An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-deminsional Tori and Elliptic Curves

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Lecture - 37
Part- A
The Weight Two Modular Form Decays Exponentially in a Neighbourhood of Infinity

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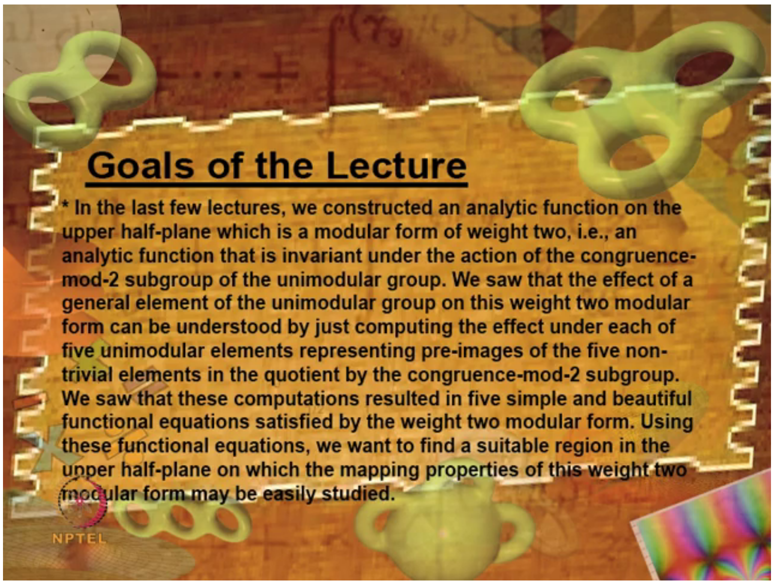

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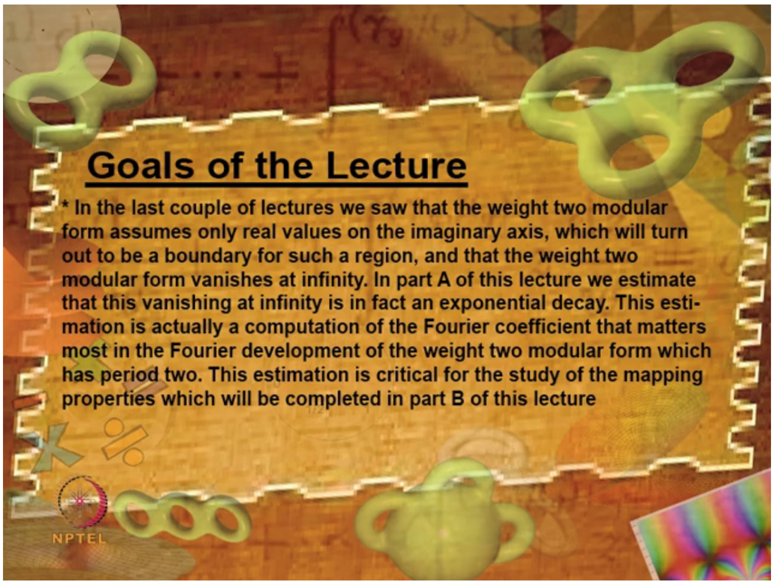

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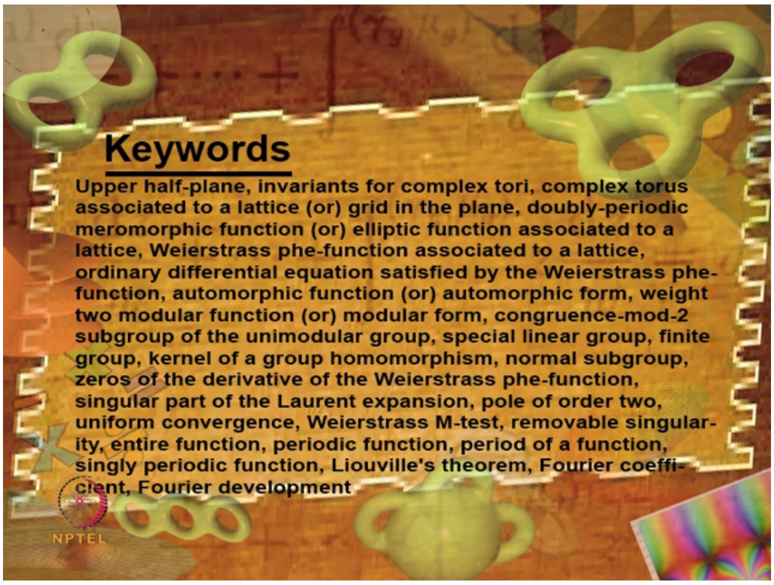

So, we are trying to understand the mapping properties of the modular function lambda, which is you know, invariant under the action of the congruence mod 2 subgroup ofPSL to z. So, let me quickly recall what we are trying to prove, have this function lambda of tau which is which is given by e 3 tau minus e 2 of tau by e 1 of tau minus e 2 of tau.

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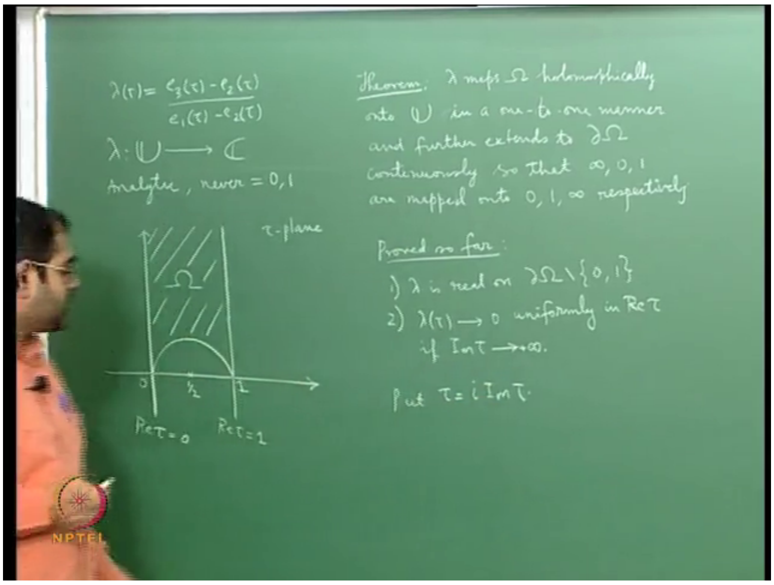

Lambda is a defined on the upper half plane, and taking values in the complex plane. And in fact, it does not seem the value 0 and 1. And is analytic never is equal to 0 or 1. And the claim was the following that if we take the following region in the tau plane, this is a tau plane. We take this line segment, which is given by real part of tau is equal to 1, this is the point 1 this is a origin. And then we draw we take this point given by half and consider the semicircle centered at the point half and radius half. We take this region omega, this region. The boundary of omega consists of this positive part of the imaginary axis, and part of this semicircle the this semicircular arc. And then this positive part of this slide parallel to the imaginary axis and then the claim was that so, the theorem we are trying to prove is that is lambda maps omega holomorphic holomorphically isomorphic, holomorphically on to the upper half plane in a one to one manner, which means which is same as that of lambda from omega to upper half plane is a holomorphic isomorphism.

And further and further extends to the boundary of a omega continuously so that you see infinity the point at infinity the point the origin and one are mapped are mapped onto 0 1 infinity respectively. So, this is the theorem we are trying to prove; that lambda lambda maps see lambda is defined in the upper half plane and it takes complex values. What you want to say is if you consider only this region, then lambda maps the interior of this region by omega. I mean interior of this region.

It maps the interior of this region on to the upper half plane in one to one manner, and you know when a when a holomorphic maps one to one you know it is a holomorphic isomorphism on to the image which is an open set. And further you can extend lambda to the boundary of omega so that this extension is continuous. And the point at infinity 0 and 1 are mapped on to 0 1 and infinity in the target plane in lambda plane, all right. So, this is the theorem we are trying to prove, and as it happened the proof of this theorem is extending to few lectures because one has to do extensive computations.

So, let me recall what we did, what we done so far. So, what you what you done so far is a following, what we proved so far is that first thing is that lambda is real on the boundary of omega, except that points 0 and 1 ok.

So, you see what I want you to understand is lambda is already defined in analytic in upper half plane. So, the really difficult points are the are really difficult points on boundary are the points 0 and 1. So, that point is that you will have show that lambda approach is a genuine limit, as you approach this point from within omega and we have to similarly show that lambda approach is a genuine limit as you approach this point from within omega. That is what you have to show all right. And you must show that this limit is independent of the way in which you approach this point. So, long as your inside omega. Once you do that then you can extend lambda to 0 and 1.

So, I will have to essentially show that if tau inside omega and tau, and also on the boundary omega. And suppose tau tends to 0 I will have to show that lambda of tau goes to 1. And if tau tends to 1, then I have to show that lambda of tau goes to infinity, but on other hand. So, you see if you for if you do not worry these 2 points, on the rest of the boundary lambda is certainly real. If you remember we proved that whenever tau is purely imaginary we proved lambda of tau is real, and then you see the and then you see this line a real part of tau equal to 0 which is the imaginary axis.

That is mapped to this line by the transformation tau going to tau plus 1, and you know lambda of tau plus 1 is related by a functional equation in terms of lambda of tau, and the therefore, fact that lambda tau is a real on this line will tell you that lambda of tau is also real on this line. On this on this half line as you may think of it. And then this half line is mapped onto this semicircle by transformation tau going to minus 1 by tau plus 1, that is 1 minus 1 by tau.

And therefore, and again lambda of 1 minus 1 by tau is can be expressed in terms of lambda of tau, because of you know functional equation or recursive formula, and therefore, the fact that lambda real on this line will also tell you that lambda is real on this line. So, we have therefore, proved that lambda takes real values on this boundary the only problems to be settled r at g what happens at 0 and 1. So, the second thing that we proved is we proved that as imaginary part of tau goes to plus infinity, then we proved that lambda of tau dies down to 0 in a way in way that uniform with respect to part tau. So, let me write that down lambda of tau tends to 0 uniformly in real part of tau if the imaginary part of tau goes to infinity ok.

So, this is what we have so far established, all right. Now what I am going to prove is now I am going to use these functional equation, and tell you why lambda of tau as tau tends 0 goes to 1, and why lambda of tau as tau tends to 1 goes to infinity. So, well that is very easy to see. So, let. So, let me explain that put tau is equal to i times imaginary part of tau. So, you see I am taking tau on the imaginary axis. So, tau is purely imaginary, all right. And of course, imaginary part of tau which was greater than 0 I mean the upper half plane. And then now look at let me use a functional equation.

So, you know I want to I want to calculate lambda of 0. Therefore, the obvious thing that I do is I try to calculate lambda of 1 by i times minus 1 by i times imaginary part of tau, and let imaginary part of tau go to infinity. That is the most obvious thing I will do. And then I have a then I have a formula. So, if you recall from these functional equations, one of them is lambda of minus 1 bye tau is 1 minus lambda of tau. So, you see lambda of minus 1 by tau is 1 minus lambda of tau.

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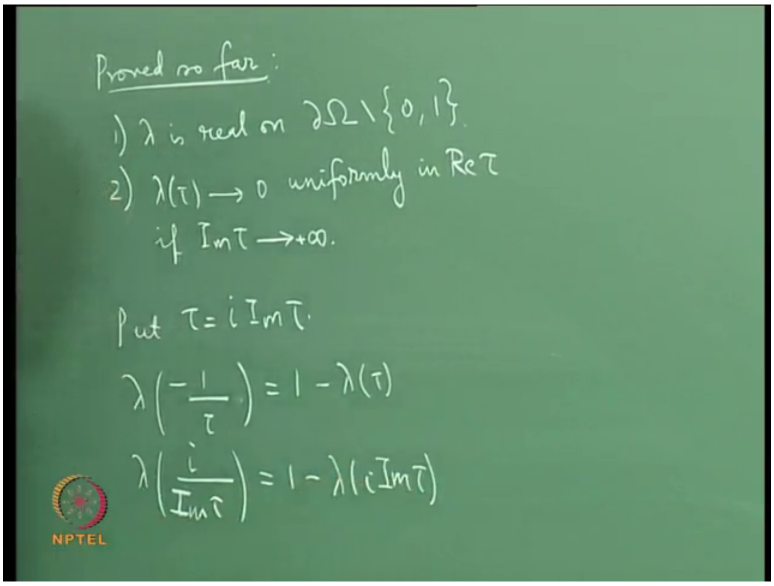

So, we have this functional equation, and what you do is in this functional equation I put tau equal to imaginary part of tau. So, I will get lambda of if I substitute this here, I will end up with minus 1 by I is of course, i by imaginary part of tau is equal to 1 minus lambda times i imaginary part of tau.

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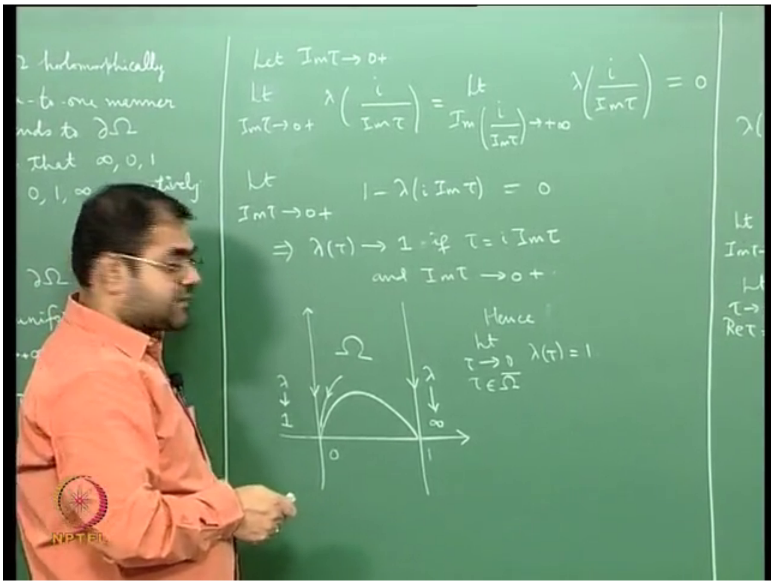

Let imaginary part of tau tend to 0 plus. Then so what you are doing is so, here letting tau to go to you are letting tau to approach 0 along the imaginary axis, all right. So, then you see lambda so, they then what will happens is lambda of i by imaginary part of tau will go to infinity. And you see, but you see lambda of lambda of this will be 0, as I let imaginary part of tau to go to 0 plus. Therefore, I will get lambda of tau to be equal to 1 all right.

So, let me write that down clearly. So, you see limit imaginary part of tau tending to 0 plus of lambda of i by imaginary part of tau, what is this? This quantity is I should say limit imaginary part of imaginary part of i by imaginary part of tau going to plus infinity of lambda of i by imaginary part of tau. But this is as we have seen, this is 0.

So, this is so, this is equal to 0, all right. Uniformly in the real part of tau of course, in case real part of tau is 0, because we are in the imaginary axis. On the other hand, you should take on this side the limit as imaginary part of tau tends to 0 plus of 1 minus lambda times i imaginary part of tau. So, you see this has to be 0, all right. So, what it tells you is that. So, this tells you that lambda of tau tends to 1 as if tau is purely imaginary, and the imaginary part of tau tends to 0 plus.

So, the moral of the story is; that you know if I let me draw the diagram again here. So, you see so, here is mine here is my region. So, as I approach along this line lambda approach is 1, that is what I told. As I come along this line lambda approach is one. And notice that so, notice that if you approach 0 by you know any sequence of points inside omega, this sequence of points whether it is a discrete sequence or even if you approach continuously, in any case because it is being caught between portion of arc of this semicircle and this and the imaginary axis.

It has to come infinitesimally close to the imaginary axis, and in the upper half plane lambda is of course, analytic. So, it is continuous therefore, what we should tell you is that. So, long as your within omega or on the boundary of the omega. If you approach, if you let tau to approach 0, then lambda will take the value 1. So, this tells you that at 0 you can there is a well-defined limit for lambda and that is one.

So, hence limit tau tends to 0 tau in omega. So, I should let me put tau in omega closure. So, I will include the boundary also. Lambda tau is actually 1. So, that settles that settles the fact that lambda maps 0 to 1, all right. Then again so, then similar trick will tell you that lambda will send one to infinity. So, let us prove that for that you make use of another you know, you make use of another functional equation.

So, what I do is I consider. So now, what I will do is that; I am I will consider so, you see here I considered tau on this line. Now let me consider tau on this line, all right.

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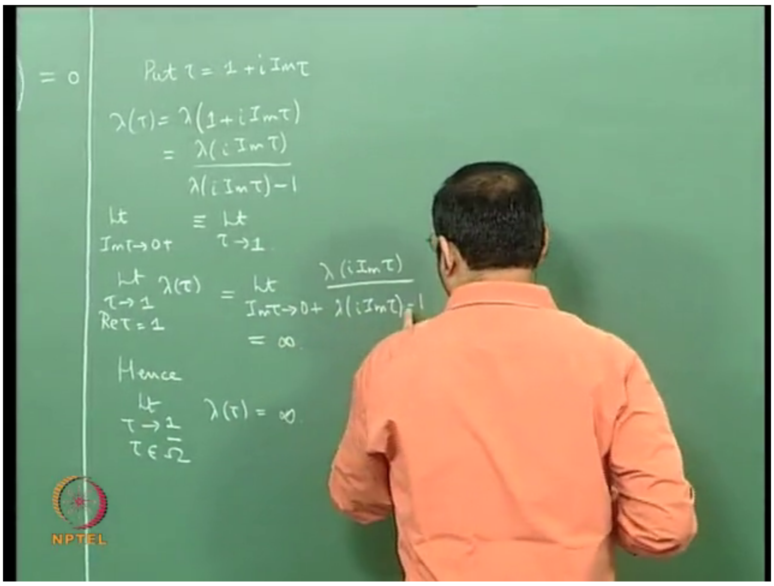

So, what you do is put tau equal to put tau is equal to 1 plus i time imaginary part of tau. That is constraint tau to be in the upper half plane with real part of tau equal to 1, all right. Put this.

And am trying to show that you see a letting tau to tend to 1 along this line is the same as trying to let imaginary part of tau to tends 0. And I am trying to calculate what is lambda of tau. So, you see lambda of tau will be just lambda of tau 1 plus i times imaginary part of tau, but then there is a recursion formula, which says that there is a functional formula which is a lambda tau plus 1 is a lambda tau by lambda tau minus 1.

So, this will be lambda of I time imaginary part of tau by lambda of i time imaginary part of tau minus 1. This is what, this is because of the functional equation that lambda satisfies. Lambda tau lambda 1 plus tau is lambda tau minus lambda tau minus 1 I am using that. And now you see therefore, you see if I take limit as the imaginary part of tau tends to 0 plus, then this is same as taking limit as the this limit is same as taking the limit as tau tends to 1. These 2 are one and the same, and of course, tau is approaching one from above from the upper half plane. And therefore, limit tau tends to 1 of lambda of tau of course, with tau with of course, real part of tau real part of tau equal to 1 is for this side I can take the limit as imaginary part of tau tends to 0 plus.

But I just now proved that you see if imaginary part of tau tends to 0 plus i times imaginary part of tau is here, and lambda approach is one all right. Therefore, the numerator approach is one the denominator approach is 0. Therefore, this tends to infinity. So, this will be limit imaginary part of tau tending to 0 plus of this quantity. And that will be 0, that will be that will be infinity that is at a point infinity. Therefore, this arguments tells you that as I approach one along this line, lambda approach is the value infinity. Lambda approach is value infinity, and now it is clear again, because of the fact that lambda is analytic in the upper half plane that if I approach along any a path which is inside omega or the boundary of omega lambda has tau tends to 1 lambda of tau will certainly tend to infinity.

So, hence so, I have limit tau tends to 1, tau in omega bar lambda of tau is infinity. So, what we have succeeded in proving is that certainly, you can extend lambda to the you can extend lambda to the boundary. The lambda at 0 being define to be equal to 1 and with lambda one define defined is equal to the point at infinity, all right. You can extend lambda right. And we must always remember that since we are since omega is an unbounded region, when we take the boundary of omega the point at infinity also has to be taken into a count. And already taken into account when we prove that literally lambda of infinity is 0 is defined as 0. This tells you that you can define lambda of the point at infinity as 0, all right. So, essentially when we thought we are drawing diagrams on the finite part of the plane, both in the tau plane and in the image plane you must always remember the point at infinity, all right. So, well so far so good, here let me draw the diagram once more.

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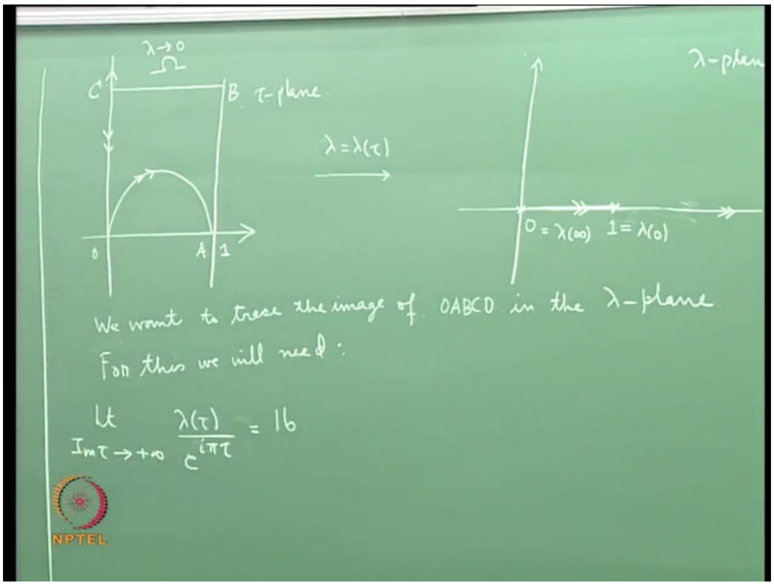

So, you see you have so, here is omega. So, let me look at let me preliminarily draw this diagram. Let me take omega. So, let me write this is lambda equal to lambda of tau.

So, what happens is like I get. So, you see this is the tau plane, and then I get the lambda plane, I get the lambda plane all right. And I know that here lambda at approach is 0, as I as I go higher and higher above, then I am making imaginary part of tau as sufficiently large, then you know lambda approach is 0. So, the point at infinity is being mapped to this point at 0. Then so, this is lambda of 0 this is lambda of infinity right, then if I come down along this line from the point at infinity.

If I come down along this line, then as lambda as tau approach is 0 lambda approach is the value 1. So, what happens is that this thing by continuity you know, and you know lambda is real on this. So, what you will trace is that you will trace, you will trace the portion of this of the real axis, and you reach a point 1 which is actually lambda of 0 and then and from here if I continue moving along the circle. So, you see let me draw the let me draw the orientation. So, you see now if I continue along this semicircular path arc and come to this point, which is 1 lambda of one goes to infinity. So, what happens is that you know. So, you know as I trace this it is going all the way it is going to go all the way to infinity is going all the way to the point at infinity.

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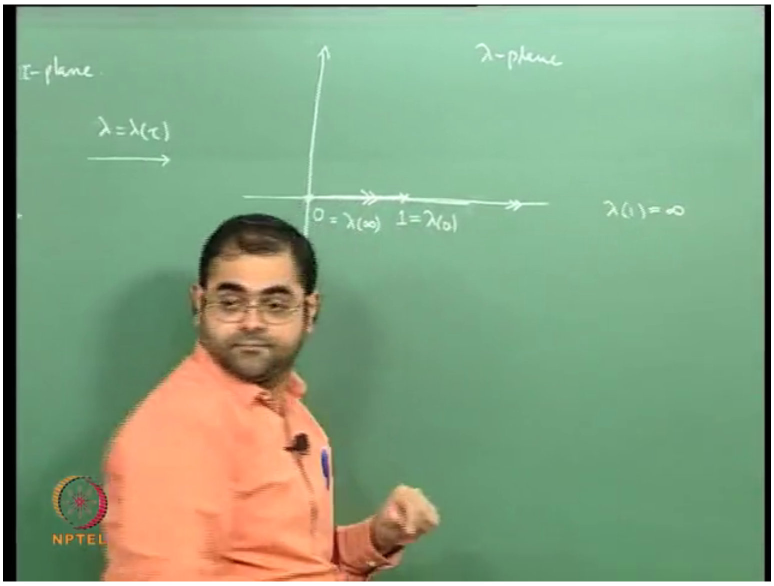

So, the information we have being able to gather is only this much.

We will have to do some computations, to really trace out the image of this region, but what we will do is that we will try to trace out by we will try to trace out the image of this closed path. By taking this line segment and assuming that this y intercept is quite large, you assume imaginary part of tau is large enough, all right. And then you one would like to draw a picture of the how the image of this restricted region, this closed bounded region, how does it look in lambda plane that is what we would like to understand, all right. So, what I wanted to understand is that this. So, this is the and actually trying to understand that will give you an idea of it will literally prove this theorem, we will we will prove the theorem in trying to understand this mapping.

So, let me first explain few things. So, what I will do is so, for that the first thing that need is I need to make an estimation, all right. So, this estimation has got to do with again with trying to expand lambda in terms of a Fourier expansion. Mainly you will you will have to involve sins and cosines of multiples of tau half integral of multiples of tau.

So, let me recall so, we will to so let me do the following thing. Let me give some names to these let me give some names to these points. I will call origin as o let me call this as A. Then let me call this as B, and I will call this as C, all right. And the question is we want, and let me put lambda omega above. Let me put omega above, and what we want to know is we want to trace to trace the image of this this curve. O A B C in the lambda plane. See this is what we want to do.

In fact, and why do we want to do that because if you do that it will explain to you why lambda is a 1 to 1 holomorphic mapping. Which takes every value in the upper half plane alright?

So, we want to do this. For this we will need a, we will need the following computation a limit as imaginary part of tau tends to plus infinity of lambda of tau is lambda tau by e power i pi tau this equal to 16. We will need this computation all right. So, see the see the point is following, the point is that you know that as imaginary part of tau tends to plus infinity, you know lambda of tau tends to you know lambda of tau tends to 0 right. As imaginary part of tau tends to plus infinity one is the lambda of tau tends to 0. As imaginary part of tau tends to plus infinity, one also know that e power i pi tau that will also tend to well e power i pi tau is going to be what it will one that also go to 0. If the imaginary part of the tau goes to infinity it will go to 0 right.

So, the point is because e power i pi tau will be if you write tau as x plus i y, all right. This will be e power i pi into x plus i y. So, the it will be e power i pi x which will be modulus 1 into e power i pi into i y will be e power minus pi y, and if you let y to tend to infinity e power minus pi y is going to go to 0. So, both these cells are going to be 0. The point is this it what this formula says is that lambda goes to 0, you know in such see it goes to 0 in a in this way this rather funny way. And we have you have this number 16 timing up. And I need this to get hold of that to trace the image of the this there, I need this estimate.

So, let us have some fun try to prove this so, recall. So, for the proof of this, we have we will have prove this.

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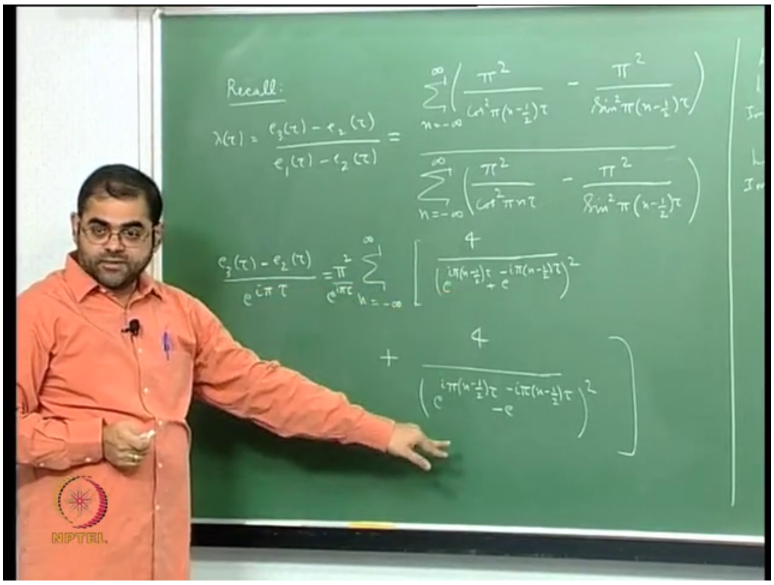

So, let me recall something. You can you may recall that lambda of tau is actually e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau. And this turned out to be the following expression the we know we arrived at this in an earlier lecture by looking at the infinite expansion for pi by sine pi is z the whole square, alright.

So, this is going to be so, it is going to be summation n equal to minus infinity to infinity pi square over cos square pi n minus half tau minus pi squared over sine squared pi n minus half tau. This is there this is what e 3 minus e 2 turned out to be. And e 1 minus e 2 turned out to be sigma n equal to minus infinity to infinity pi squared by cos squared pi n tau minus pi squared by sine square pi n minus half tau.

So. In fact, this is how we proved that as imaginary part of tau goes to plus infinity, you see the numerator goes to 0, and the denominator will go to pi squared. Because only the n equal to 0 term will survive, all right. Because all the moduli of all the denominator they will all go to infinity. So, long as n is involved. So, the numerator goes to 0, denominator goes to pi squared. So, this whole expression is run out of therefore, go to 0. And this convergence was uniform in real part of tau, because these series are all uniformly convergent.

So now you see, now let us compute the following thing. I want to compute what happens to lambda t by e power i pi t as imaginary part of tau goes to plus infinity. But you see the as imaginary part of tau goes to plus infinity, the you see I am just comparing the orders to which these 2 vanish at infinity. And I am saying that there ratio to each goes to each vanishes at infinity, but the ratio is finite. And that is that what is fixed.

So, but you see the vanishing of lambda tau is essentially due to the vanishing of the numerator. So, what I will do is let me compute the limit as imaginary part of tau tends to infinity of the numerator divided by e power i pi tau. Because any way we know that the limit as the imaginary part of tau tends to plus infinity of the denominator is pi squared. Because that is contribution you get from term n equal to 0, corresponding to this term which is only term that will not involve n when I put n equal to 0 right.

So, yeah so, limit so, let us calculate e 3 of tau minus n e 2 of tau over e power i pi 2, let me calculate this. So, this is going to be well I am going to get sigma n equal to minus infinity to infinity maybe I can maybe it will help to remove a pi squared outside all right. And then you know I have used the usual formula that cos theta is you know e power i theta plus e power minus i theta by 2 and sine theta e power i theta minus e power minus i theta by 2 i.

So, I use that to rewrite these in terms of e power i pi n minus half tau and e power i pi so, the numerator in terms of e power i by n minus 1. So, what I will get so, I will have an e power the. So, there is an e power i pi tau outside, there is an e power I tau outside, right. And I have pull the pi square outside I get. So, the first time I get is 1 by you know e power i pi n minus half tau plus e power minus i pi n minus half tau the whole squared and I will get 4 numerator plus. So, then I will get here I will get e power i pi n minus half tau minus e power minus i pi n minus half tau. And I will get squared. And you see there is a 2 I squared and the denominator will go to the numerator that will become a minus 4. Because there is a already minus. So, it becomes plus 4. So, this is what I get.

Now, what am going to do is am going to write this e power i pi tau, as e power i pi by 2 tau that is e power i pi tau by 2 the whole square and push into inside I am going to push it inside the squares of the denominators.

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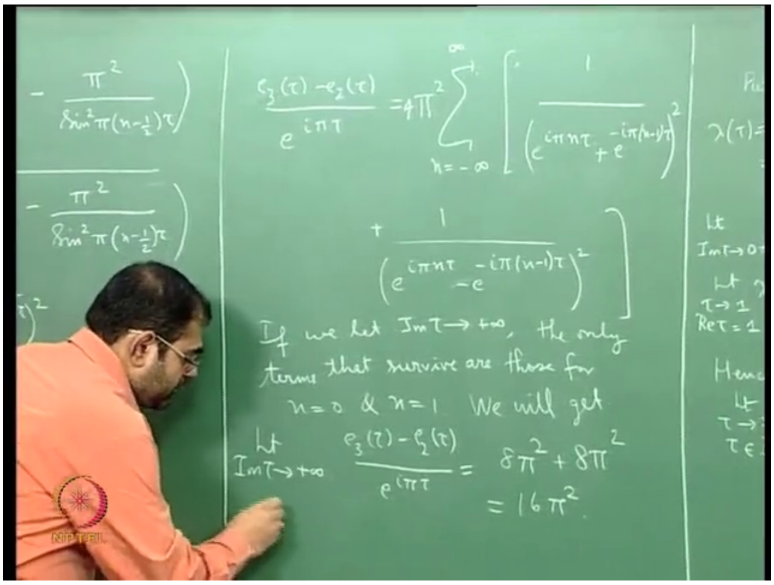

So, that will lead to the following. So, e 3 tau minus e 2 of tau divided by e power i pi tau will be I am going to push this e power i pi tau inside, and I am going to when I push it inside it shows up as e power i pi tau by 2 alright.

Hence then what I will get is I will get pi squared sigma n equal to minus infinity to infinity I will get 4 by so. In fact, I think I can get the 4 out, as well right. And I will get 1 by. So, you see if I add an e power i pi tau by 2 then so, what I will get is a I will get an I will get e power i pi. So, I will get an e power i pi tau by. So, so this this half tau is going to go away.

So, I am going to just get e power i pi n tau all right, here because, I have an e power i pi minus half tau that is going to cancel with e power i pi half tau all right. And here I am going to get minus e power minus i pi. So, you see it is minus of minus half tau it is a plus half tau and another plus half tau. So, I am going to end up with n minus 1 times tau, this is what I get right. And then there is a plus here, this is what I get all right. And the second term will be same kind of term 1 by I will get here again I will lose this half tau. So, I will get e power i pi n tau plus and here, I will this minus half it becomes minus 1. So, I will get minus e power i pi n minus 1 times tau is the whole square this is what I am going to get.

If I push this 1 by e power i pi tau inside this square all right. And you see now you see that if I let the imaginary part of tau to go to infinity, the only terms that will survive are going to be the terms in which no argument appears in the power. So, you see only if I the only terms that will survive is the terms corresponds to n equal to 0, and n equal to 1 if I take a value n different from a 0, and 1 then all those terms are going to go to 0. So, you see if we let imaginary part of tau to tend to plus infinity the only terms that survive are those for n equal to 0 and n equal to 1. These are the only terms that will survive. So, long as n is not 0 or 1 each of those terms is going to be a 0 all right.

We get only a contribution if you take the limit as imaginary part of tau tends plus infinity, you are all going to get only 2 terms 2 finite one 0 terms. And what will you get? You see we will get limit imaginary part of tau tends to plus infinity of e 3 tau minus e 2 tau by e power i pi tau, what you will get? See if I put n equal top 0, I this this will become 1. And the I will get I get e power i pi tau, and e power i pi tau is going go to 0, as imaginary part of tau goes to infinity. So, essentially, I am going get 1. And similarly, for this also when I put n equal to 0 I will get 1. So, I will get 2 into 4 pi squared, which is 8 pi squared, alright?

And then when I put n equal to 1, the reverse is going to happen. When I put n equal to 1 you see this will become e power i pi tau, this also become e power i pi tau that is going go to 0, and this is going to become when I put n equal to 1 this is going to become 1. So, again I will get one here I will get one here, I will get another 8-pi squared plus 8 pi squared. So, I will get 16 pi square.

Now, combine this with the fact that as imaginary part of tau goes to plus infinity e 1 tau minus e 2 tau goes to pi square. Because here this will go the only contribution you will get as I told you was from the term n equal to 0, and in that case, we will get a pi square alright.

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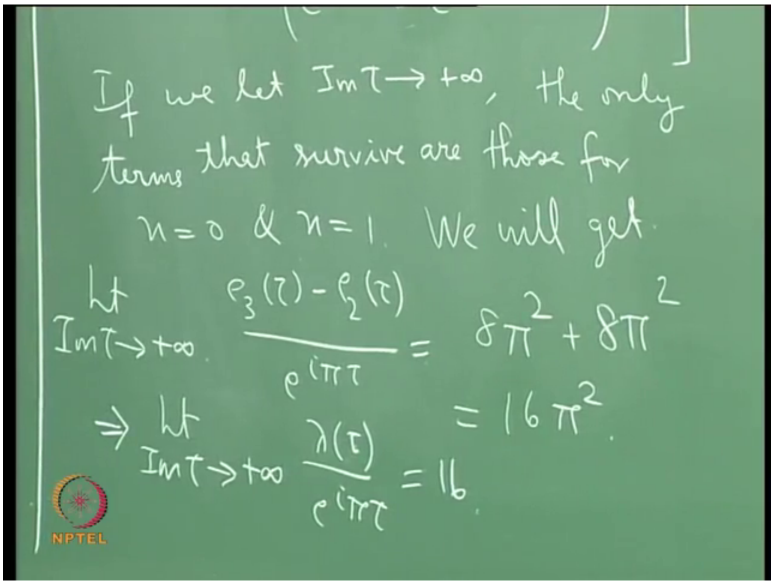

So, this will tell you that the limit as imaginary part of tau tends to plus infinity of lambda of tau by e power i pi tau is 60. So, you that is how you get that estimate. So, this this estimate is very, very important right. Now having done that I try to go and trace the image of trace the image of this there. So, that again involves a couple of simplifications.