An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-Dimensional Tori and Elliptic Curves

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Lecture – 18
Classifying Complex Cylinders Riemann Surfaces with Universal Covering the Complex Plane

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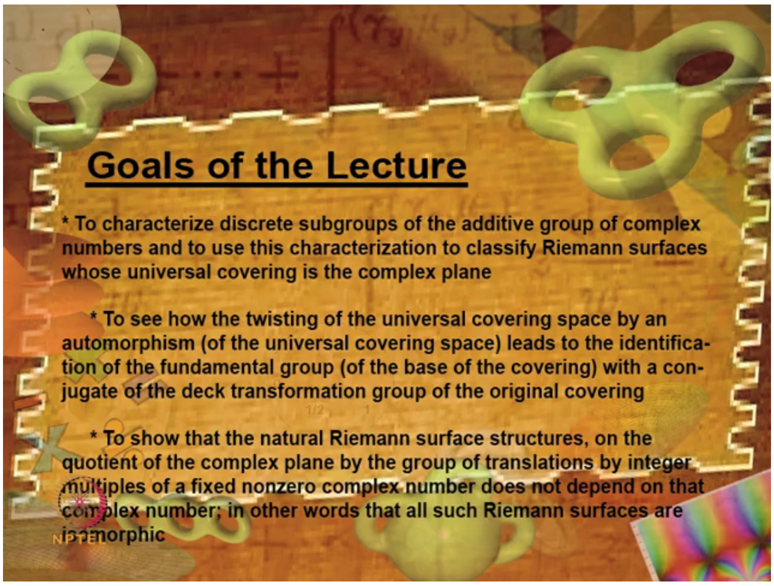

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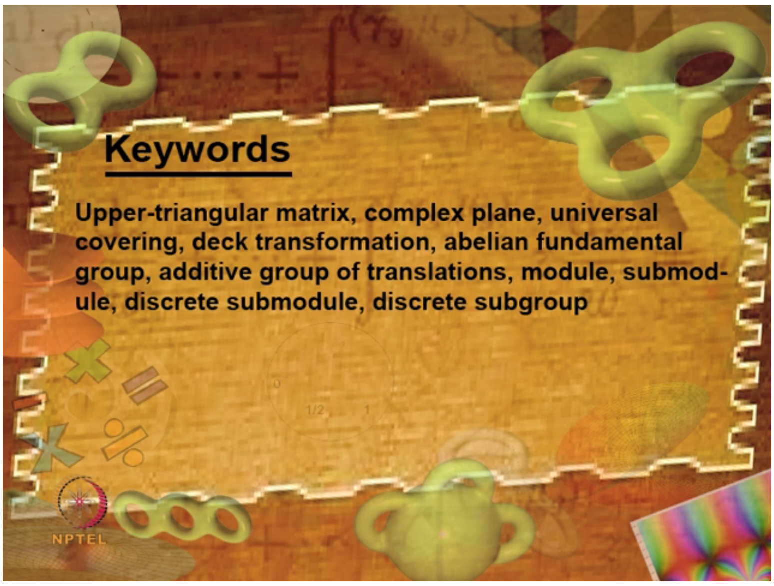

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So let us continue with our discussion. So, let me remind you where we were at the end of the last lecture.

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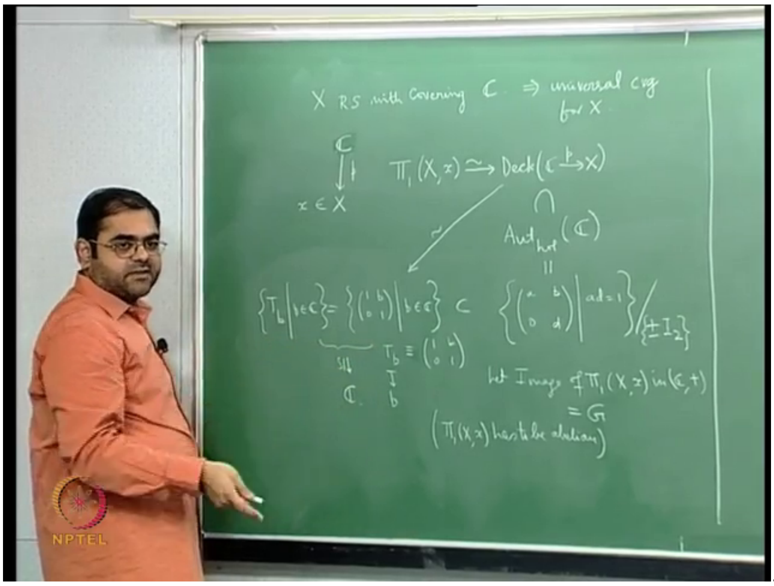

So, we had taken x Riemann surface with covering the complex plane. So, of course, this implies that . So, if I call the covering as, I am I call the covering map as p from C to X, since the complex plane is simply connected this is also the universal covering. So, this implies it is a universal covering, for x and if I fix a point small x in capital X, then you know that the fundamental group of capital X be it is small x can be identified with the deck transformation group of the covering .And the deck transformation group is a subgroup of the holomorphic automorphisms of the universal cover, which in this case is the complex plane.

And this the automorphisms, the holomophic automorphisms is the complex plane are upper triangular mobius transformations, those mobius transformations which when represented in matrix form are upper triangular. So, they will be given by this kind matrices. So, so a d a d equal to one and of course you have to go mod plus or minus the identity matrix a sub grouped two element sub group ok .

In fact, what happens is that ,the deck the deck transformation group actually lands even in a smaller subgroup ,then this namely the subgroup of translations that is because you know you know that any deck transformation cannot have fixed points, unless it is a identity transformation you cannot have even a single fixed point therefore, if you take a non trivial deck transformation it should be it should be a mobius transformation, that goes from C to C and has no fixed point .And of course as a gentle mobius transformation it should fix infinity the point at infinity. So, it has to be a translation. So, the moral of story is there is a subgroups of translations which is which can be written actually in the form ah 1 b 0 1 b in C this is a subgroup here ok.

And this is actually the this is just the set of translations by complex numbers, where of course translation by b is just the map that sends z to z plus b .And the image of this will actually will actually land inside this. So, this image here what you will get is it will this the image of the subgroup will actually land inside this subgroup ok.

And you know I can further identify this with, I can identify this with complex numbers by just sending T sub b to b. So, you just sent T sub b to b ,and of course T sub b is being thought of as the mobius transformation, it is a mobius transformations z going to z plus b which has this matrix representation given by 1 b 0 1. So, and this will give you an isomorphism of groups and of course, here it is going to be a group at addition and therefore, first of all the first observation that one makes is that pi 1 has to be abelian,because it the pi 1 is a subgroup of this group of translations and translations commute with one another and therefore, this an abelian group and therefore, pi 1 is abelian. So, this condition that X is a Riemann surface with universal covering C automatically forces that there fundamental group of X has to be abelian ok.

So, and well. So, a image of pi 1 of capital X comma small x in C comma plus, let the image b equal to G, let us call it as G. So, of course, we get this pi 1 of X has to be abelian ok.And what else did we see last time, last time we prove that this G which is the subset of the complex numbers, it is a subgroup under addition ok ,this G is actually as a set of complex numbers it is discrete .

So, G is discrete and another thing that we also noticed was that G is a z sub module of C. That is if a translation if you take an element of G which corresponds to a translation, then all integer multiples of if you compose a translation. So, many times you are going to just get another translation which is integer multiples of translation by integer multiple of the original number. So, what I am saying is T sub b is in G, then T sub n times b is also in g ,which is just T sub b composed n times ,assuming n is positive of course, if n is negative then you have to compose T minus b, T sub minus b minus n times ok.And of course T sub b composed 0 times is to be taken as T sub 0, which is just the identity map right.

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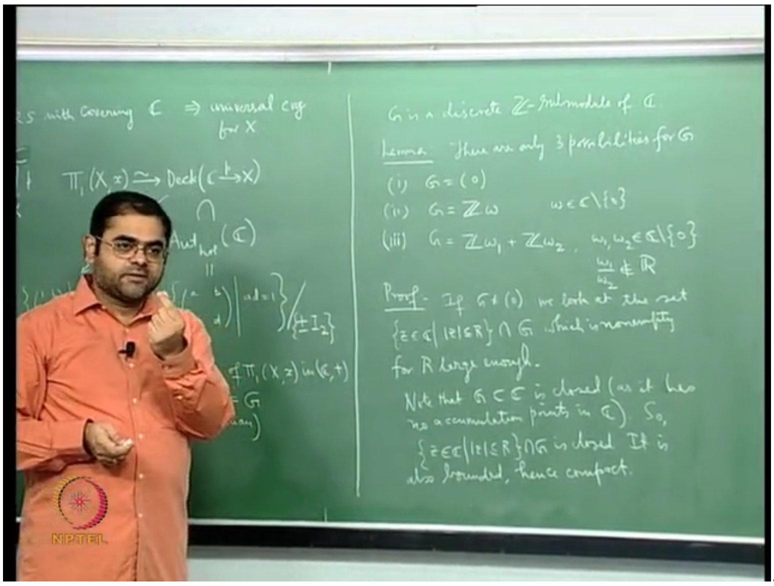

So,. So, what we G is a discrete is at z module of a C, we have seen this and then I was give you telling you about this lemma which says that there are only three possibilities for G, number one of course, is a trivial possibility that G is just the 0 group, the trivial group that means it consists of only the identity mapping translation by 0 ok

The second thing is G is just integer multiplies of a single non 0 complex number omega a non 0 complex number ,and then the third possibility is the G is a integer it is an integer linear combination of two complex numbers ,where both the complex numbers are non 0. And their ratio is not a real number that means, here linearly independent as elements of linearly independent over r over reals as elements of the field of complex numbers ok.

So, these are the only three possibilities and. So, let us take establish this proof . So, the first thing I want to say is of course, let us assume G is not 0, if G is not equal to 0 ,we look at the set of all Z in C ,such that mod Z is less than or equal to capital R intersection G ok, which is non empty for R large enough ok.

So, G is not 0 then G has some element. So, all you have to do is you have to choose capital R greater than the modulus of that element and then this intersection will continue in that element. So, that is you can choose R sufficiently large. So, that this is non empty now, the point is that this intersection will turn out to be finite .The point is that this intersection will turn out to be finite why is that so, that is because first of all note that G in C is closed ,ok the set of point G is closed, because you know you get the closure of a set by adding to the set accumulation points, ok but G has no accumulation points. G is just a discrete set. So, G is equal to it is closure. So, it is closed. So, the G is a closed set as it is it has no accumulation points in C, that is what we proved last time ,that is how we proved G is discrete ok.

And then this set here this is just a disc of a radius less than or equal to R it is a closed disc. So, it is a closed set, ok it is already this is also a closed set of the compact close subset of the compact this intersection of two closed sets and therefore, it is closed also. So, this set the set of all Z in C such that mod Z less than or equal to R, intersection G is closed, but it is obviously bounded, because it is a subset of this close disc of bounded radius. So, it is closed and bounded and therefore, you can conclude that it is compact ok. So, it is also it is also bounded hence compact, therefore it is a compact set ok.

Now, you see take every point see every point of G is an isolated point. So, we can find a small you know disc surrounding that point, which open disc if you want surrounding that point that does not contain any other points of G ok. Now you take all those discs correspondent to the points in this intersection, that will be an open cover for this intersection, but it is compact. So, there must be a finite sub cover and that will force that this has to be finite ok.

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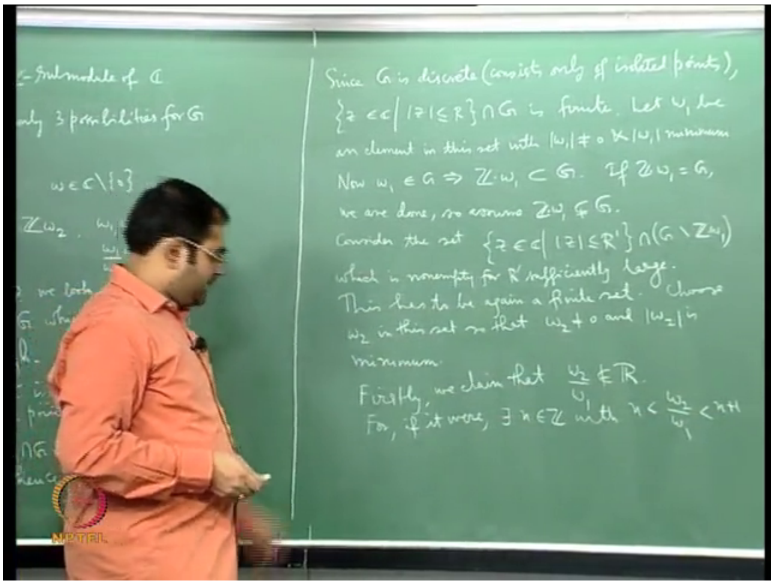

Since, so let me write that here since G is discrete, consist only of isolated points, this set of all Z in C such that mod Z less than or equal to R, intersection G is finite it is a finite set.

In fact, you know that this close disc is compact ok, and you know that if there is a if you take any infinite subset it should have an accumulation point. So, that will also that is also another way of saying that this intersection has to be finite ok.

So, it is finite. So, what I can do is I can choose a member here, which is closest to the origin. It is a finite set of complex numbers and is non empty, and it contains a non 0 complex number ok, because G I assume G is not 0. So, and you know I have chosen R in a such a way. So, that there is at least one non 0 complex number in this set G.

So, I can choose a complex number in this intersection such that it has minimum modulus and let me call that as omega 1. So, let omega 1 be an element in this set with mod omega 1 not equal to 0 and mod omega 1 minimum, because it is a finite set I can of course choose one such element.

Now, omega 1 is an element of G, this implies Z dot omega 1 will also be subset of G ok. That is because G is a Z sub module of C alright, and of course if Z dot omega 1 is equal to G we are done, we have come to case two. So, if Z dot omega 1 is equal to G we are done .So, assume Z dot omega 1 is properly contained in G ok, then I will have to show that we will be in the third case of the lemma alright.

So, if this is true again we play the same game ,what you do is consider ah, the set again you take the set of all Z in C, such that mod Z is greater less than or equal to sum R prime now, and intersect it with G minus the compliment of Z dot omega 1 in G. So, what is this set this is all those elements of G, which are not an integer multiple of omega 1 ok ,that is what the set is and then I am intersecting it with this close disc ok and since the other elements of G which are not integer multiples of omega 1, this is a non empty set and therefore, if I choose R prime sufficiently large this intersection will be non empty. So, consider the set which is non empty for R prime sufficiently large.

Ok consider this set, now again the same argument will tell you that this set is also finite, why because you see the ah what will happen is that you see this is a disc this G is discrete ok, every subset of a discrete set is continues to be that continues to be discreet and so this is a discrete set in this closed and bounded disc ok. So, it has to be finite and of course again the argument will be if it were not finite, then you know every infinite subset of a compact set like this will contain a an accumulation point, but we know G has no accumulation points therefore, this has to be again finite set ok, this has to be again a finite set alright and ah choose omega 2 in this set. So, that you know omega 2 is not 0 and modulus of omega 2 is minimum ok. Now in this set you choose an omega 2, which means you know it is an element of G which is not an integer multiple of omega 1 ok.

And of course, since there are only finitely many I choose one with minimum orders that I can do alright of course omega 2 is non 0. Now we are going to now I am going to show that G is actually Z omega 1 plus Z omega 2 ok, that is what I am going to prove. So, for the first firstly, we claim that ah omega 2 by omega 1 ok is not a real number ,the first claim I am making is that omega 2 by omega 1 is not a real number, or if it were then that would be exist an integer n with you know n strictly less than omega 2 by omega 1 strictly less than n plus 1 you can get this.

See if you have a real number then of course, you can it has to lie in some interval ok and you will have strictly equality here because omega 2 is not an integer multiple of omega 1, omega you cannot have n equal to omega 2 by omega 1, because that will mean omega 2 is n times omega 1 that is not possible ,because omega 2 has been chosen outside of integer multiples of omega 1, and for the same reason omega 2 by omega 1 cannot also be equal to n plus 1. So, it is this two are both strictly (Refer Time: 19:52), but this will immediately give a contradiction.

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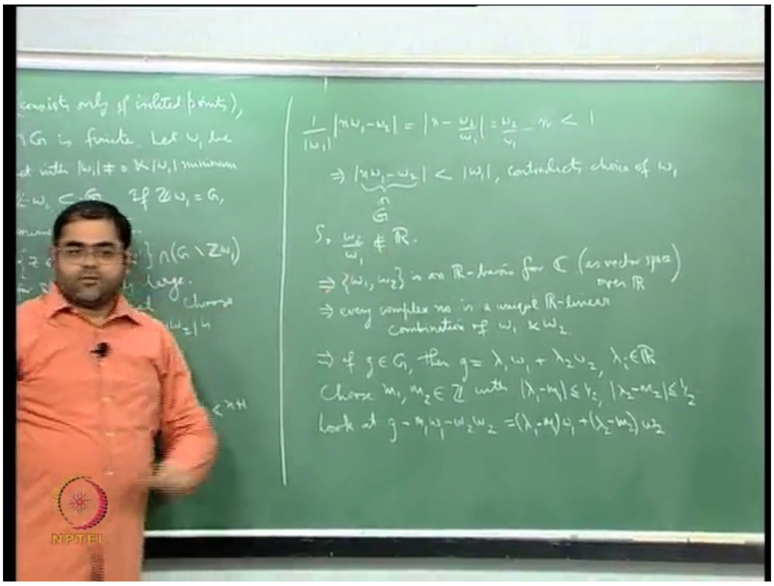

That is because you see, what will happen is you know if i write if I consider 1 by mod omega 1 times modulus of n omega 1 minus omega 2 ,if I calculate this then this will be modulus of n minus omega 2 by omega 1 ok ,and you see n see of course, n minus omega 2 by, but omega 2 by omega 1 is greater than n. So, this will be actually be omega 2 by omega 1 minus n ,that is what it will be because omega 2 by omega 1 is a real number greater than n alright

And of course you see, but any way this is lying in an interval of length 1, therefore this has to be less than 1. So, this is less than 1. So, what this will tell you it there it will tell you n omega 1 minus omega 2 is strictly less than mod omega 1. That is what it will tell you, it will tell you n omega 1 minus omega 2 is strictly less than mod omega 1, but you see this the this element this belongs to G ok. So, what you have done is you have found an element of G, whose modulus is less than mod omega 1, but mind you mod omega 1 is chosen with minimum modulus ok mod omega 1 was chosen with minimum modulus.

See does not see once I have once I have chosen mod omega 1 I can still increase R ,I can let R to code infinity it is not going to change the choice of this mod omega 1. So, mod omega 1 is kind of smallest it is one with smallest modulus, but you see now I have got hold of an element of G with modulus lesser than mod omega 1, that is the contradiction. Ok so contradicts choice of omega 1 ok. So, this contradiction will tell you that omega 2 by omega 1 cannot be real. So, omega 2 by omega 1 is not a real number.

So, you get that, now the next thing is the movement omega 2 by omega 1 is not a real number, it means that omega 1 and omega 2 are they form a basis for the complex numbers as a vector space over real numbers, because you see they are there is since the ratio is not real they are linearly independent over R ok, and it is linearly independent there are two elements and the dimension of the complex numbers as a field over real numbers is two dimensional.

So, the moment you have a linearly independent set with cardinality equal to the dimension it has to be a basis. So, so this will imply that you know ah omega 1 comma omega 2 is an R basis for the complex numbers C, as a vector space over R ok, now so what does that that implies every complex number, can be is a is a unique is ah a unique linear combination a unique R linear combination of omega 1 and omega 2 ,that is what basis makes, every element can be expressed as a linear combination of omega 1 omega 2 ,and the coefficients of omega 1 and omega 2 are unique ok.

So, but this happens for every complex numbers. So, it happens for every element of G, after all G is a sub set of complex numbers. So, this implies if you take a small g and capital G, then G has to be writable in the form lambda 1 omega 1 plus lambda 2 omega 2 where lambda I are real numbers ok. I should be able to write any element of G in this form as a real linear combination of omega 1 and omega 2, and the coefficients lambda 1 and lambda 2 are unique, that is because of linear independence of omega 1 and omega 2 ok.

Now, well again lambda 1 and lambda 2 are real numbers. So, I can choose for both lambda 1 and lambda 2, I choose an integer that is closest to lambda 1, and I choose an integer that is closest to lambda 2 ok. So, I will write that down choose m 1 comma m 2 integers with modulus of lambda 1 minus m 1 less than or equal to half modulus of lambda 2 minus m 2 less than or equal to half I can do this of course, you see lambda 1 is a real number it has to lie in some interval. So, it is an interval with integer n points. So, if it lies to the closer to the left end point you take that as your m 1, if it lies closer to the right end point you take that as your m 2, in any case you can get m 1 m 2 with this (Refer Time:25:25) alright.

Then ah and once I do this and you know now I want you to look at G minus m 1 omega one minus m 2 omega 2, I want you to look at this difference this turns out to be lambda 1 minus m 1 into omega 1, plus lambda 2 minus m 2 m 2 into omega 2 ok by our definition alright.

And now let us compute the modulus of this difference and try to use it for triangle inequality.

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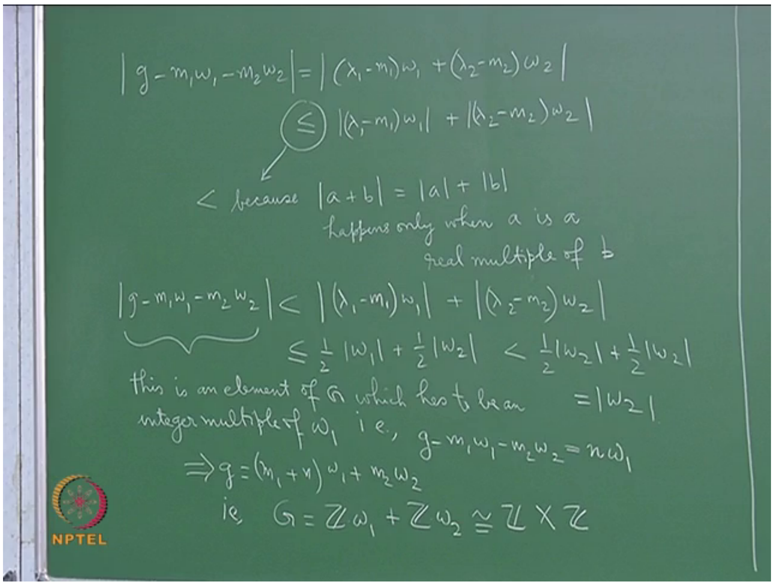

So, modulus of G will be modulus of modulus of sorry G minus m 1 omega 1 minus m 2 omega 2 will be modulus of lambda 1 minus m 1 into omega 1, plus lambda 2 minus m 2 into omega 2 and now if I apply the triangle inequality this is less than or equal to modulus of lambda 1 minus m 1 omega 1, plus modulus of lambda 2 minus m 2 into omega 2. I will get this by the triangle inequality mod of a plus b less than or equal to mod a plus mod b ok.

And what I want to say is that I want to say that this is actually strict inequality ,why is this a strict inequality because you see ah this is a strict inequality because you see mod of a plus b is equal to mod a plus mod b for two complex numbers a and b occurs only when a and b are real multiples of one another there is a co linearity condition for the three sides of a triangle to that makes the for the that makes a triangle degenerate into a straight line and then your triangle inequality becomes an equality that is a only case. So, because this happens only when a is a real multiple of b ok.

So, you know if this does not happen if there is equality it will tell you that you know it will tell you that the you know if there is an equality here it will tell you this a real multiple of that, but that then these are real coefficients. So, it will tell you omega 1 is a real multiple of omega 2, but that contradicts the fact that the ratio omega 2 by omega 1 is not a real number ok. Therefore, this in this triangle inequality this has to be a strict inequality ok. So, what I can write is that modulus of g minus m 1 omega 1 minus m 2 omega 2 is strictly less than mod lambda 1 minus omega 1 m 1 lambda 1 minus m 1 times omega 1 plus mod lambda 2 minus m 2 into omega 2, and this is strictly less than or equal to half mod omega 1 plus half mod omega 2 that is because mod lambda 1 minus m 1 and mod lambda 2 minus m 2 are each less than or equal to half.

Now, you see let us recall how omega 1 and omega 2 were chosen, omega 1 was chosen to be an element of g with least modulus which is non 0 ok and omega 2 was chosen to be again of least modulus of course, non 0 modulus among those elements of g which are not integer multiples omega 1 . Now, therefore, I can write that this is less than half omega 2 plus half omega 2 which is equal to omega 2 ok therefore, you see this quantity on the I mean this element on the left side g minus m 1 omega 1 minus m 2 omega 2 you see this cannot be an element of g ,which is not an integer multiple of omega 1 because if it were then it would have modulus less than mod omega 2 which is against the choice of ah omega 2 therefore, the conclusion is that this quantity has to be an integer multiple of omega 1 ok and that will tell you that g is therefore, an a lean integer linear combination of omega 1 and omega 2.

So, let me write that down here ,this is an element of G, which has to which has to belong which has to be an integer multiple of omega 1 ,for if it is not an integer multiple of omega 1 then it will be an element with modulus less than that of omega 2 which is against the very choice of omega 2. So, it is an integer multiple of omega 1 that is so, if I write it down I will get g is equal to g minus m 1 omega 1 minus m 2 omega 2 is actually say some n times omega 1, and this will tell you that g is actually m 1 plus n times omega 1 plus m 2 times omega 2 and in other words what you will get is therefore, g is an integer multiple of omega 1 and omega 2, and it is an integer linear combination of omega 1 and omega 2 and therefore, you will get g is actually Z dot omega 1 plus Z dot omega 2 as we wanted. So, that finishes the proof of this claim that G is just the group of integer linear combinations of omega 1 and omega 2 ok and of course, you must realise that this is isomorphic to Z cross Z as a group under addition. So, that finishes the proof of the claim.

Now, what we have going to do is use this lemma to get all the results that we want. So, what I will do is ah let me rub this off and look at a each of these cases.

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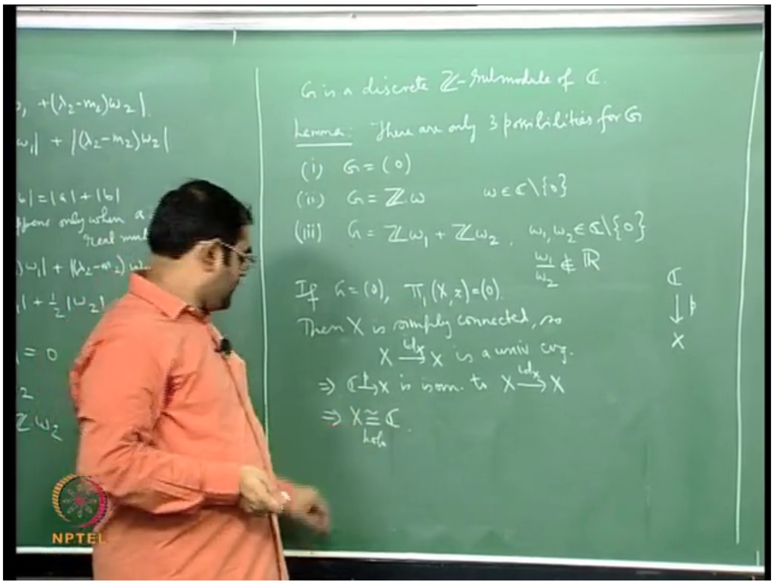

So, you see remember that our situation was we had C to p ,C to X, p is covering map this universal covering this was our situation, if ah G is 0 then what happens see if G is 0 that means, pi 1 of x capital X comma small x is 0 ,after all G is the image of pi 1 if you remember pi 1 the fundamental group of the base was identified with the deck transformation group. The deck transformation group was identified with the subgroup of translations and the image of the deck transformation group is what we had called as G. G is isomorphic to pi 1 ok if g is just the image isomorphic image of pi 1. So, if G is 0 then pi 1 is 0, if pi 1 is 0 it means X is simply connected, and you know if X is simply connected then you know X to X itself the identity mapping it itself is a covering map and you know by unique by the uniqueness property of universal coverings it will tell you that X has to be holomorphic to C ok.

So, then X is simply connected. So, X to X identity map is a universal covering, and this will imply that therefore, you know C to P is isomorphic to the covering given by the identity map and this will imply that X is holomorphic to C ok so, if G is 0. So, what we have proved is if we have a if the universal covering of X is C and if the fundamental group is 0, then that corresponds to this case and then X has to be just C, these are simplest case ok

Then of course, you know what I expecting other two cases. So, let me write it down.

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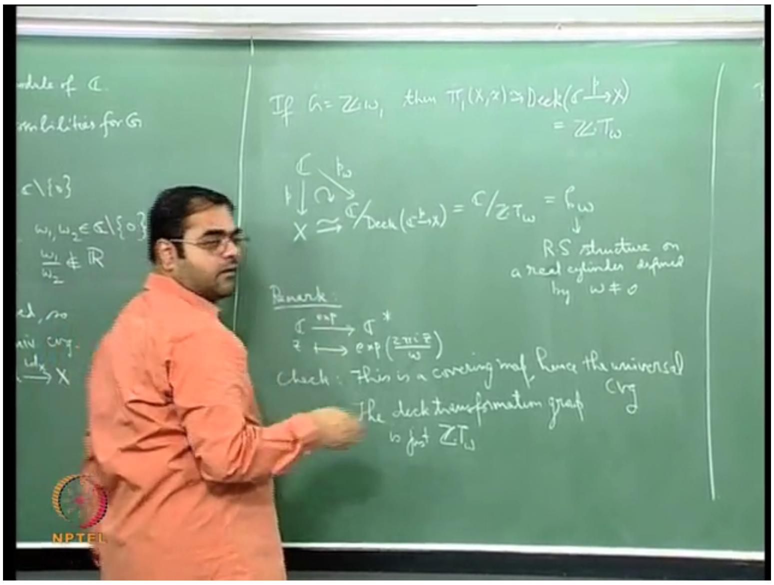

So, if G is Z dot omega 1 ok, if G is Z dot omega 1, then actually what it means is then pi 1 x comma x is identified with the deck transformation group of this covering and this is just Z times translation by omega ok, mind it that is how we identified a deck transformation with a translation .We found that all possible deck transformations can only be translations ok.

And the complex number omega corresponds to translation by omega. So, this is what it is. And then what happens we have that we have see let me recall another fact that I told you last time,see we have to C to P, C to X this is the covering, I told you C to C modulo the deck transformation group ok of this cover these two can be identified, such that this diagram commutes so. In fact, you know I told you that ah the if you take the fundamental group of the base space below that gets identified with a subgroup of automorphisms of the base of the covering space ok that subgroup is nothing, but the deck transformation group and if you go modulo the deck transformation group what you will get back in the identified with the base space ok.

So, this can be identified with X, but then you know this is just C modulo Z dot T omega ok ,and if you remember this was C sub omega in one of our other example, this is the a Riemann structure surface on the cylinder defined by this non 0 complex number omega. So, this is Riemann surface structure on a cylinder defined by omega not equal to 0 ok.

So, what you are saying is that if the universal covering of X is C, and if the fundamental group is isomorphic to Z, then we are in case two and your Riemann surface X is nothing, but a holomorphic structure namely a Riemann surface structure on the real cylinder ok .So In fact, if you want else a real cylinder on a real cylinder and ah. In fact, you know i. So, this first tells you that X has be just a Riemann surface structure on a cylinder, then I gave you if you go back and recall I gave you a theory I am saying that all these various holomorphic structures that you can get on a real cylinder, if you change omega they all do not change they are all the same ok.

So, I made that statement also. So, let us try to prove it. So, what I want to say is that if you change omega here still this the various the various X you are going to get for different omega ok ,they are all going to be the same. In fact, I told you that there is one special representative that is given by the exponential map as a covering map from C to C star; I told you that was the special representative for all the Riemann surface structures on a real cylinder. So, let us try to prove that. So, first of all I what I want to say is that. So, I want to first make I want to make a remark, see let us take from C to let me take the map from C to C star ,which is given by Z going to exponential of 2 pi I Z by omega ok.

Let us take this map C to C star, Z going to e power 2 pi I Z by omega I can divide by omega because omega is not 0 alright, now what you can do is that you can easily check that this is a covering map therefore, it is universal covering ok, and you can check that the you can check that the deck transformation group is just translation by integers ok, because that is that is what see the kernel of this map is exactly the integers alright. So, when integer times omega. So, what I want to say is that. So, let me write that down check this is a covering map hence the universal covering, because the space above C is simply connected you check that it is a covering map. For this you just have to study the properties of the exponential function Z f of Z equal to e power Z, if you study properties of exponential function you will see that mind you omega is a constant omega is to be treated as a constant Z is a variable alright.

So, property is a exponential map will tell you that this is a universal covering alright .Then that is one thing you will get the second thing is you will you can see that the deck transformation group is just Z dot translation by omega, that is translation by integer multiples of omega. So, if Z is an integer multiple of a omega alright ,then you can see that this is going to be one ok this is going to be one, if Z is n omega I will get e power 2 pi I n that will be 1 alright.

So, what these two things will tell you is that this is now that put together with this remark will tell you that C modulo Z dot translation by omega is just C star. So, in other words what it will tell you is that X has to be just holomorphic to C star ok. So, let me right that down.

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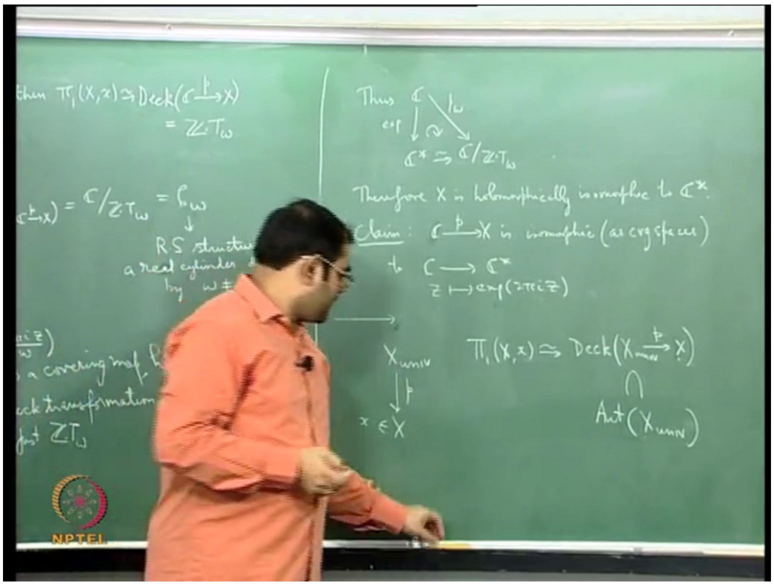

So, thus C to C star by this map can be identified with C to C mod Z dot T sub omega this is the group of integer translation by integer multiples of omega, this can be identified this is our map well if you want let me call this as this is p sub omega ok alright and if I call this map as X alright. So, this is still X and this is a P sub of omega ok.

So, what these two diagram together will tell you is that this covering C so, it will first tell you that X is just homeo is actually not only homeomorphic, but it is actually holomorphically isomorphic to C star and it will tell you that C to X this universal covering is as same as this universal covering C to C star.

So, therefore X is holomorphic I mean by holomorphic holomorphically isomorphic to C star ok, X is holomorphically isomorphic to C star alright.

In fact, that I told you that you can scale this omega and just make this map the exponential map which is just Z going to e power 2 pi I Z that that is what I gave as the canonical representative for all in the holomorphic isomerism class for all these Riemann surfaces. So, let me explain how that happens because there is something there is a little bit of covering theory that comes into the picture. So, so let me make this claim C p X is isomorphic as covering spaces to C C star Z going to exponential of 2 pi I Z ok. So, this is a statement that I made you take all possible Riemann surface structure on a cylinder, then they are all isomorphic and they are all isomorphic just to C star, that is the theorem that I stated and now I am saying that even at the covering space level ah not only is not only is X isomorphic to C star holomorphically isomorphic this covering itself is isomorphic to the exponential map ok.

So. In fact, I told you that this was this can be taken as a special representative for all these ah holomorphic structures on the cylinder, Riemann surface structures on a cylinder. So, let us see how this is true. So, for this I am going to do something there is a little bit of discussion. So, let us again go back to our let us go back to the topological category let us take a covering let us take a topological space X now. So, do not confuse this X with our X forget our X for the present. Let us assume that we have a topological universal covering and you fix this point small x capital X then you know that the fundamental group of the base space can be identified with it can be identified with the deck transformation group of the universal cover ok, which is subgroup of automorphisms let me just say homeum I will just put automorphisms of the universal covering space.

Of course you know if this is a holomorphic covering ok, then this should be holomorphic automorphisms and everything will be holomorphic, if it is just a topological covering then these are all just homeomorphisms all right and now what I am going to do is I am going to twist the covering by a by an automorphism of the universal cover. So, what I am going to do is a following.

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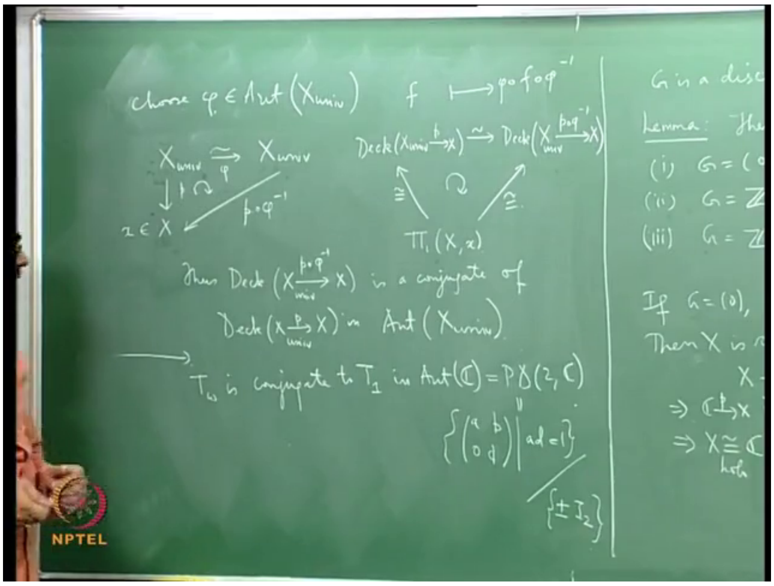

So, what I am going to do is the following choose a phi which is an automorphism of the universal covering ok, then you see X to X sub univ p this is a covering alright then what I will do is I put this phi here and I will consider this map which is the composition first apply phi inverse then apply p ok.

Now, what will happen is that this will also be a covering map this will also be a universal covering after all this is an isomorphism and this is the covering an isomorphism followed by a covering is also a covering.

So, this is also a covering, but this is again simply connected. So, it is also the universal covering. So, this is also another avatar of the universal covering alright .Now what will happen is because of this the fundamental group of X will also be identified here and there is a difference in these two identifications ,that differences arises because of this automorphism phi and what that difference is it is actually conjugation by phi. So, that is what I want it to realise. So, so this will give rise to a diagram like this. So, I have the fundamental group of capital X base it is small x so i fix a point small x in capital X and this is identified with the deck transformation group of this covering X univ p X and this identification is because of this cover ok.

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Now, this covering will give me another identification. So, what will I will have here is I will also have deck transformations of the other covering that is X sub univ to now this covering is p followed by phi inverse. So, that is also an identification like this ok. So, this covering gives me identification of the fundamental group of X with the deck transformation group of this cover, and similarly this covering gives me identification of X, identification of fundamental group of X with the deck transformation group of this covering and what is this map this map is just isomorphism that sends any f to you know phi inverse followed by f followed by p ok.

So, it is varies clear see if you give me a deck transformation here something that permutes the decks here, how do I get one here I go by phi inverse apply this deck transformation and then apply phi, that is how this is related. So, what this tells you is that if you change the universal cover ok by an automorphism of the universal covering space ok.

The top space the covering space then the deck transformation group will change by a conjugate where it will change by a conjugate subgroup in the group of automorphisms of the universal cover. So, what this will tell you is thus deck the deck transformation of X to a p p composite phi inverse X sub univ to X is a conjugate of you see the deck transformation group of the original cover in T group of automorphisms of the universal cover ok. So, the in other words you know this is nearly one should say philosophically reflection of the fact that you see the fundamental group will change by a conjugate if you change the base point ok.

So, what you do is the same kind of thing happens above, if you change the covering by an automorphism of the top space then the deck transformation group will change by a conjugate ok. So, if you remember this it is very easy to actually see that this translation by the translation by any omega is conjugate to translation by 1, in the full group of automorphisms of C. So, what I want to say is that translation by omega is conjugate to translation by 1 in automorphisms of C, which is p delta 2 C, which is of course you know the set of all metrics of the form a b these are all just translations, a b 0 c such that a b 0 d such that a d equal to 1, modulo plus or minus identity 2 identity matrix ok.

And how do you prove this it is very simple. So, let me write that down. So, see you just solve for you solve for such a transformation ok.

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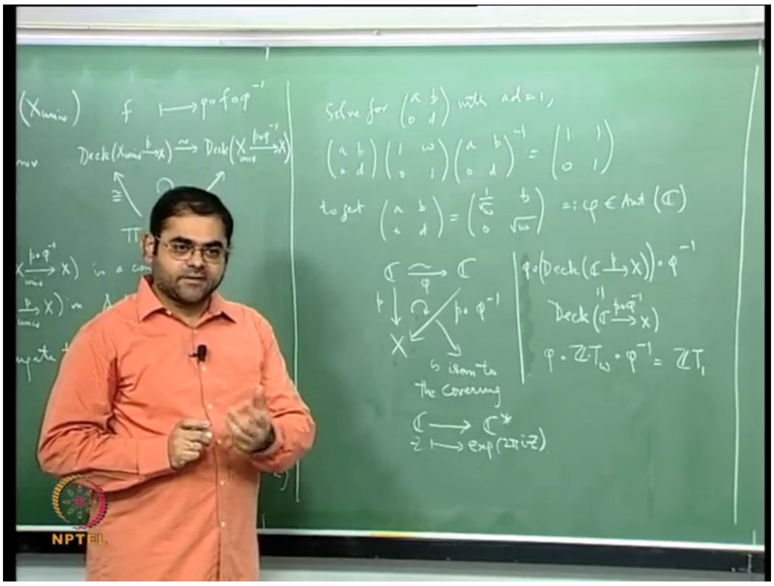

So, what you do is solve for a b 0 d with a d equal to 1, a b 0 d if you conjugate that with translation by omega which is given by 1 omega 0 1, and if I put a b 0 d inverse I want to get 1 1 0 1 which is translation by 1 ok. See you can solve for this you can you can you will get a b 0 d to b of a you can it is a simple calculation what you will get is let me write it down it is just going to be, 1 by root omega b 1 by root omega b 0 root of omega ok, where b is any complex number any such mind you omega is non 0. So, it has a square root choose any one square root that is two square roots and you take 1 by root omega and root omega here ok , now this is clearly determinant one and this a mobius transformation ok.

So, now let me call this as a let me call this mobius transformation as phi ,let me call phi to be this mobius transformation then phi is what phi is an automorphism of C is the universal covering alright. So, now, look at this you have C and you have p and you have X, and then I apply this phi here I have this automorphism of C and here I get I get the other covering just like that I get p circle phi inverse this diagram commutes ok now (Refer Time:56:05) and this diagram commutes you are going to have an identification of deck transformation group with the deck transmission of group with this and what it will tell you see that will tell you that deck transformation group of a C p X is this group this group if you conjugate it by phi you get that group ok. So, phi. So, let me write that. So, I will write it as phi circle deck ,circle phi inverse is actually deck transformation of C to this is p circle phi inverse I will get this and in other words what I will get is phi circle this is Z dot T omega, is just Z dot T sub 1 ok.

So; that means, that for this covering the your actually going modulo translations by ,, and if you were going modulo translations a I mean translation by integer multiples of 1, which is a translation by integers and if you are going modulo translations by integers then this covering is the same as is isomorphic to the covering Z going to e exponential of 2 pi I Z. Because this is the map when you are going modulo translations by integer multiples of omega, if omega is 1 then this covering map will be just Z going to expensive of 2 pi I Z. So, what this will tell you is that this guy is isomorphic to the covering C to C star, this is given by Z going to exponential of 2 pi I Z ok, because here the deck transformation group is just translation by integer multiples of 1, translation by integers. So, the moral of the whole story is take any Riemann surface whose universal cover is C the complex plane and assume that the fundamental group is isomorphic to Z.

Then that covering it iself not only is that Riemann surface isomorphic to C star, but the covering it iself is isomorphic to this specific covering. So, this is just unique representative for all possible covering spaces of holomorphic structures on a real cylinder, Riemann surface structures on a real cylinder ok. So, so that ends it and finally, let me look at the third case the last and final case namely when G is isomorphic to Z dot omega 1 plus Z dot omega 2 you know it that in this case we are going to get a complex torus ok.

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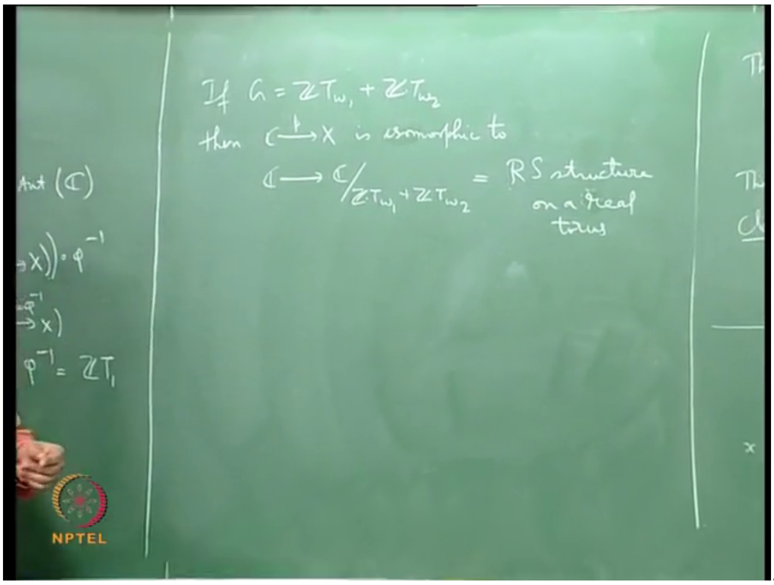

So, if G is Z dot translation by omega 1 plus z dot translation by omega 2, then C p X is identified is isomorphic to C mod Z dot translation by omega 1 plus z dot translation by omega 2, which is Riemann surface structure on a torus on a real torus. Ok so, this is the third case. So, the moral of the story is whenever the universal covering of a Riemann surface is a complex numbers, then either it is that Riemann surface is either the complex numbers itself or it is C star and the covering is just given by the exponential map, or it is a Riemann surface structure on a complex torus ok. And these three cases corresponding they correspond to the fundamental group of the Riemann surface being 0 or isomorphic to Z or isomorphic to Z cross Z.

So, that is the that was a classification theorem that I gave earlier and this is the proof. So, you can appreciate the elements of topology and covering spaces coming into the picture yeah. So, I will stop here.