements are the same; it will tell you that these two elements are the same. So, what will tell you is p circle beta; tilde is equal to beta, these two are the same. So, what it means is they have to be a F e p homotopy; so, in particular it means they are endpoints are the same. So this will tell you that endpoint of p circle beta tilde will be the endpoint of beta.

Now, you see the endpoint of beta; so see you have the endpoint of beta is C x. So, I will get C x on this side and this p circle beta at 1 is actually; so, what is this p circle beta? So, what is p circle beta tilde? You see this is p circle beta and what is p circle beta tilde? It is the path that I get by shrinking, I just shrink this p circle beta along itself and that is a path I get above. And this shrinking at 0, it is the constant path at X and at 1; it is a path p circle beta is circle beta itself.

So, p circle beta of 1 is actually p circle beta; is just p circle beta homotopy class because for a moment if you let me again redraw that the diagram that I just erased, see this is X, this is my alpha, this is alpha of 1; what is the lift above? The lift above is alpha tilde and this alpha tilde, which is a lift of alpha is how is it given; alpha tilde at t is just alpha sub t.

So, alpha tilde at 1 will be alpha tilde at t is just alpha sub t homotopy of class of alpha sub t. So, alpha tilde of 1 is going to be alpha sub 1, but what is alpha sub 1? Alpha sub 1 is just alpha because you see alpha sub t was defined from I to X by s going to alpha of t s. So, alpha sub 1 will be just s going to alpha of s; so, this will be just alpha. So, alpha tilde of 1 is just alpha, so p circle beta tilde of 1 is just p circle beta.

But what does this tell you? This tells you the thing on the right is p lower star of the constant path at C x and the thing on the left is p lower star of beta and p lower star being injective, will tell you that this beta has to be equal to the homotopy class of beta is equal to the homotopy class of the constant path at C x. So, that is it; so that tells you that given any element or the fundamental group above it is homotopy to the constant path; that means, the fundamental group above is trivial. And since the space above X univ is arc wise connected, all the fundamental groups at various points are all going to be trivial and therefore, it is simply connected.

So, again let me repeat what is happening; I will start with a loop base at C x; I push it down and then if I take the unique lift of this again to C x; by the unique lifting property I am going to just get back this and if I take the end points, that is going to tell me that the homotopy of beta is the same as the homotopy type of the constant map at this point. And therefore, the fundamental group is trivial; that is the whole point.

So that finishes the construction of the universal covering space, what one needs to know is; next is the following, one needs to understand why the fundamental group of x can be identified as a sub group of auto morphisms of X sub univ. So, this will involve some work.