An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture - 21
Torsion-freeness of the Fundamental Group of a Riemann Surface

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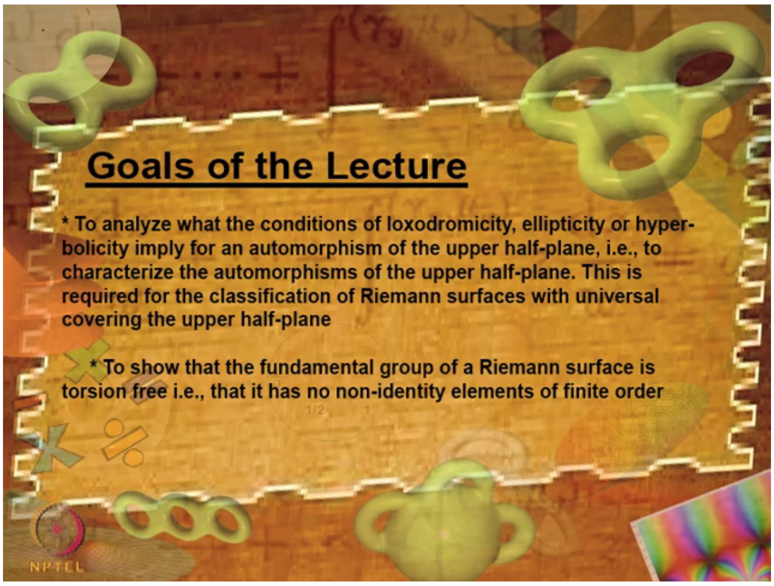

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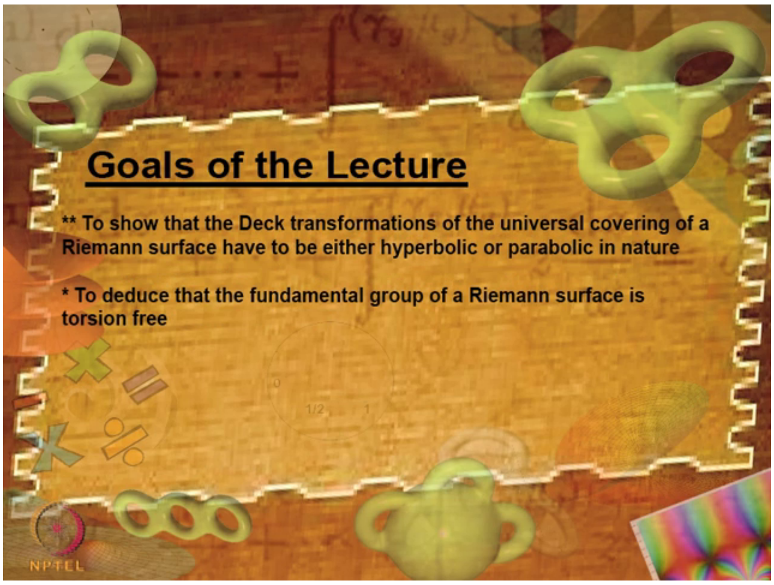

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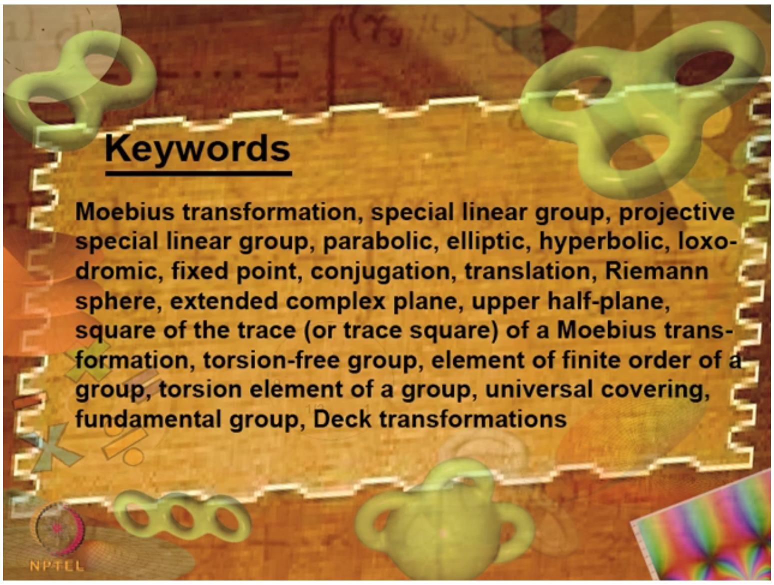

So, let us continue with our study of Moebius transformations. So, last time I explained to you the classification of Moebius Moebius transformations into you know parabolic, elliptic, hyperbolic and loxodromic Moebius transformations. So, just to recall that you see we started with a Moebius transformation.

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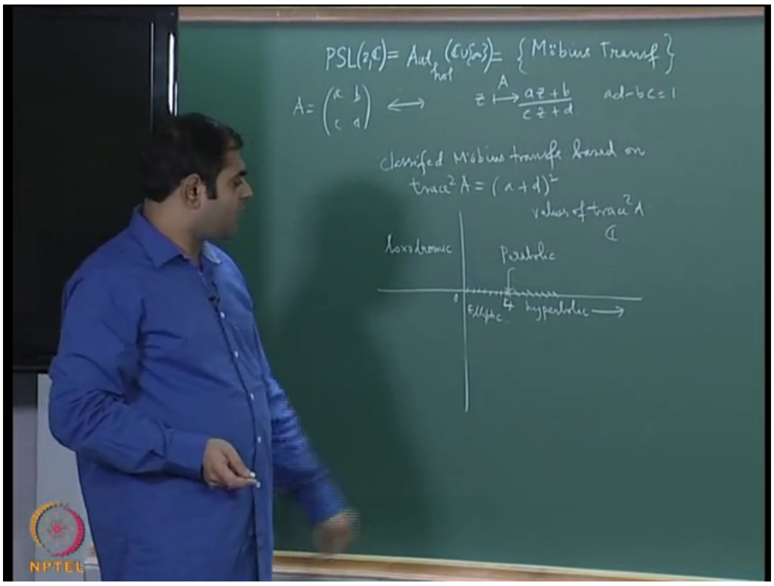

So, you take the set of all Moebius transformations that is a group under composition and that is identified with the holomorphic automorphisms of the Riemann sphere which is C union infinity which is the same as C union infinity via the stereographic projection and well these the holomorphic auto morphisms is written, there is a metric matrix representation which is given by the PSL 2 C. So, this is determinant one matrices with complexentries 2 by 2 matrices, go and you will have to go modulo the subgroup given by plus or minus the amenity matrix.

So, and you know the way we do this identification is on this side if you have Moebius transformation e z is e z is it going to az plus b by cz plus d. Aee this is identified with the matrix a b c d and we usually put the condition that ad minus bc is 1. So, that this matrix is already an element of SL 2 alright and therefore, this represents this Moebius transformation and of course, the other element that will represent this Moebius transformation is putting a minus sign to each of these entries and you see if I put a minus sign to each of these entries this transformation still is the same. So, what we did last time was we classified Moebius transformations based on trace squared.

So, you know the value of trace squared of A which is, if I call this Moebius transformation as A or if I call the representative matrix as A the only ambiguity is that I could have had minus sign, but that is a reason I am taking trace squared A and this is a plus d the whole squared. And well the classification went as follows you look at values of trace squared A which are which can in general be complex values and what we said was well you see if trace squared is a real number and it lies between 0 and 4, so 4 then that is if it is lied if it lies in this region we said that it is elliptic and if trace square is exactly equal to 4 we said that it is parabolic we found we saw that it is parabolic which is corresponding to the condition that it has only one fixed point in C union infinity alright. And of course, in which case if it is parabolic it is it has to be conjugate to a translation.

And then if trace squared is real and greater than 4 we called it as hyperbolic here and for trace squared values outside of this segment 0 comma 4 we call it we called the Moebius transformation as loxodromic. So, this was our classification, I mean this was at least this was a classification based on trace squared this was the definition alright and then we found that if you take. So, barring the Moebius transformations which are translations which are conjugated translations which are parabolic the other ones namely elliptic hyperbolic loxodromic. So, hyperbolic is a special case of loxodromic also because everything outside this outside this line segment 0 to 4 is supposed to be loxodromic.

So, we found that if you take a non parabolic Moebius transformation which means that it has two fixed points parabolic corresponds to the case when it has one fixed point then you can further make a conjugation and bring it to a special form. So, what was that special form, let me write that down.

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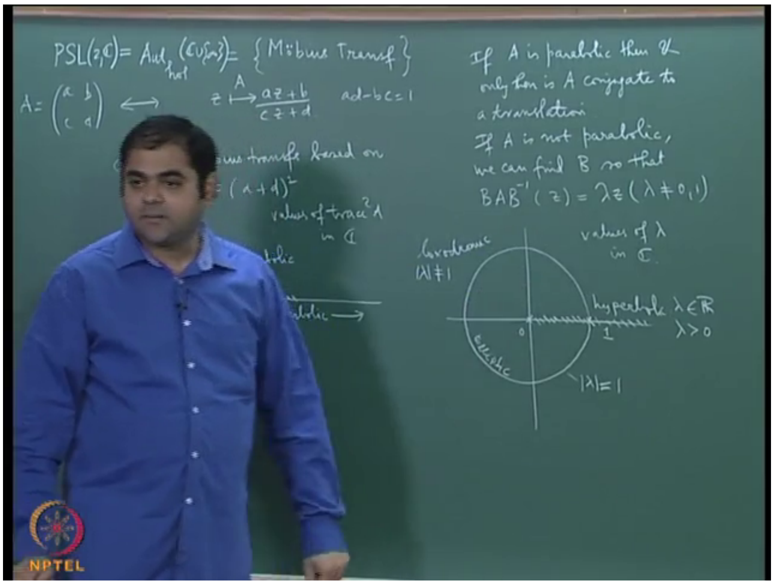

So, let me write this here if A is parabolic then and only then is A conjugate to a translation, if A is not parabolic we can find B. So, that you see BAB inverse b is another Moebius transformation and you have this transformation B. So, you this means first apply B inverse then apply A then apply B this is composition of functions if you think of it as functions here. On the other hand if you also think of it as matrices it will be just conjugation of the matrix A by B. So, this identification here under this identification the composition of mappings will correspond to multiplication of matrices here alright.

So, so we can find a B so that this is of the; is of this form e z going to lambda z and then you get various values of lambda which will tell you when it is parabolic elliptic hyperbolic I mean when it is elliptic, hyperbolic, loxodromic of course, not parabolic. Of course, lambda is not 0 or 1 because if lambda is 0 then it is z going to 0 that is not at all a Moebius transformation and lambda equal to 1 corresponds to the identity transformation which we are not considering that at all we are trying to only classify non-trivial transformations right.

So, now what are the values of lambda? So, if I draw a diagram like this. So, this is values of lambda. So, in C, here also its values of trace squared in C. So, what happens is well you know the value 0 and the value 1 are forbidden the value is 0 and 1 are forbidden then you have this unit circle which is given by mod lambda equal to 1. And this is the condition for lambda so that the original A you started out with was elliptic. So, this circle gives you; these are the elliptic 1s and of course, one is not included mind you these are the elliptic 1s.

Then you have and if mod lambda is real I mean if lambda, that is if lambda is real positive if lambda is real positive and of course, lambda is not equal to 1 then you get hyperbolic and so the real line the whole of the positive real line excepting 0 and 1 that corresponds to the case when lambda is real positive and lambda not equal to 1 that corresponds to the case when A is hyperbolic mod lambda equal to 1 is the case when A is elliptic alright and all the rest of it is loxodromic, is a loxodromic non hyperbolic case. So, in general I will just write it here as loxodromic mod lambda is not equal to 1.

So, hyperbolic is lambda is real lambda positive of course, lambda is not equal to 1. So, this is the picture that is based on values of lambda. And what is that lambda? Lambda is this special form e z going to it belongs to the special form e z going to lambda e z which you get after conjugating A by A suitable Moebius transformation B. After all if you remember B was chosen in such a way that BAB inverse has fixed point 0 and infinity and that is why BAB inverse had to be of the form e z going to lambda z alright. So, this is to recall what we did last time.

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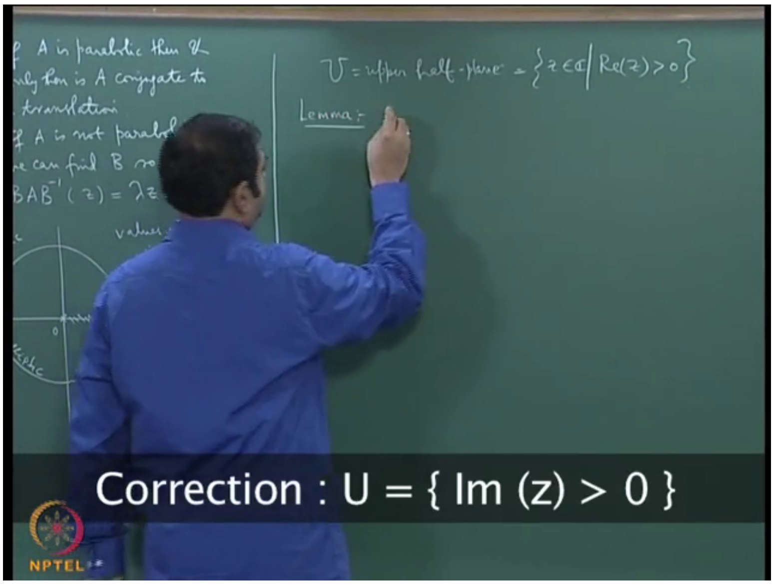

I am going to now look at, as you know we have we have studied Riemann surface is universal covering Riemann's sphere and we know that in that case it is just the Riemann surface has to be some of it is a Riemann sphere with fundamental group trivial. And then you looked at the case when the Riemann surface is having its universal cover as the complex plane and in that case of course, we have seen that its either the complex its by holomorphic either the complex plane or it is a complex structure on the cylinder which is the same as a complex structure on c star. And the third case is when it is a complex to us.

The only case that we have not seen so far is when the universal covering is the upper half plane. So, I am going to look at next what are the Moebius transformations that leave the upper half plane invariant that is automorphisms upper half plane. So, U is upper half plane which is by definition the set of all z in c such that real part of z is positive is the upper half plane and well, so here is a lemma.

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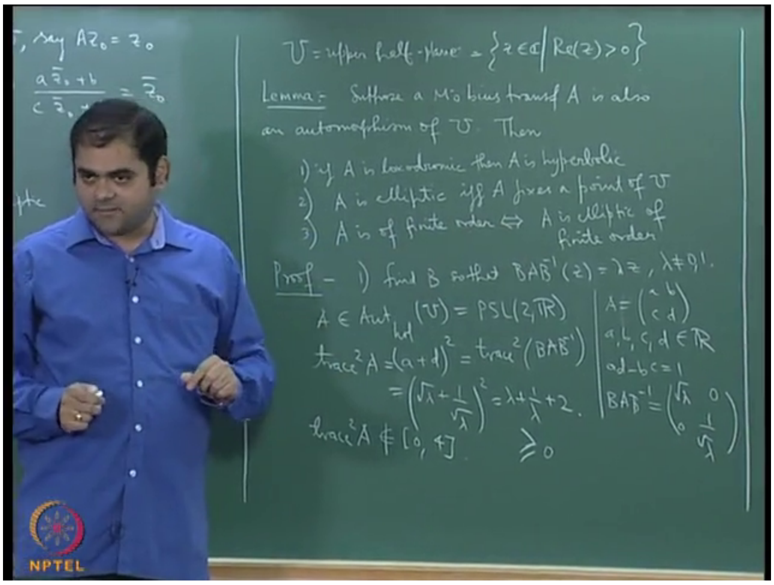

Suppose a Moebius transformation A is also an auto morphisms of U. So, I am looking at Moebius transformations which map the upper half plane isomorphically onto itself alright. Then there are special you get more special conclusions. So, the first thing is if A is high if A is loxodromic then it has to be hyperbolic. So, if A is loxodromic then A is hyperbolic. So, please remember hyperbolic was the case was a special case of loxodromic alright. The condition for loxodromic was that the trace squared is not in this it is not a real number line between 0 and 4 alright.

So, what that lemmas the first statement says is that the upper half plane does not have any non hyperbolic loxodromic elements, every loxodromic element also has to be hyperbolic. So, you can also say this you can also state this as you know loxodromic is same as hyperbolic as far as Moebius transformations that preserve the upper half plane that is a first one.

Then the second thing is A is elliptic if and only if A fixes a point of U. So, the ellipticity condition on a Moebius transformation that preserves upper half plane is equivalent to the transformation fixing a point of the upper half today alright. Then the third one A is of finite order this implies that A is this is true if and only if. So, A is elliptic of finite order. So, in other words if you have an auto morphisms of U if it is finite order it has to be elliptic.

So, let us try to prove these things the proofs are all quite easily doable. So, the first one. So, you see I can now use this criterion and I can find B so that BAB z, BAB inverse of z is lambda z. So, find B so that BAB inverse is lambda z, BAB inverse of z is equal to lambda times sine lambda not equal to 0 or 1. I can do that alright by what I have stated here alright.

Now, see the condition that, so you see first of all A is an auto morphisms of U it is a holomorphic auto morphisms upper half plane and mind you this is the subgroup PSL 2 R see the a general Moebius transformation is an element of PSL 2 C alright. So, you will have a matrix with entries a b c d like this and the entries are complex numbers, but then you see if it is going to preserve the upper half plane then these entries have to be real numbers alright and that the reason for that is because if it preserves upper half plane then it also has to preserve the boundary of the upper half plane which is a real axis. So, it has to take the real axis onto itself and that will force all the entries to be real alright.

So, it is all these entries are real alright. And now you see; what is the condition that A is loxodromic. So, you see trace squared A, trace squares A turns out to be trace squared A turns out to be a plus d the whole squared alright, if I write A as if, I write A in this form A is equal to a b c d alright if I write it like this then trace squared A is a plus d the whole squared alright. And this is also equal to trace squared of BAB inverse because you see the trace is not going to be changed if you conjugate and trace of BAB inverse see BAB inverse the matrix representative for this is. So, let me write it let me put a small box here and say A is a is just a b c d with a comma b comma c comma d belonging to R and ad minus bc equal to 1 and BAB inverse is identified with z going to lambda z the matrix for that matrix representative in SL 2 is root lambda 0 0 1 by root lambda this is B matrix representative.

It is actually, if you just blindly write it as a matrix and invertible matrix it will be lambda 0 0 1 alright, but the determinant of that is lambda, but we want determinant 1 alright. So, one important thing to remember is that before we calculate the trace squared you normalize it to this, you normalize the condition that the determinant is equal to 1 otherwise this definition cannot be applied this definition assumes that you have assumed ad minus b c equal to 1 alright. So, you see BAB inverse will be this and well I will get this is equal to I will get this is equal to root lambda plus 1 by root lambda the whole squared which is lambda plus 1 by lambda plus 2. Now, notice that the condition is trace squared trace squared A is not a real number lying between 0 and 4 that is a condition for a to b loxodromic alright. But notice a and d are real numbers therefore, a plus d the whole squared is a non negative real number alright and it cannot be 0 because a plus d whole square well it probably well in fact, if it is 0 it is elliptic alright, but I mean it is not supposed to be 0 any real number between 0 and 4.

Now, if you remember I told you that see the from this I want you to conclude that lambda has to be real and lambda has to be positive I claim that lambda has to be real and lambda has to be positive alright and then it is clear that if lambda is real and lambda is positive then it is clear that its hyperbolic by the by this condition here lambda is real and lambda is positive then it is hyperbolic. So, I want to say that lambda is lambda is actually real and I think that is probably quite obvious.

So, we have already seen this last time you know a plus d the whole squared is the condition that the a plus d the whole squared is real already means that either lambda is real all mod lambda is 1. That we got that condition by saying that lam a plus d the whole squared is real if and only if lambda plus 1 by lambda is real then we and lambda plus 1 by lambda is real if and only if you can equate it to lambda bar plus 1 by lambda bar and then if you collect it out collect the terms out and it will then it will tell you that either lambda is real or mod lambda is 1, but if mod lambda is 1 its elliptic, but you have assumed its loxodromic so mod lambda is not 1. So, lambda has to be real. So, let me write that down, let me write that down here.

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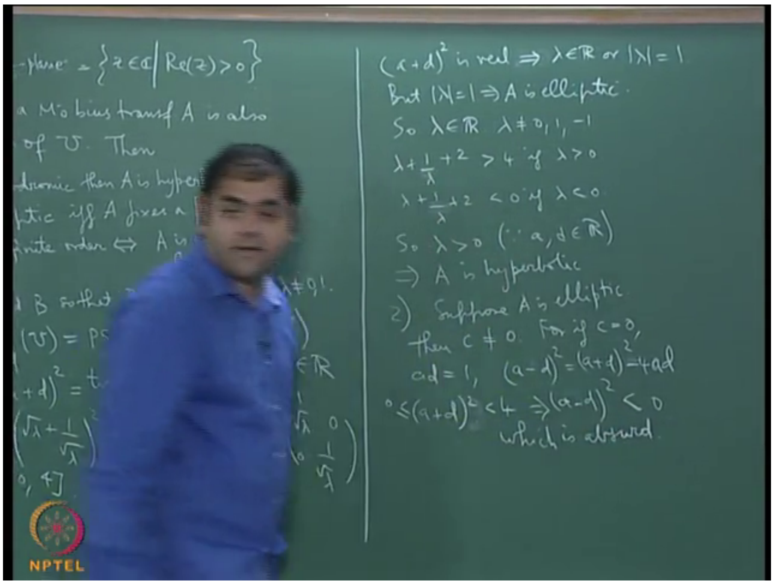

So, a plus d the whole squared is real if and only if implies you know lambda is real or mod lambda is equal to 1, but mod lambda equal to 1 implies A is elliptic, but which is which is not true I have assumed A is loxodromic alright.

So, lambda has to be real. So, lambda has to be a real number it is a real number and lambda cannot be 0 or 1 or minus 1. Now, you see we that is that is another thing there is another remark that I made last time that you know this number I mean this function lambda for lambda real lambda plus 1 by lambda plus 2 that is always greater than 4 if lambda is positive and it is always less than 0 if lambda is negative and why do you exclude 1 and minus 1. So, lambda plus 1 by lambda plus 2 is greater than 4 if lambda positive and lambda plus 1 by lambda plus 2 is less than 0 if lambda is negative, excluding the cases lambda equal to plus 1 and lambda equal to minus 1 alright.

And well what I have here is that lambda plus 1 by lambda plus 2 you see on the one hand this is greater than or equal to 0, this is greater than or equal to 0 because you know a plus the a and d are real numbers a plus d is whole square the square of real numbers, it is greater than or equal to 0. So, the only possibility is that lambda is positive. So, you see. So, lambda is has to be positive since a and d are real. So, lambda is positive and you know lambda is real and lambda is positive. So, by this characterization a has to be hyperbolic. So, this implies a hyperbolic. So, that finishes off the proof of the first claim alright. So, there is no difference between hyperbolic and loxodromic for auto morphisms of the upper half plane right. So, in other words there is no non hyperbolic loxodromic auto morphisms of the upper half plane right.

Then the second one, the second statement is A is elliptic if and only if a fixes a point of U. So, suppose A is elliptic suppose A is elliptic alright then I claim that you know first of all I claim that c is not 0, then c is not equal to 0 and why is this true because you see, I will have to say just a moment thought. So, you see for if c is 0 then ad equal to 1 alright and you see a minus d the whole squared is a plus d the whole square minus 4 ad minus 4 ad and this is going to be well what is a plus d the whole squared. So, that a plus d the whole squared is a trace squared and I have assumed that trace squared is going to be greater than or equal to 0 and strictly less than 4 that is the condition for it to be elliptic. So, you see a plus d the whole square is a real number which is greater than or equal to 0 and strictly less than 4 this is the condition for ellipticity.

So, this will tell you that a minus d the whole squared is negative because you see a plus d the whole squared is strictly less than 4. So, it will be 4 minus 4 ad, but you see ad is 1. So, it is 4 minus 4 it is 0. So, it will be strictly less than 0 which is impossible which is which is observed it is observed because a and d are real numbers and I am getting a square of a real number as negative. So, c cannot be 0 alright.

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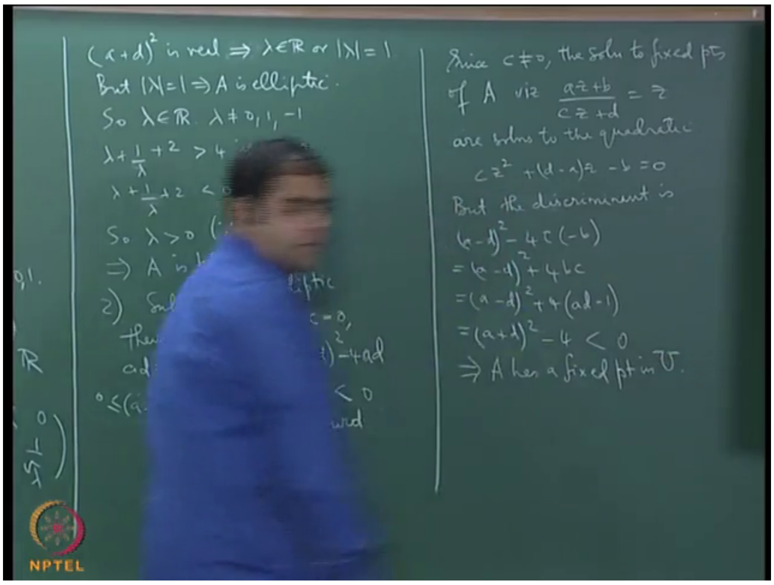

Now, if c is not 0, since c is not 0 the solution to fixed points of a namely a e z plus b by c e z plus d equal to z is the solutions, are solutions to quadratic to the quadratic - c z squared plus d minus a times z minus b equal to 0 alright. Now, this is a honest quadratic because c is nonzero alright

But again you see d minus, but you see a minus d, but the discriminant is you see a minus d the whole squared minus 4 into c into minus b which is you know it is a minus d the whole squared minus plus 4 bc, but you see ad minus ad minus bc is equal to 1 alright, so I can use that. So, I will get a minus d the whole squared plus 4 into this is ad minus 1 alright. So, and this is going to be a plus d the whole squared minus 4, but you see a plus d the whole squared minus 4 is negative because it is elliptic a plus b the whole squared minus 4 is negative. So, what is happening? The fixed points turn out to be the roots of a real quadratic equation with negative discriminant therefore, you have imaginary roots and you know the two imaginary roots are conjugates one of them has to lie in the upper half plane the other one is going to lie in the lower half plane namely its conjugate therefore, you get a fixed point which belongs to you. So, this implies a has a fixed point in U and its conjugate will be the fixed point in the lower half plane alright and those two fixed points will be roots of this quadratic equation, real quadratic equation with imaginary with negative discriminant right.

So, we have proved that if A is elliptic then it fixes a point of U. Conversely suppose A is such that a fixes a point of U, we will prove that it is elliptic. So, let us go to that. So, let me rub off this side.

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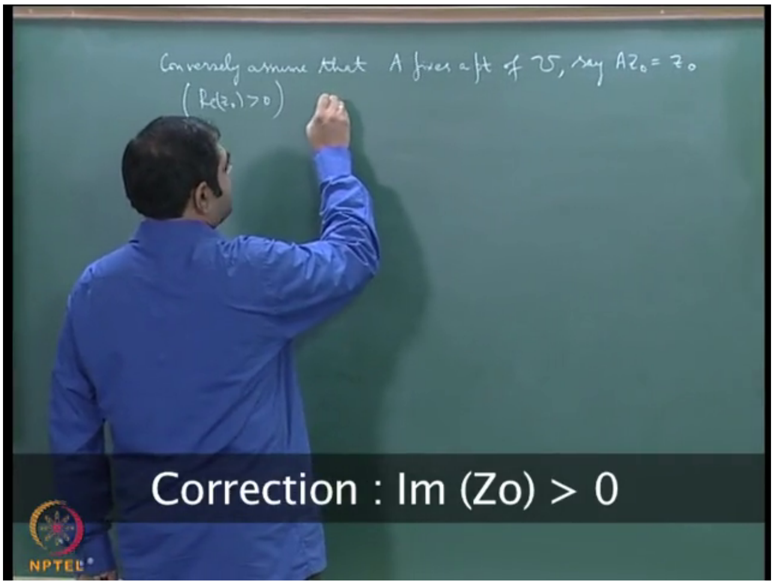

Conversely assume that A fixes a point of U say A of z naught equal to z naught real part of z naught positive. So, z naught is a point of view and A fixes z naught.

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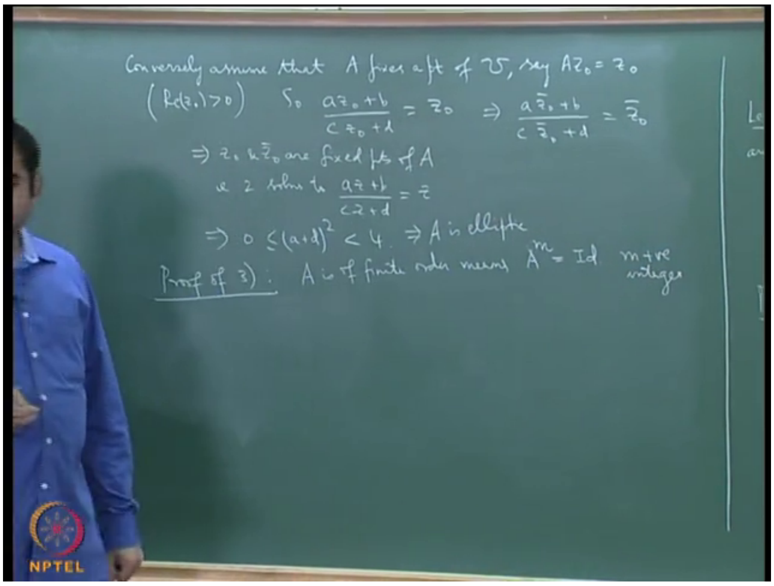

Now you see, you know what this means is it means a z naught plus b plus b by c z naught plus d is equal to e z naught alright. Now you know if I take conjugates on both sides and remember that a b c d are now real I will get that z naught bar is also a fixed point. So, this will imply that a z naught bar plus b by c z naught bar plus b is also equal to z naught bar just simply take conjugates on both sides. So, what this will tell you is it will tell you that e z naught and e z naught bar are fixed points of A, e z naught and e z naught bar or fixed points of A alright.

So, you have two fixed points one being the conjugate of the other. So, from this you should conclude I conclude that you know these are solutions these are two solutions to the equation a e z plus b by c z plus d equal to z that is two solutions to a is z plus b by c is it plus d is equal to z. So, this should force that and mind you if I rewrite this as a quadratic in equation in z it is a quadratic with real questions and now you have two solutions which are conjugates of one another and that will happen if and only if the discriminant is negative and that condition will tell me that 0 is less than or equal to a plus d the whole squared is strictly less than 4. So, this will imply that A is elliptic.

So, essentially the only principle that you are using is that if you have a quadratic equation with real coefficients then it will have imaginary roots and they are conjugates of one another if and only if the discriminant is negative nothing more than that right.

Then , let us try to prove the third statement A is a finite order if and only if is, so I will have to prove I will have to assume A is finite order and I will have to prove is A is elliptic alright. So, let me do that proof of 3, A is a finite order, order means a power m you apply the transformation Moebius transformation A m times then you get the identity transformation where m is a positive integer this is what A is a finite order.

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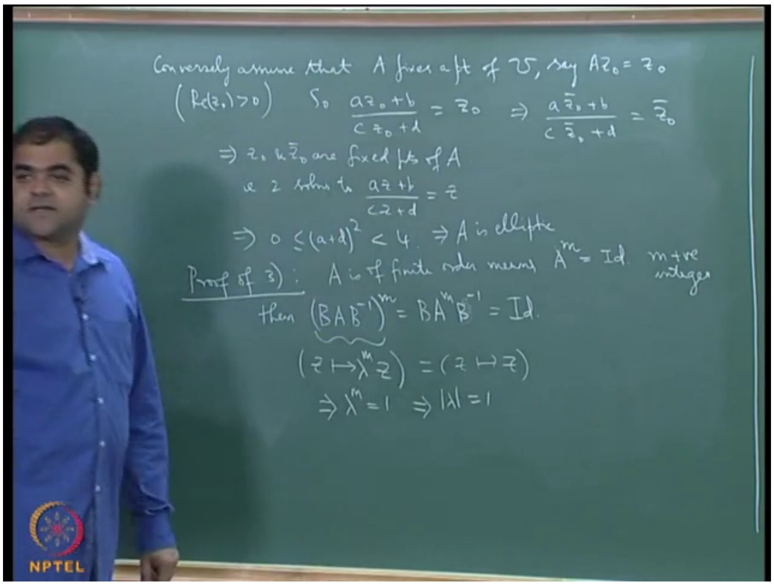

Then you see BAB inverse power m is also going to be B is that going to be BA power m B inverse if you right it out alright and this is going to BA power and B inverse and this is going to be just identity. So, what this will tell you, but on the other hand BAB inverse is supposed to be the map BAB inverse is supposed to be the map that takes z to lambda z. So, BAB inverse whole power m is going to be the map that take z to lambda power m z because it is composition alright and, this is the same as z going to z which is identity map. So, this will tell you that lambda is the root of unity. So, it will tell you that the lambda power m is 1 and this will tell you that mod lambda is 1 and of course, mod lambda is 1 is the condition for a to b elliptic. So, this implies A as elliptic, so that finishes the proof of this lemma.

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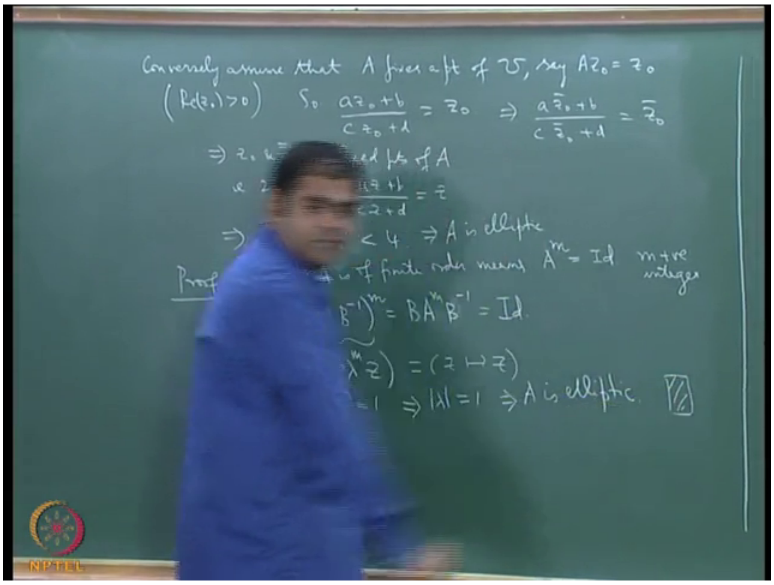

So, there are some, so up to whatever we have learnt from these lemma we can deduce a couple of facts about the deck transformation group which is the same as a fundamental group of the Riemann surface. So, let me write those facts down.

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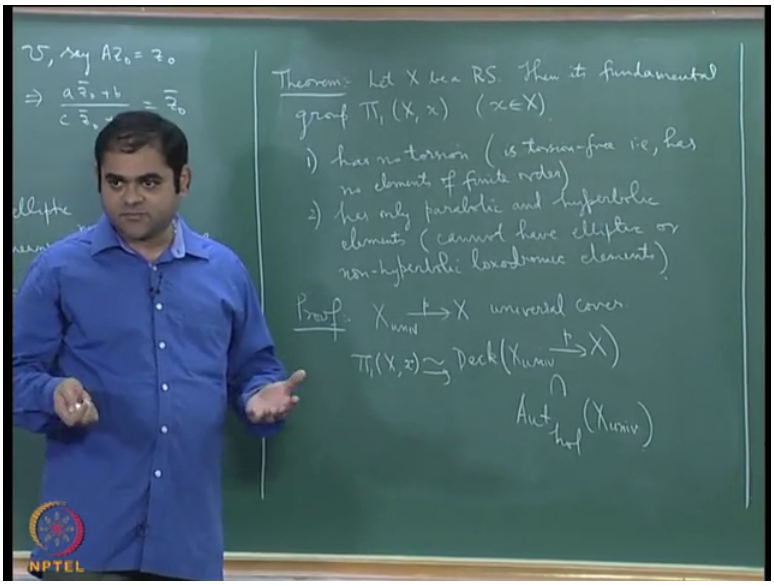

So, I will say, this is an theorem let X be a is Riemann surface, let X be as a Riemann surface then its fundamental group, group pi 1 X comma x, x belonging to X. So, if you fix a base point small x in capital X and take the fundamental group then the fundamental group number one has no torsion which is the same as saying that it is torsion free is torsion free and by that I mean it has no elements of finite order that is has no elements of finite order.

Number 2, has only parabolic and hyperbolic elements. In other words it cannot have elliptic elements it cannot have non hyperbolic loxodromic elements, cannot have elliptic or non hyperbolic loxodromic elements. So, you are able to get this information about the elements in the fundamental group of the Riemann surface and what is the proof the proof is you see we take a universe universal cover for the Riemann surface. So, X sub x sub unit p X universal cover and because of this universal curve what happens is that you have identification of the fundamental group of the base with the deck transformation group of the cover which is a subgroup of the holomorphic automorphisms of the universal cover.

So, let me write that neatly. Now, what are they let us exhaust the three possible cases for the universal cover, for universal covering space and then both statements claimed the theorem will be easy.

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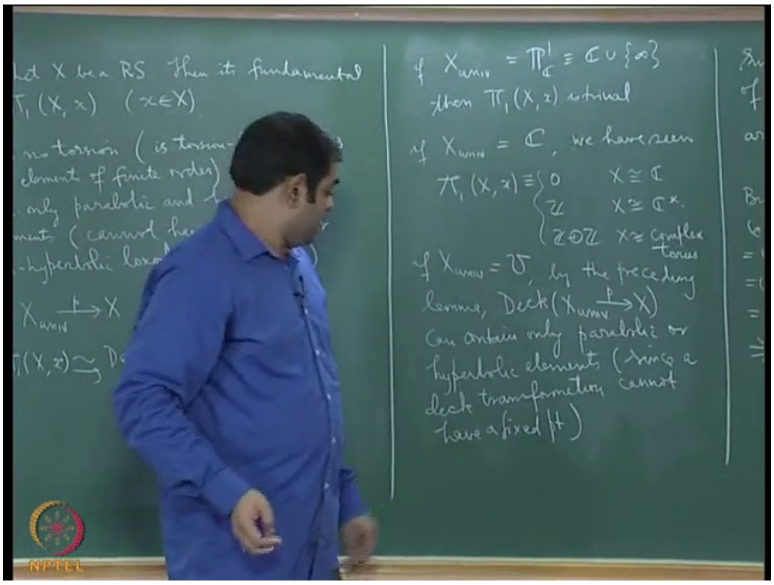

So, if X universal is p 1 which is the same as C union infinity that Riemann's sphere then you know the deck transformation you see then the fundamental group of x has to be trivial, then y 1 of X comma x is trivial. So, there is nothing to, there is nothing to prove. And its and you know why its trivial that is because its anyway going to consists of Moebius transformations and Moebius transformations you know a Moebius transformation will have at least one fixed point or it will have two fixed points you C union infinity alright for deck transformations are not supposed to have any fixed points, deck transformations are not supposed to have any fixed points. Therefore, the only possibility is that it has deck transformation group contains only the identity element and; that means, that the fundamental group is just a trivial group alright.

So, the other case is if X sub univ is the complex plane if X sub is the complex plane we have already seen that pi 1 is a billion pi 1, we have seen pi 1 of x comma x is either 0 or is isomorphic to 0 or psi directs some psi these are the only three cases and these three cases correspond to X isomorphic to C here, X is isomorphic to C star and this corresponds to X isomorphic to a complex torus these are the three cases.

And if you go back to the proof of this result what we did was we identified the deck transformation group as just translations by making a suitable conjugation if necessary, we identified the deck transformation group as translations and then you know that translations are all parabolic. So, in this case the fundamental group is either trivial or it can contain only parabolic elements. So, in this case it is translation by a single nonzero complex vector and in this case it is translations by in this case its translations by integer multiples of a single ground 0 complex vector and in this case, it is translations by integer linear combinations of 2 nonzero complex numbers whose ratio is non real. So, in this case you are going to get only parabolic elements in the fundamental group alright.

So, the only other case is the universal covering is actually the upper half plane now you see if the universal covering is the upper half plane notice that. So, the first thing I want to say is that you see a deck transformation cannot have fixed points and we have seen that Moebius transformation that leaves the upper half plane invariant is elliptic if and only if it fixes a point of U and since the deck transformation is not allowed to have fixed points the elements cannot be elliptic and then, so therefore, the only possibility is either parabolic or hyperbolic and mind you hyperbolic is same as loxodromic if you are working if you are considering auto morphisms U. So, in this case the only thing with the only deck transformations you will get are going to be either parabolic or hyperbolic, by the previous lemma by the preceding lemma pi 1 of X comma x can contain or other let me rather write it as deck which is isomorphic to pi 1, deck can contain only parabolic or hyperbolic elements.

So, the point that I am using here is that a deck transformation cannot have a fixed point because if you remember what was the reason if I deck by that I mean a non trivial deck transformation. If a deck transformation has a fixed point then it is agreeing with the identity transformation of the universal covering space at that point, but then both of them are lifts of the covering projection and uniqueness of lifting will tell you that it has to be the identity. So, a deck transformation cannot have even a single fixed point that is what I am using it.

Since a deck transformation cannot have a fixed point. So, this is the information you are able to get about the fundamental group of Riemann surface. So, what is very interesting is that you see your the fundamental group was defined in a very abstract way, but trying to get abstract properties of the group. One is able to do that by studying Moebius transformations because these are finally, only some among these are going to be members of the deck transformation group which is identified to the fundamental group. So, that finishes the proof of this lemma, alright.

So, I should also remark the following thing. So, you see I have literally proved both of the statements the only thing is I have to make remark suppose there is an element in the deck transformation group which is a finite order then it cannot be this case, it cannot be this case because here the groups are psi and psi directs psi no element is a finite order alright. Whereas, it has to be only in this case and in this case an element of finite order is elliptic that is what we approve, but the elliptic, but if it is elliptic then it has to fix your point of U, but a deck transformation is not supposed to fix any point. Therefore, it is not allowed therefore, you can see that in all the cases you cannot have a deck transformation which is a finite order; that means, the fundamental group of Riemann surface is torsion free it is a non trivial statement that comes out of this study, right.

So, let me stop here.