An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1dimensional Tori and Elliptic Curves
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Lecture – 33
The Construction of a Modular Form of Weight Two on the Upper Half-Plane

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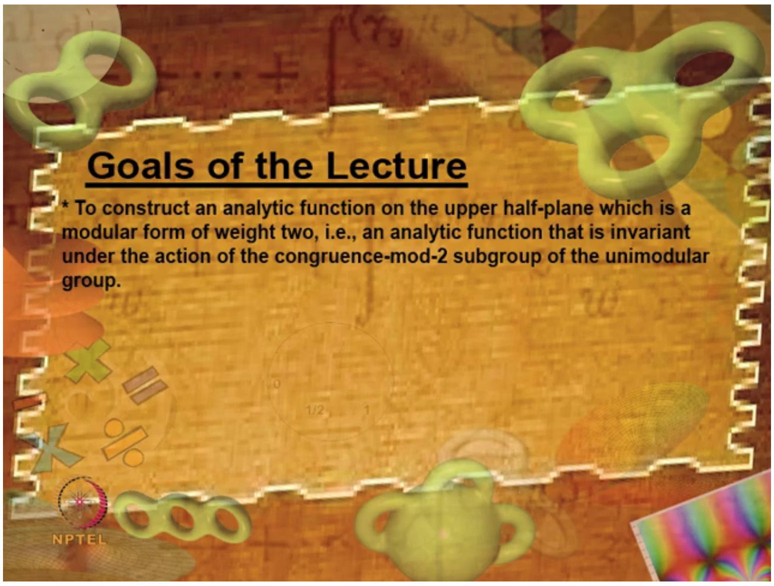

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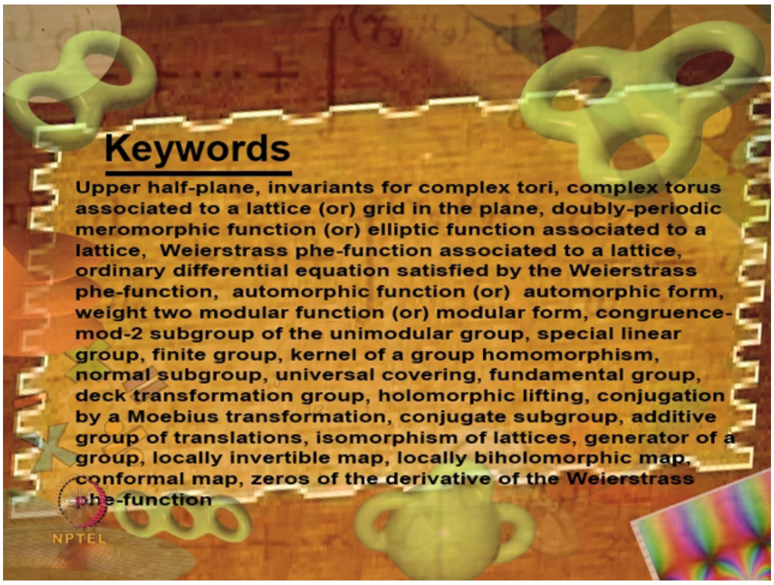

So, we are trying to prove in this lecture that this modular function weight to modular function lambda that we constructed last time using the phi function. The Weierstrass phi function is indeed weight to modular form namely that it is invariant under the action of the congruence two congruence mod 2 sub group of the group PSL 2 z of any modular vectors of unimodular matrices which we think of as Moebius transformations.

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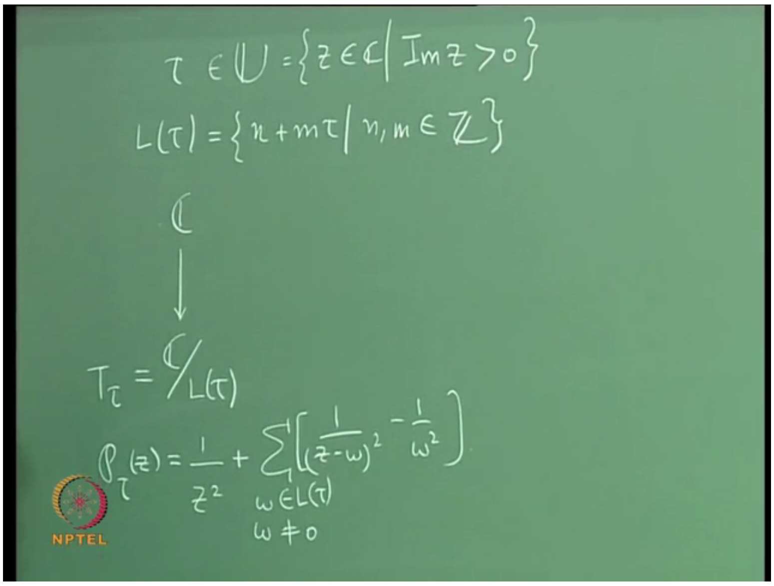

So, let me recall, so we have the following situation, we have just again recall the notation see we fix we fix tau in the upper half plane this is the set of all z; all those complex numbers such that in the imaginary part of z is positive. And then associated with tau we have the lattice L of tau which is all those complex numbers of the form n plus m tau, where n and m are integers. And then we have the torus associated to this tau which is just so you have the complex plane and then you have to go modulo the action of L tau which means that you think of each L tau each element of L tau as translation by that element of L tau. So, and of course, you know these translations are certainly Moebius transformations they are automorphic, they are holomorphic automorphisms of the complex plane.

And so this map is just the map that sends every complex number to its equivalence class under this action or I should say to the orbit under this action. So, if you want to think of it is an equivalence of course, two complex numbers here are equivalent that is they go to the same point below if and only if their difference is an element of L tau. So, this is our t tau and we were trying to get hold of invariants for you know these complex one dimensional Tori. And therefore, we were first look at functions on them and there are no holomorphic functions. So, we constructed the associated Weierstrass phi function which is given by this explicit formula. 1 by z square which is the singular part at the origin and summation over omega in the lattice omega is not equal to 0 of 1 by z minus omega the whole square minus 1 by omega square, this is the; this is the Weierstrass phi function. And then well we be found that the Weierstrass phi function satisfied a differential equation which is so let me write down properly it is I guess.

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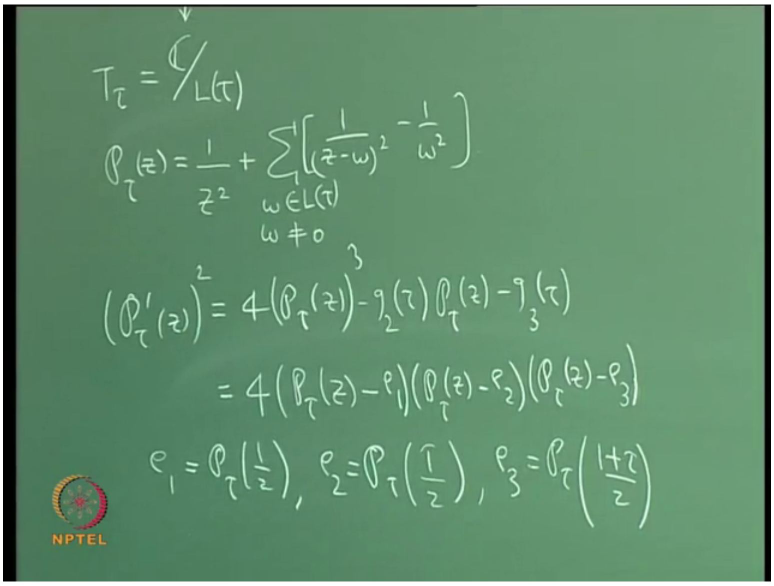

Yeah here, P tau prime z the whole square is equal to 4 times P tau z the whole cube minus g 2 of tau into P tau of z minus g 3 of tau where g 2 and g 3 are certain numbers that depend on tau. And then we also factorise this as thinking of the right side as a polynomial in the variable P tau of z, we factorize z as 4 times P tau of z minus e 1 into P tau of z minus e 2 into P tau of z minus e 3. And of course, and we found that and of course, e 1, e 2 and e 3 are well they are the zeros of the right sides, so they are the zeros of the left side. And that is the method we used to find out what the; what e 1, e 2, e 3 are they are exactly the zeros of this elliptic function.

So, in fact, we set e 1 to be P tau of half, e 2 to be P tau of tau by 2, and e 3 to be P tau of 1 plus tau by 2. So, this is what we took for these three values up to permutation this is the choice that we can make. And then we also we then cooked up now all this depends on tau which is varying in the upper half plane. So, what we did is that we realise that even all these three no matter what your value of tau in the upper half plane is all these three are distinct.

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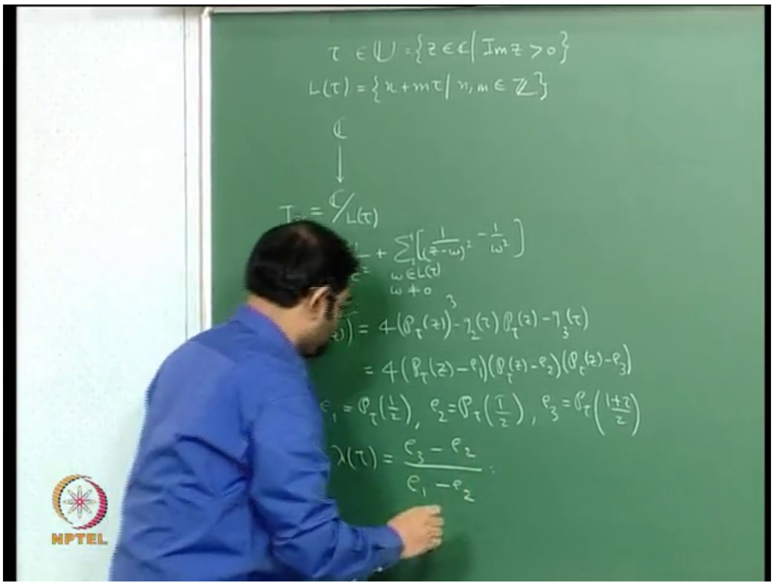

And therefore, we constructed the function lambda of tau which was which was I guess. So, let me write it properly it is e 3 minus e 2 e 3 minus e 2 by e 1 minus e 2. So, we got hold of a function lambda.

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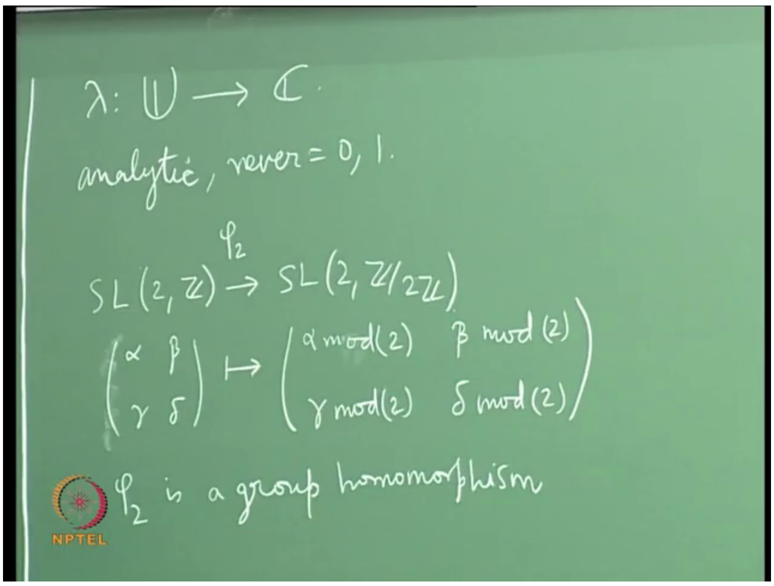

From on the upper half plane taking values in the complex plane, and we found that this function is in fact analytic and it is never equal to 0 and it never takes the value 1. So, this is analytic which means by which I mean holomorphic never equal to 0 or 1. So, we constructed this function. So, I mean all this the whole point of this the argument so far was to get hold of some function on the upper half plane, but the story has to continue because we want a function which is which depends only on the holomorphic isomorphism class of the Torus. That means that function has to be in variant under the action of PSL 2 z. And therefore, you want a function which is invariant under that on the group of on the subgroup of Moebius transformations.

And then the point is that this is only a first step in the sense that this is not in variant under the wholes of whole group PSL 2 z. But it is invariant only under the subgroup of all those elements of PSL 2 z whose coefficients if you reduce them on two you get the identity matrix two by two identity matrix. So, so let me so I will have to so the purpose of this lecture is to prove that this is indeed a function that satisfies those properties. So, usually an analytic function or a meromorphic function which is invariant under group of Moebius transformations is called an automorphic form or an automorphic function. And in particular if the group is a group of is a subgroup or a related group of the uni modular group PSL 2 z, then we say that the function is a modular function or a modular form and. So, we have to prove that this is the modular form of weight two.

So, let me make a few comments. So, the first thing I want to say is that you see we have from S L 2, z to S L 2, z mod 2 z morph 2 z that is this phi 2 this is the. So, we have a map like this which just sends well any alpha beta gamma delta any element of S L 2, z to simply this same element, but with each entry red mod 2. So, this is alpha mod 2 beta mod 2 and then gamma mod 2, and this will be delta mod 2. It goes to this element and mind you z mod 2 z is just 0 or 1, it is exactly the integers red modulo two.

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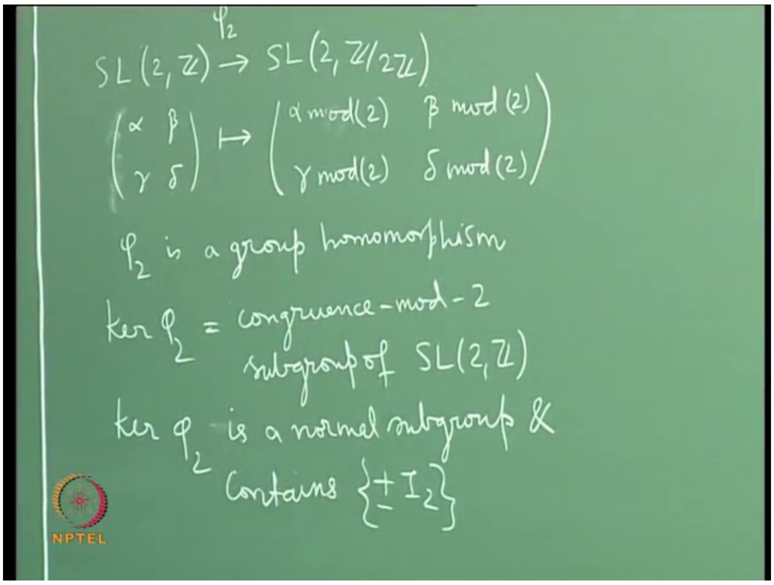

And the fact is that this map phi 2.is a group homomorphism namely it respects the multiplication on the left side these are matrices with determinant one, and under multiplication the left side forms a group. And there similarly the right side also forms a group the only thing is that the coefficients are taken from z mod 2 z. And this map is a homomorphism of groups that is very, very simple because reading modulo two will respect addition multiplication, it is a ring homomorphism. So, you can see that this is a group homomorphism. And you see; what is the kernel of this group homomorphism, the kernel of this group homomorphism is precisely the congruence mod 2 subgroup. So, the kernel of phi 2 is precisely the congruence mod 2 subgroup of a S L 2, z namely it is all those elements of this form S L 2, z.

So, of course, you must remember that alpha beta gamma delta are integers, and alpha delta minus beta gamma is equal to 1. And all those elements here which mod 2 look like 1 0 0 1 are precisely in the kernel of this map and you know the kernel is a normal subgroup. So, you see this the kernel of phi 2 is a normal subgroup it is a normal subgroup and contains and contains the subgroup given by plus or minus identity I 2 is the 2 by 2 identity matrix 1 0 0 1 row wise. And of course, this map maps both of these guides on to the identity matrix there you must remember that mod 2 minus 1 is a same as plus 1. So, it is a normal subgroup, so the point is that if you take the quotient of S L 2, z by plus or minus I 2 you get P S L 2, z. And that will contain the quotient of S L 2, z that will contain the quotient of the kernel of phi 2 by plus or minus I 2 which will give you the congruence mod 2 subgroup of P S L 2, z.

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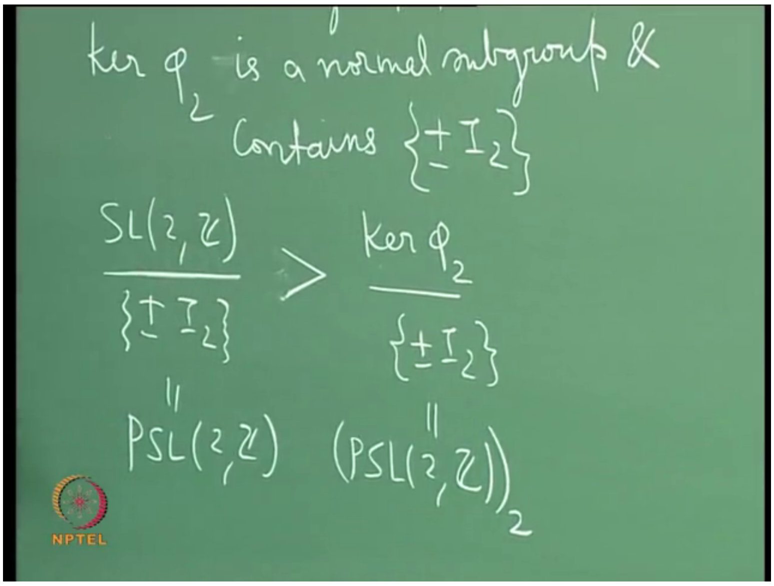

So, let me write that S L 2, z mod I should say yeah plus or minus I 2 which is which by definition is P S L 2, z contains as a subgroup kernel of phi 2 modulo plus or minus I 2 and this is precisely the subgroup P S L 2, z. I will put subscript two to say that this is the congruence to subgroup congruence mod 2 subgroup of P S L 2, z and of course, this symbol is just to tell you that this is a group of this right. So, what is the claim the claim is that for every, now, you must understand now you must realise that you should what we are doing is we think of elements of P S L 2, z as Moebius transformations.

And then you know that P S L 2, z is a subgroup of P S L 2, r which is precisely the set of holomorphic automorphisms in the upper half plane. So, this is a subgroup of holomorphic, these are the subgroup of Moebius maps of the upper half plane onto itself. And what you want to say is that this function which is defined in the upper half plane its actually invariant under this subgroup not under the whole uni modular group, this is the uni modular group, but under this subgroup.

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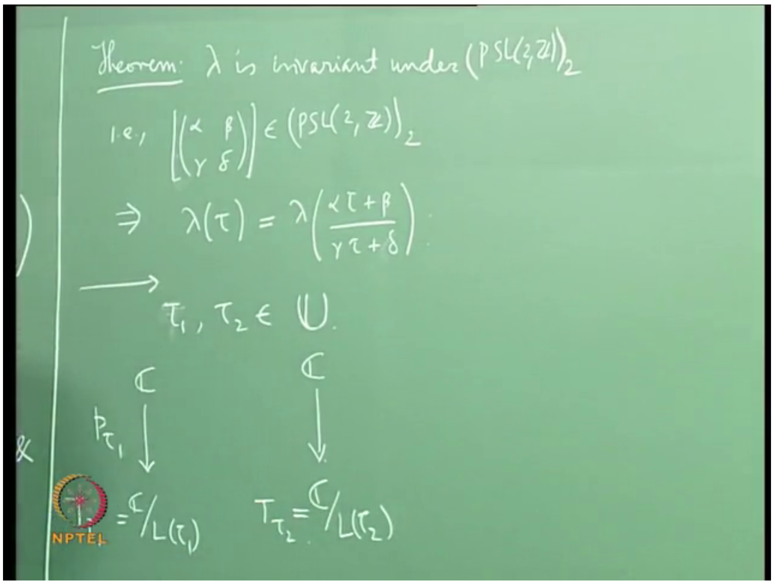

So, let me precisely write the claim the theorem is lambda is invariant under the action of P S L 2, z congruence mod 2 subgroup. So, this is this is the theorem. So, in other words that what does this mean this means that you know if you take an element alpha beta gamma delta in P S L 2, z sub 2? When I write something like this, I mean of course, you know I mean an equivalence class I have put a square bracket outside to tell you that this is a representative of an equivalent class in the quotient, so it could vary by it could change by sign. And this implies that you know if you if you take lambda of a of a tau, you evaluate lambda at a point tau on the upper half plane, this is going to be the same as evaluating lambda on the image of tau under the Moebius transformation that is defined by this element of P S L 2, z.

So, this is equal to lambda times alpha tau plus beta by gamma tau plus delta, so that is the statement this is the this is the statement. So, this is the statement. So, in particular what is the importance of this importance of this is this is a very good example of an automorphic function certain analytic function; in this case is actually analytic lambda is an analytic function and so it is a modular form and it is invariant its invariant under the congruence mod 2 subgroup. So, this whole lecture is devoted to trying to prove this statement.

So, and of course you know what next is once you know that you have gotten hold of a function that is invariant modulo the subgroup somehow you will try to extend you try to construct an another function which is will be invariant to under the action of the full P S L 2, z and that function will give you that function will be constant on orbits of P S L 2, z. And you know orbits of P S L 2, z in the upper half plane are precisely isomorphism holomorphic isomorphism classes of complex tori, so you get an invariant that is a quantity that depends only on the holomorphic isomorphism class of the torus. So, let us try to so this is so the proof of this is what we are going to do.

So, to do this I will have to recall I will use the same notation that I used in an earlier lecture. So, I will have to recall what I did in lecture 24. So, this was you know I mean this was how we prove that this two complex tori, which are associated to tau 1 and tau 2 in the upper half plane are holomorphicly isomorphic if and only if tau 1 and tau 2 differ by an element of P S L 2, z. So, I will recall exactly what I wrote down in that lecture part of it which I need for our calculations. So, you see and you know that was the statement that we proved in order to show that I mean essentially that statement showed that finally after we proved it. It showed us that you know taking the upper half plane and going modulo P S L 2, z in namely taking the P S L 2, z orbits the upper half plane is exactly bijective to the set of holomorphic isomorphism classes of complex tori of the form t sub tau.

So, well, so I will recall what we did. So, you see so we had we took tau 1. So, so take tau 1 and tau 2 in the upper half plane. And then you have these so you have piece of tau 1 this is the projection from c to c mod L of tau 1 that which gives you the torus associated to tau 1. And then you also have from c to c mod L of tau 2 that is the torus associated with tau 2, and if you remember what we did was you know we did the following thing.

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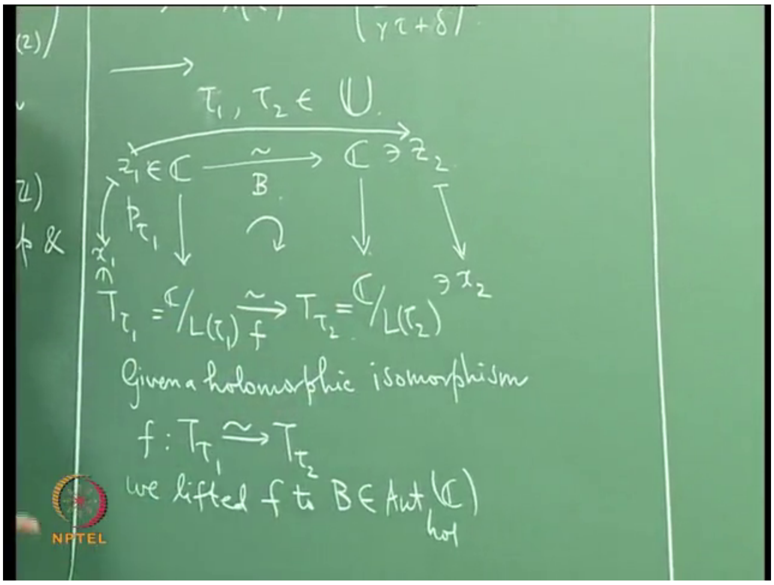

Suppose you are given a holomorphic isomorphism f from this torus to that torus suppose these two tori where holomorphically isomorphic. Then what we did was you see we use the theory of covering spaces, mind you these two are universal covers. We use the theory of covering spaces to lift this map all the way to give a Moebius transformation B, so that this diagram commutes we got we lifted f to a moebius transformation B, which is which as an automorphism of the complex plane holomorphic automorphism of the complex plane.

And in fact, what we did was well the diagram like this induced a diagram at the level of fundamental groups. So, what we did was well I guess if I go back to that lecture I think we took a point z 1 here and assume that z 1 goes to the point z 2 here under B. And then you assumed that z 1 goes to a point x 1 in the torus below and z 2 well goes to a point well x 2 in this torus T tau 2 and of course, f takes x 1 to x 2.

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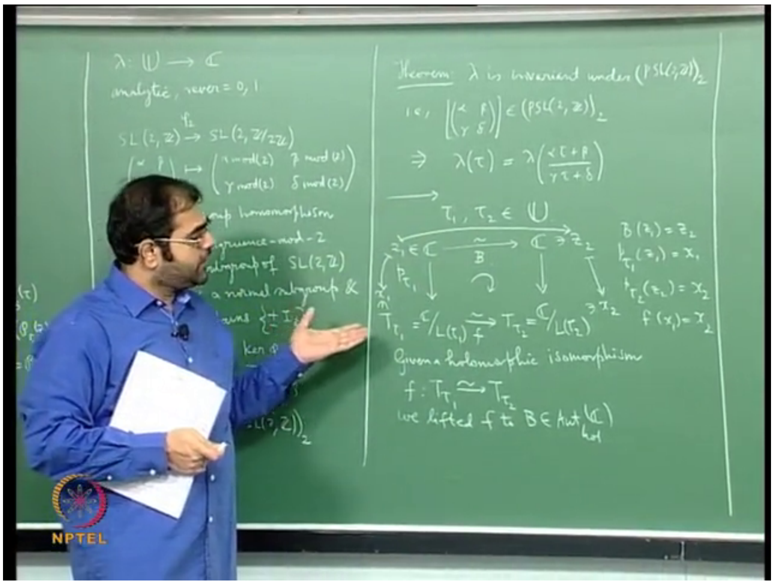

So, let me write that down here B of a z 1 is equal to z 2 P tau 1 of z 1 is x 1 P tau 2 of z 2 was x 2 and then f took x 1 to x 2. I mean the whole the point is we are fixing base points in an nice way, so that we can identify fundamental groups based at those points. So, this because taking the taking the fundamental group is (Refer Time: 22:13) operation this gave us an identification of the fundamental group of this torus at this point with when the conjugate of this becomes the conjugate of becomes the fundamental group here. So, let me write that down. In fact, you see you know that whenever you have a covering like this if you take the map induced by you know yeah. So, I should say that the fundamental group of the base below is exactly the deck transformation group. And the deck transformation group is canonically identified with naturally identified with L of tau 1.

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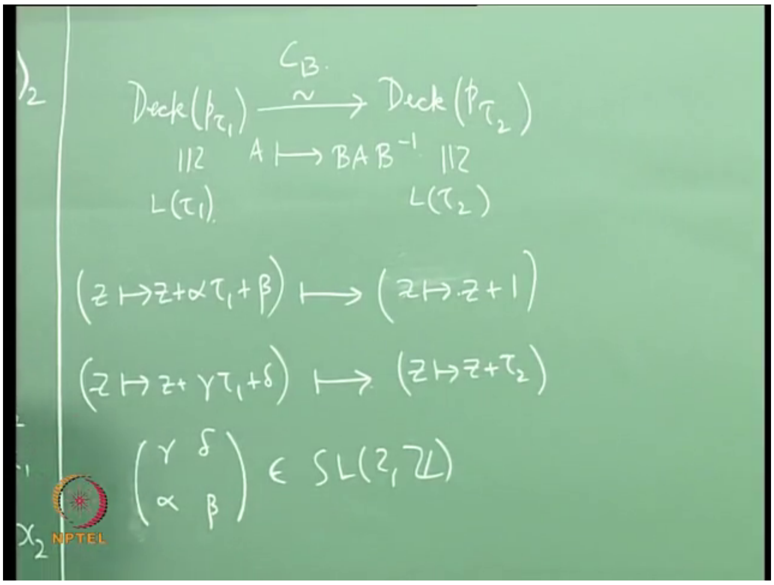

So, you see what happened is so we got so you have deck of the deck transformation group of P tau 1 P sub tau 1 to the deck transformation group of P sub tau 2, we had an isomorphism. And this was conjugation by B that is this was by the map a going to B A B inverse we got this because of this diagram, we got this because of this diagram all right.

And well so, but mind you that the deck transformation group of P sub tau 1 is canonically identified with L sub tau 1. And the deck transformation of group of P sub so this map is P sub tau 2 that is canonically and the fundamental group below is canonically identified with the deck transformation group of the of this cover all right. So, by looking at this diagram we got this. And that this deck transformation group here is actually L of tau 1. So, I should say isomorphic two and this is isomorphic naturally to L of tau 2 in what sense the deck transformation group in P tau 1 consists precisely of translations by elements of L tau 1. And the deck transformation group P tau 2 consists precisely of translations by L by elements of L tau 2 and then you are identifying a translation with the element by which you are translating. So, this is the isomorphism.

We always tend to identify the group of translations group under addition because composition of translations affectively is addition of the translating vectors. So, this is an additive group of a translations and that is identified with this with this group, and it is in fact, this is a z model, it is a discrete z module we have we have seen all that earlier. So, this is a natural identification. So, what it tells you is that you know this group here you know this lattice is generated by one and tau 2. In other words we what I am trying to say is that this deck transformation group is generated by the translations by one and translations by tau 2. And this transformation deck transformation group is generated by translations by one and translations by tau 1.

And because of this what you can write is that you can write that so we can write is you assume that you know so you know if I take the element z going to if I take the element. So, let me write on the right side z going to z plus 1 the element z going to z plus 1, it is a translation it is an element of this deck transformation group. And because this is an isomorphism it comes from a deck transformation here, and that deck transformation just not to confuse notations I will continue with that the old notations that I used in that in lecture 24, it is z going to z plus alpha tau 1 plus beta. So, this was the map.

And then similarly the other generator of this is z going to z plus tau 2 that is the other translation and that is the other generator the deck transformation group. And we assume that that comes from z going to z plus gamma tau 1 plus delta. So, we assumed, this is exactly what I used in that lecture. So, I am continuing to use this. So, this is how we got these integers alpha, beta, gamma, delta we got these four integers. And then we found that in fact, we found that we got this matrix we got this element gamma delta alpha beta we got this element in S L 2, z we got this element in S L 2, z, we proved that gamma beta minus delta alpha is equal to 1. And we prove that this element actually takes tau 1 to tau 2. In other words this element considered as a Moebius transformation namely you consider it as a Moebius transformation z going to gamma z plus delta by alpha z plus beta for that Moebius transformation if you apply it on tau 1, the image is tau 2.

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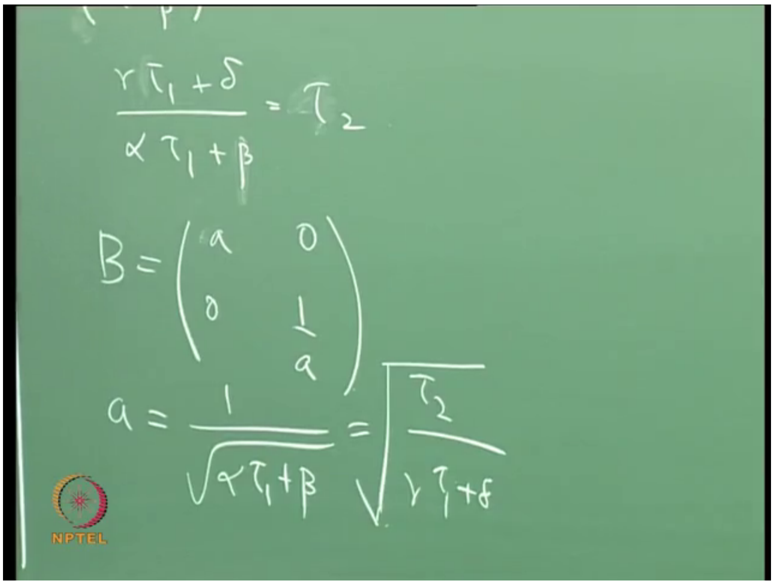

So, we found that gamma tau 1 plus delta by alpha tau 1 plus beta is equal to tau 2 this is what we got. So, with this we proved that you know if there is an isomorphism between the tori defined by tau 1 and tau 2 then there exists an element of S L 2, z. And therefore, you can also take its image in P S L 2, z that element takes there is an element which moves tau 1 to tau 2 that means, tau 1 and tau 2 are in the same orbit for the action of P S L 2, z on the upper half plane.

Conversely we said that the whole argument can be reversed and how could you reverse it. In fact, you could reverse it because there is a nice formula for B, B turned out to be A. So, B turned out to be the following it turned out to be A 0 0 1 by a as an element of written as an element of P S L 2, z. In fact, so you know a representation for a Moebius transformation whenever we represent Moebius transformations by to by matrices we insist that the determinant is 1. So, B turned out to be this with A B, a square root of a alpha tau 1 plus beta and which also turned out to be equal to the corresponding square root of tau 2 by gamma tau 1 plus delta that is I mean this is just because of this.

So, the point is that given f, you get this B and then from the by looking at the fundamental groups that the covering groups, you get these element of S L 2, z. And you can conversely suppose I am given an element of this of S L 2, z, which takes tau 1 to tau 2. Then you see I can define B by this by these equations. And then you can check that this P will give me a Moebius map, it will give me a holomorphic automorphism of C and that B will take you see that B will take the lattice here L tau 1 to the lattice there.

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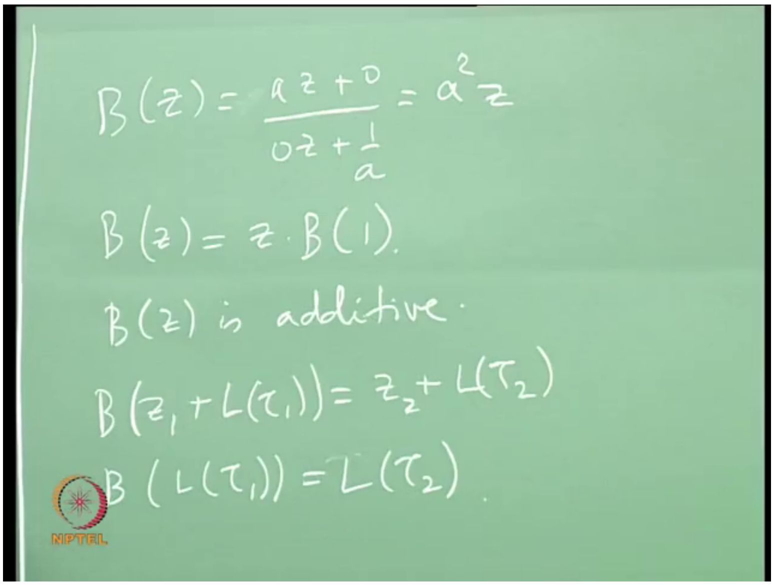

See because you can you can see what is happening is that see B if you write B of z, you see B of z is just B of z is just a square times you see it is just a z plus 0 by 0 z plus 1 by a. So, it is just a square z, it is just multiplication by a square. So, you see so B of z you see is in fact B of z is an additive map. In fact, you can see that B of z is well z times B of one because after all a squared is B of 1. So, B of z is z times B of one and B of z is B of z is additive namely B of z 1 plus z 2 B of z 1 plus B of z 2 because it is just multiplication by a scalar namely a square.

So, what happens is that you see B of z by our assumption what B does is well you see, it takes z 1 to z 2, but then you see what does that mean it means that you see z 1 goes down to x 1. So, if I take the inverse image, I will get all translates of z 1 by L tau 1. And well x 1 goes to x 2 and if I take the universe image here, I will get all translates of z 2 by L tau 2. So, what is actually tells you is that B takes z 1 plus L tau 1 to well z 2 plus L tau 2. In fact, B takes L tau 1 to L tau 2. So, I know that B is just multiplication by a square and so B take 0 to 0. So, you know that will tell you that B takes L tau 1 to L tau 2, because 0 goes to 0. Whereas 0 here, if 0 goes to a certain point then all the elements of the lattice go to that point and that will go to the point here to which 0 goes to 0 again.

Therefore, all the elements of this lattice have to go to that lattice and mind you B is an isomorphism. So, the moral story is B takes L tau 1 to L tau 2, B of 0 is 0.

So, the point is that because of this if I am if I am already given an element of S L 2, z like this which takes tau 1 to tau 2, I can cook up B like this. And B will take L tau 1 to L tau 2 therefore, B will go down to a map f from a T tau 1 to T tau 2, and this map f will be holomorphic because you see this map f is locally this followed by this, this map f mind you this is holomorphic covering. So, it is locally by holomorphic. So, locally the maps P tau 1 P tau 2 are locally invertible namely if you take an admissible neighbourhood below then P tau 1 is invertible it becomes there is an inverse which is a holomorphic map. So, actually this map f locally is this map followed by this map followed by this map, which is the composition of holomorphic maps and therefore, f is locally holomorphic. So, it is holomorphic because holomorphisity is a local property.

And well the just exactly the way I got f from B I will get f inverse from B inverse and it will tell you that f is a holomorphic isomorphism. So, the point is that so this is this was the essence of the proof that you know two elements of the upper half plane define the same to define isomorphic tori if and only if they are in the same orbit of P S L 2, z. Now, I need to have these calculations. So, you see the point I am going to make is that I am going to bring in the Weierstrass phi functions. So, you see well so this is what I wanted to recall.

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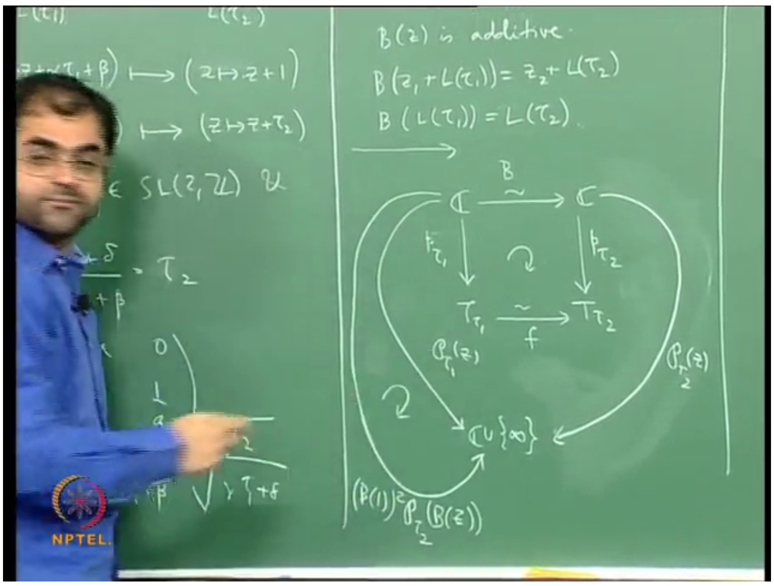

So, again I draw another diagram. So, here is the complex plane. So, here is B again and well this is P sub tau 1, and here is P sub tau 2, and this is T tau 1, this is T tau 2. And this is an isomorphism, this is the isomorphism f and this diagram commutes. Now, you see for tau 1 we have defined Weierstrass phi function. The Weierstrass phi function goes all the way from I mean its defined on C, I takes values in c union infinity I have to include infinity because it is a meramorphic function. So, that is the value I am going to assume that is the value that I am going to assign to the function at a pole and this is this is my p. So, this is my P sub. So, let me write it somewhere here P sub tau 1 of z. So, this is my phi function all right.

Similarly, on the target, I have another phi function, which is going to be well phi sub tau 2 of z this is another Weierstrass phi function. Well, you see you expect that you know some kind of commutativity of the diagram should hold. And what I am trying to say is you compose B with phi sub tau 2 of z only then you will get a map from this cover of T tau 1 to this. And the relationship is between that and phi tau 1 itself. So, in fact, so let me draw one more here and write this as so this is going to be first apply B of z then apply P sub tau 2 to that all right, so that I get a map I get I go all the way here. And then so I get again a map from here to here, but then I have to modify it by a correct by a certain multiple and that turns out to be B 1 the whole square, B of 1 the whole square and the fact is it this diagram commutes namely this is the same as this.

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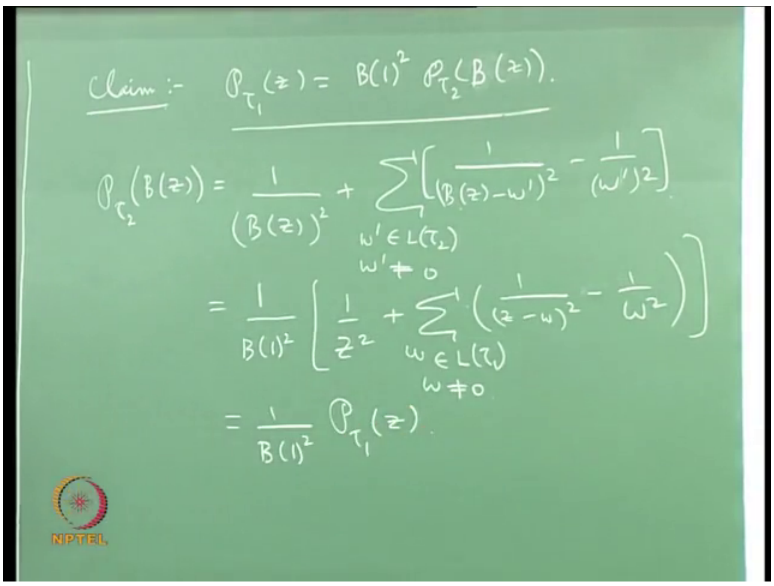

So, P tau 1 z is actually B 1 the whole square, P tau 2 of P outside. So, this is the claim, this is the claim. So, how does one prove this, it is pretty easy. In fact, if you see P p tau 2, so you just plug into the formula and use the linearity of B. See, after all you see P tau 2 of B z is by our formula 1 by B z the whole square plus 6 summation over omega prime in L of tau 2 because you see it is related to tau 2 omega prime not equal to 0 1 by B of z minus omega prime the whole square minus 1 by omega prime omega prime the whole square. This is by the direct definition of the tau function, but then notice that you see B of z can be written as B of 1 times z because of this equation because after all B is just multiplication by a square.

So, if I do that I can write this whole thing as 1 by B of 1 the whole square into 1 by z square plus summation over C, write this omega prime as B of omega because you see B gives an isomorphism of L tau 1 with L tau 2 and B takes 0 to 0. Therefore, B takes all non-zero elements of this lattice precisely to non-zero elements of that lattice, so I can re label this omega prime as B of omega and let omega vary over nonzero elements of L tau 1. So, I can write this is as summation over omega belonging to L tau 1 omega not equal to 0. And I can write this as 1 by z minus omega the whole square plus 1 by omega square minus 1 by omega square, which you know is just 1 by B 1 the whole square into the phi function associated to tau 1 evaluated at z. So, it is a very simple calculation, if you notice that B of z is z times B of 1, it is a straight forward calculation. So, this is so that establishes this claim.

Now, what I am going to next is look at what happens to the you know from the phi function, we constructed this modular function, I mean this function which we want to show is modular for the congruence mod 2 subgroup. So, we will have to look into that. So, you see we have. So, you see from this see the first thing is what you do is that you try to differentiate this because that will tell you that the whole point is I have to keep track of these e 1, e 2 and e 3 as tau varies because tau is changing from tau 1 to tau 2 all right. So, I have to keep track of e 1, e 2, e 3, but you know the way of looking at it is to think of e 1, e 2, e 3 as zeros of P prime of tau. So, I need to look at the derivative.

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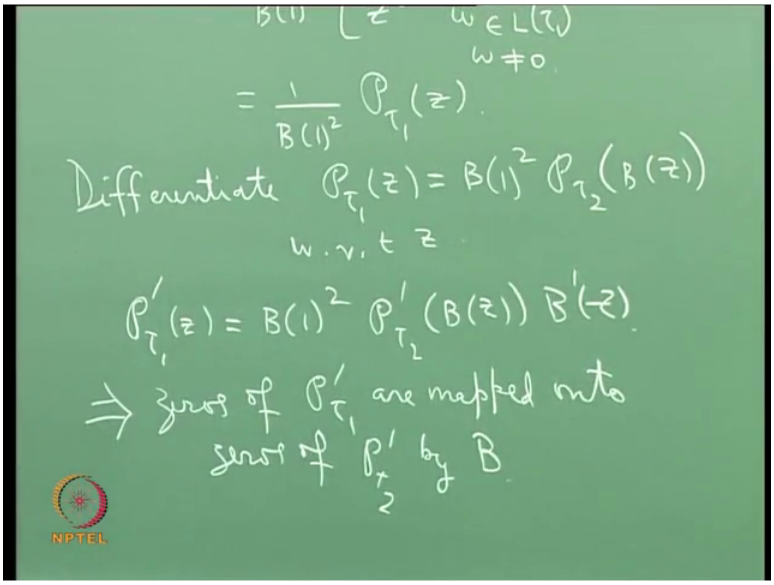

If you differentiate this thing P tau this equation P tau 1 of z is equal to B of 1 the whole square into P sub tau 2 of B of z with respect to z. What I will get is I will get P prime tau 1 of z; and on the right side, I use the chain rule for differentiation. So, what I will get is I will get P prime tau 2 of B of z into B prime of z this what I will get. Now, what does this equation tell me, this equation tells me that you see the zeros of P prime tau 1 are precisely the zeros of P prime tau 2 I mean if z is a zero of P prime tau 1 then and only then is B z zero of P prime tau 2? That the reason is because this B one is non zero and B prime z you see it is a derivative of a; of a Moebius transformation and that is always non zero because the Moebius transformation is always a conformal map its derivative never vanishes. So, if this vanishes if and only if this vanishes. So, what this tells you is that the zeros of P prime tau 1 are precisely map by B onto the zeros of P prime tau 2.

So, and you know B is a Moebius transformation. So, it is one to one and one two. So, these three distinct zeros of P prime tau 1 or precisely map to the three distinct zeros of P prime tau 2. So, what this tells you is that if I write down this lambda of tau 1, and if I write down lambda of tau 2, the only problem is that it might juggle around with this e 1, e 2, e 3 that is the only freedom that you have. And the observation is if you put the further restriction that this element that I started with is in the congruence mod 2 subgroup namely that all these entries we read them mod 2 I get the identity matrix then there is no freedom. In other words, lambda tau 2 becomes the same as the lambda tau 1 that is what we will see now. So, you see so let me write this here this implies that zeros of a P prime tau 1 are mapped on to zeros of P prime tau 2 by B that is what it says.

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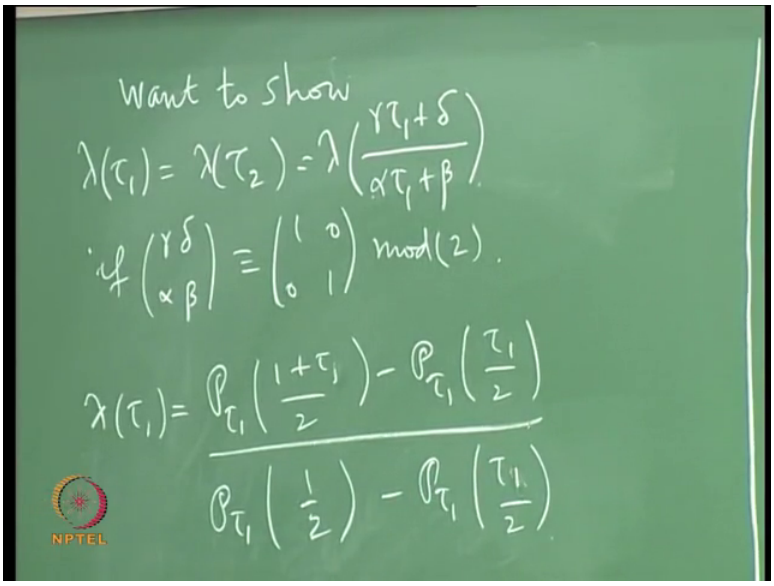

So, now let us bring in the congruence mod 2 condition here. Let me write down what we want to prove. We want to prove that lambda of tau 1 is equal to want to show lambda of tau 1 is equal to lambda of tau 2, which is lambda of because tau 2 is you have assume tau 2 is gamma tau 1 plus delta plus divided by alpha tau 1 plus beta. If gamma delta alpha beta is in the congruence mod 2 subgroup namely if this is congruent to 1 0 0 1 identity matrix mod 2 this is what you want to show.

So, you see what is see what is lambda of tau 1. If you remember lambda of tau 1 was well I have raised I think it was phi sub tau 1 applied to what was it was 1 plus tau 1 by 2 or was it tau 1 by 2 yeah 1 plus tau 1 by 2 tau 1 by 2 minus phi sub tau 1 applied to tau 1 by 2 divide by phi sub tau 1 applied to well a half minus phi sub tau 1 applied to tau 2 prime I mean tau 2 by 2 sorry tau 1 by 2. This was this was lambda of tau 1 all right. And now you see if I use the fact that P tau 1, now let me use again this result P tau 1 of something is B one square P tau 2 of B applied to that thing.

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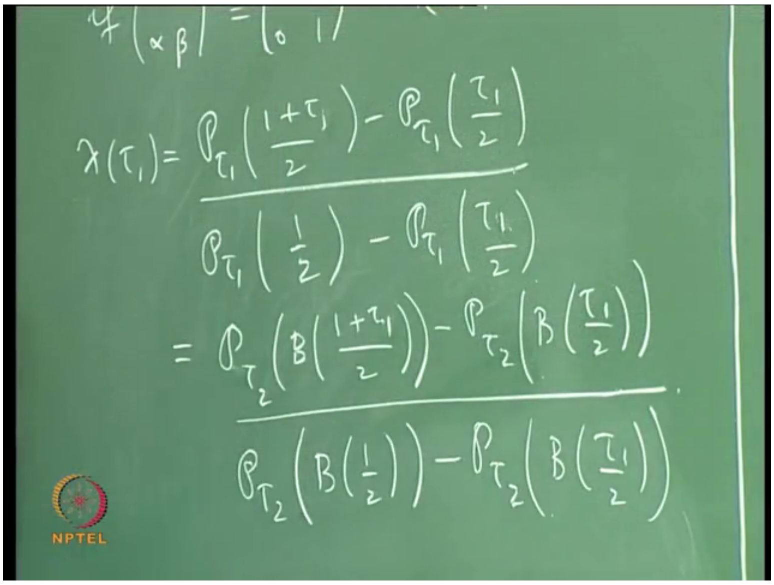

So, you see this is going to be P tau 2, P tau 2 applied to B you see the constant B 1 the whole square it is going to come out and get cancel, so I am not going to write it down. So, I am simply going to get P tau 1 of B of 1 plus tau 1 by 2 minus P tau 2 of B of tau 1 by 2 divided by P tau 2 of B of half minus P tau 2 of B of tau 1 by 2, this is what I will get. Here I have used that all right. And of course, as I said B 1 whole square it just gets cancelled of. Now, and I told you that B of half, B of tau 1 by 2 and B of 1 plus tau 1 by two cannot, but B other than half tau 2 by 2 and 1 plus tau 2 by 2 up to a permutation. Now, what I want to say is that once you have this congruence condition then it has to then you do not have any freedom at all. And how does one see that that is very, very easy to see. You see if see let us try to write out let us do it with B of half see B of half.

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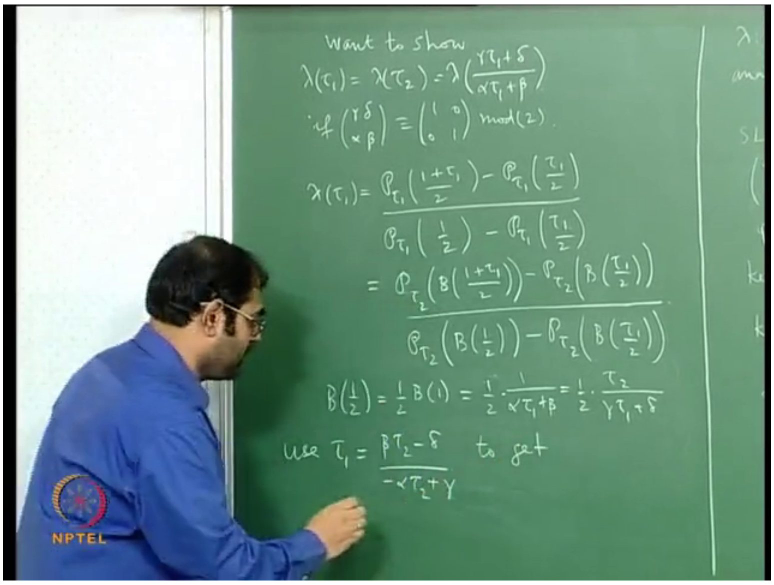

What is B of half, B of half is if I look at this, if I look at this it is half times B of one all right it is half into B of 1. And that is equal to well and what is B of 1 you see, so B of 1 is a yeah B of 1 is a square and it is 1 by alpha tau 1 plus beta and that is also equal to tau 2 by gamma tau 1 plus delta. So, let me write that down. So, it is half into a B of 1 is a square is 1 by alpha tau 1 plus beta this is also half into tau 2 divided by gamma tau 1 plus delta.

Now, what you do is that now what you do is try to write tau 1 in terms of tau 2, see because this takes tau 1 to tau 2. So, it is reverse will take tau 2 to tau 1, you write that. So, what you will get is essentially if you write that down you will get the following thing. You see you will get tau 1. So, let me write it down tau 1 is just beta tau 2 minus delta divided by minus alpha tau 2 plus gamma, because you know beta minus delta minus alpha gamma is precisely the inverse of this matrix gamma delta alpha beta the inverse of this matrix is just simply beta minus delta minus alpha gamma. That is precisely what I have written here that that will take tau 2 to tau 1. Now, you plug this in the last one and write everything in terms of tau 2. So, you will get use this to get you will get well you will get B of half.

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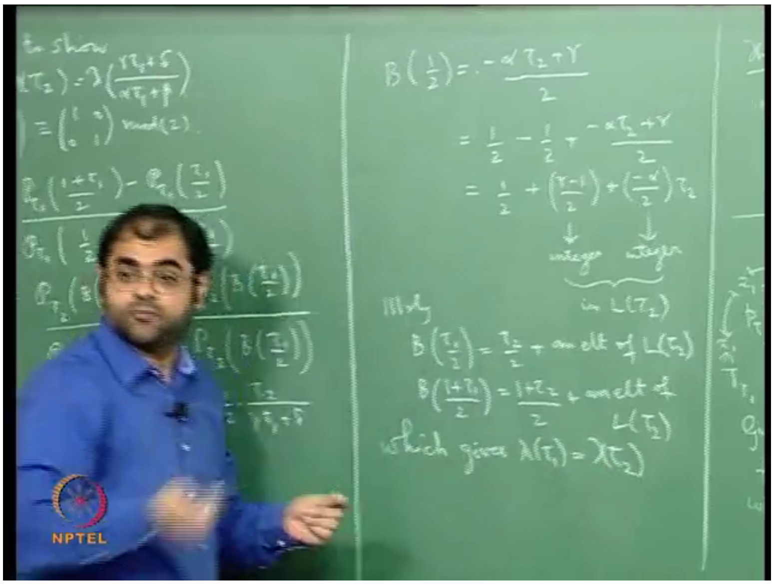

So, let me try to write it here to get B of half is equal to well what you will get is the following you will get minus alpha tau 2 plus gamma by 2, this is what you will get. If you simplify it, you will get this all right. And I use this use this one and also make use of the fact that the determinant is 1, you will get this. But then you see you can write it as well you know I can write it as half minus half plus minus alpha tau 2 plus gamma by 2, then I can write it as half plus well you know I can write it as minus of yeah I mean it does not matter minus of alpha plus 1 correct minus of alpha plus 1 into minus of, no, I will get I will get gamma minus 1 gamma minus 1 by 2 plus minus alpha by 2 times tau 2. See, this what I will get.

What I will get is I will keep this half as it is and then I write this I observe this minus half inside and then write it as gamma minus 1 by 2 plus this. Now, you see if this condition holds then you see gamma minus 1 is divisible by 2 that means, this is an integer. And you see alpha is 0 mod 2 that means, alpha is divisible by 2. So, alpha by 2 is also an integer. What does this tell you this tells you that this guy here on this side is an element in L of tau 2, this is an element in L of tau 2? So, what it will tell you is B of half is half plus an element of L of tau 2.

So, you see P tau 2 of B of half will be just P tau 2 of half plus an element of L tau 2, but P tau 2 is periodic with respect to elements of L tau 2. So, you will simply get P tau 2 of half, you do the same carry over the same argument to the others. So, what you will get similarly you will get B of tau 1 by 2 is tau 1 by 2 plus an element of L of tau 2. And you will get so in fact I should write tau 2 by 2 and you will get B of 1 plus tau 1 by 2 is equal to 1 plus tau 2 by 2 plus an element of L of tau 2. Now, if you plug in all these things here and realise that P tau 2 is periodic with respect to the lattice tau 2. You will see you will simply get lambda tau 2 so which gives lambda of tau 1 is equal to lambda of tau 2 which is what we wanted to prove. So, you see it is a very simple calculation all right. The only thing is you have to keep track of this earlier I mean all these formulae that we got in this earlier proof, then it is kind of a very easy to write down and to see that this lambda is indeed a modular function modular function for the congruence mod 2 subgroup.

So, the next job will be to somehow use lambda to cook up another function which is modular for the whole uni modular group and that will give rise to essentially an invariant that in variant is called the j invariant of the elliptic curve. I mean, I am going to tell you how this differential equation satisfied by the Weierstrass phi function, which I remarked looks like a cubic equation. So, it is actually an elliptic curve all right, and the fact is that you are getting an invariant for that elliptic curve. So, therefore, the function that you are going to cook up using lambda which is going to be invariant under the full uni modular group that is will be called the elliptic modular function. And it is called the j function it will give you what is called the j invariant of an elliptic curve and elliptic curve is as the same as complex one dimensional tori. So, we will see that in the forthcoming lectures.