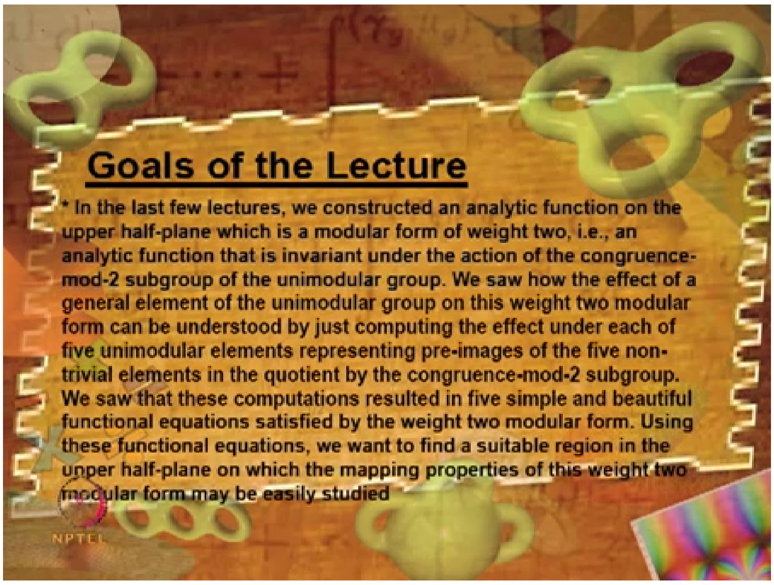
An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 - Tori and Elliptic

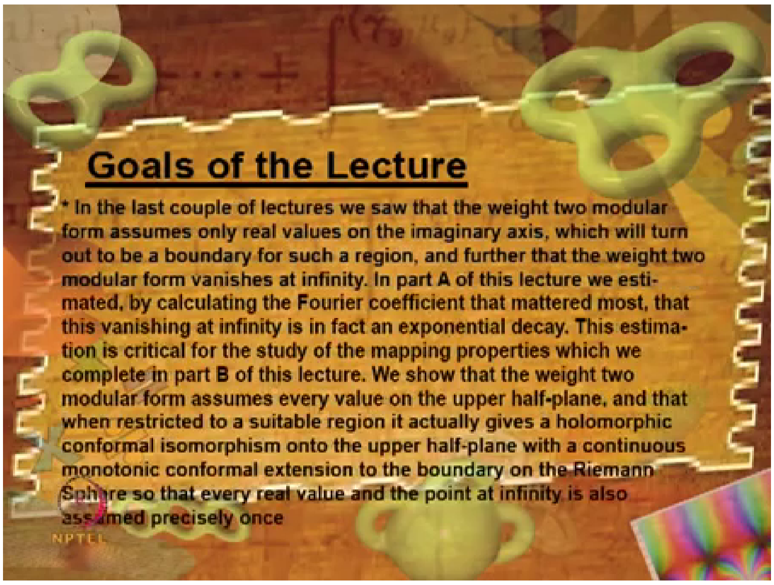
Dr. Thiruvalloor Eesanaipaadi Venkata Balaji
Department of Mathematics
Indian Institute of Technology, Madras

Lecture – 37
Part - B
A Suitable Restriction of the Weight Two Modular Form is a Holomorphic Conformal Isomorphism onto the Upper Half – Plane

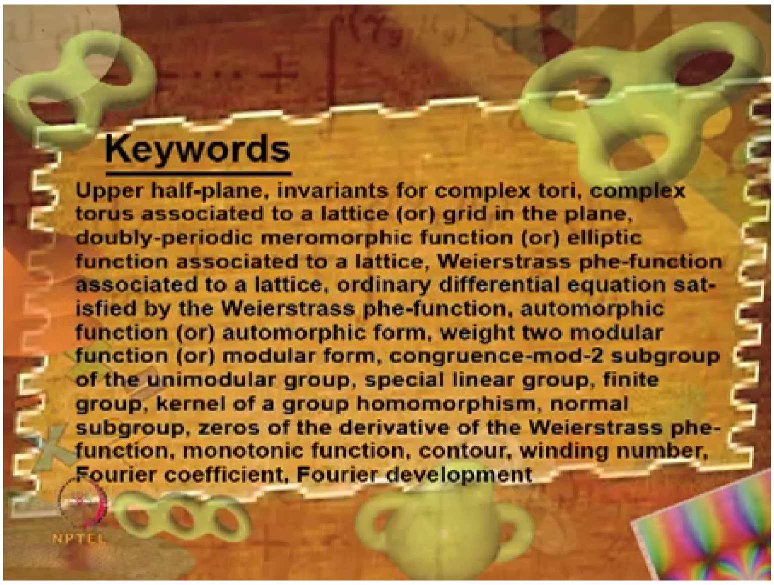
(Refer Slide Time: 00:10)



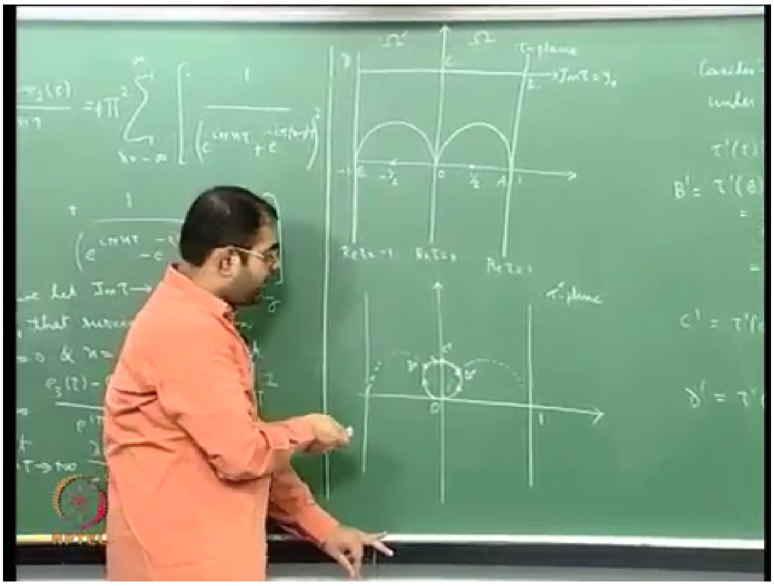
(Refer Slide Time: 00:15)



(Refer Slide Time: 00:21)



(Refer Slide Time: 00:31)



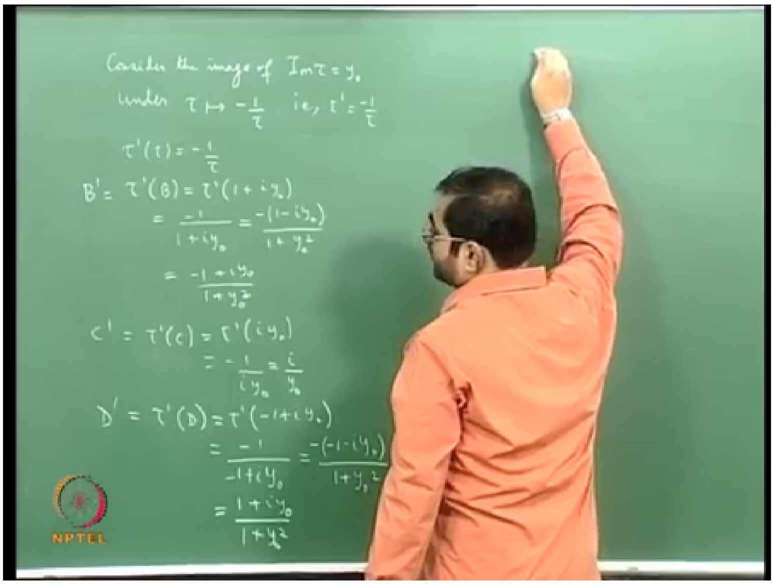
So, we take again the tau plane. And again, we take we take these 2 lines. So, let draw them little big. So now I am also considering the part of the region. That is a reflection about the imaginary axis of the region omega. So, this is real part of tau is equal to 1, this is real part of tau is equal to 0, and this is real part of tau equal to minus 1. And so, this is 0 this is 1, this is minus 1 all right. And I draw this semicircle center at half radius half. And then I have omega here. And let us call the reflection of omega as omega prime. In fact, in the very same methods that prove that that we will show to prove that lambda maps omega holomorphically isomorphically on to the upper half plane we will show that it will map omega prime holomorphically isomorphically on to the lower half prime.

So, it is certainly not out of place to consider omega prime. And so, you see I have my situation like this. Now what I am going to do is I am going to take a line segment, and I am going to take a line segment here or rather let me take a line, here parallel to the real axis. And with imaginary part of tau is equal to say some y naught. So, this so, length is y naught right. And let me give a names for these for these points. So, this is o, this is A all right and then B C D. This is again E I guess yeah. So, you see this is O A B C D and E right. And I am going to look at; first of all, I am going to for reasons that will be clear because I want to make use of the you know the functional equations. I am going to first study the effect of the transformations, tau going to minus 1 by tau and tau going to minus 1 by tau plus 1 on parts of this diagram. So, what I am going to do is I will draw another piece of this of the same diagram, another copy of it rather.

So, you see so, here I am in the tau plane right. And maybe I will call this. Maybe I will call this as tau prime plane right. And again let me draw this. So, you see so, this is again one this is 0. And then I am I just want to show this circle I just want to show these 2 circles here. So, I am going to draw them as dotted lines, because these circles these semicircles are in the source plane. And well let me draw these dotted lines like this. And well I am I am going to do the do the following thing. I am going to look at the first I want to look at what is the image of this line under 2 transformations; which is tau going to minus 1 and tau going to minus 1 by tau plus 1 all right.

So, this will be involve little bit of repetition. So, the first thing I want to say is well, so, you see consider the image of this line.

(Refer Slide Time: 05:24)



Imaginary part of tau is equal toy naught, this line which is parallel to the real axis. Consider the image of that under. Under let us say tau going to minus 1 by tau. So, which means so, that is my transformation is tau prime is tau prime is equal to minus minus 1 by tau this is my transformation ok.

See you see a tau going to minus 1 by tau is actually you see it is a moebius transformation. And you know a moebius transformation maps straight lines in circles on to straight lines in circles.

Now, you can see that you see if I if I put what is the image of B. So, B is the point if I calculate tau prime. So, so tau prime of tau is minus 1 buy tau. So, if I calculate tau prime of the point B, it will be just tau prime of well, the point B is 1 plus it is 1 plus I times y naught. This is what it is. And tau prime of tau is supposed to be minus 1 by tau. So, it is this is going to be minus 1 by 1 plus iy naught. And so, this is going to give you if I multiply by the conjugate I will get 1 minus iy naught by 1 plus y naught squared. So, I am going to get the point minus 1 plus iy naught by 1 plus y naught square.

This is what I am going to get. And this is and where will this point lie. So, you see so, this so, this point notice. So, B has to go to this point all right. Here, and well if I calculate also notice that this line, this whole line is mapped by tau going to minus 1 by tau onto this circle on to the same circle. So, the claim is the following the claim is that you see the image of this line all right, under these map is going to be, it is going to be a circle here. And let me draw it with a correct orientation, then I can explain to you B to C, yeah.

So, it is going to be a circle like this. This is what I am point to get. And in fact, you see B C D will go to these 3 points. So, this is the image of so, this will be prime, this will be C prime, and this will be D prime. This is what you will get. This is what you will get. So, B prime will be top prime of B. So, it is this point it is a point with a negative real coordinate minus 1 by 1 plus y naught squared and if you calculate what is C? Prime C prime will be tau prime of C it is going to be tau prime of what is C, C is just iy naught. So, it is going to be tau prime of iy naught, but tau prime of tau is supposed to be minus 1 by tau all right. So, I will get minus 1 by iy naught. And so, I am going to get I by y naught all right.

So, what I will get is I will get this point. So, this C prime will be this point. On the on the imaginary axis i by y naught. So, and what will D prime be? D prime is the image of D. So, it is and what is D? D is minus 1 plus iy naught. So, I will get minus 1 plus iy naught, if I plug it in here I will get minus 1 by minus 1 plus iy naught, and that is going to if I multiply and divide by the conjugate, I am going to get minus of minus 1 minus y naught by 1 plus y naught square.

So, I am going to get 1 plus iy naught by 1 plus y naught squared so that, is this point D, and you can see that you see, the if you look at see first of all B C and D are being mapped to this 3 points. C is the on the imaginary axis, and B and have the same y coordinate. You see, B the y coordinate of I mean B prime and D prime have the same y coordinate y coordinate of B prime is y naught by 1 plus 1y naught squared the y coordinate of D prime is a also y naught by 1 plus y naught square.

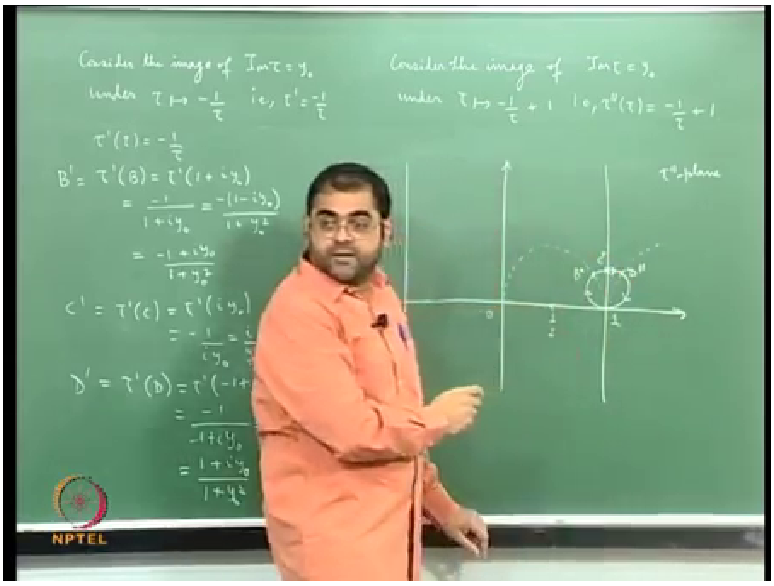
So, they are at the same height, and then you know if I let tau to tend to infinity. Tau now you see my tau is varying on this line, if I let tau to tend to infinity all right. Then essentially, that means, you know every point on this line the imaginary part is fixed to bey naught. So, I will have to let x to tend to infinity, but if I tend let tau to tend to infinity minus 1 by tau will go to 0. Therefore, the point at infinity on this line is going to be a map to 0.

So, therefore, what you will get is the image of this whole line, will be this will be this circle. And by symmetry you can see that this circle has to be tangent to the real axis.

Because of conformality all right. So, this is how this is what you will get. So, the important thing that I need is you see that as you go from B to C, the image in the tau prime plane will trace B prime to C prime. And as you go from C to D, the image in the tau prime plane will trace C prime to D prime all right. This is what will happen.

Now, then I also want to look at what is going to happen, if I take the tau double w prime. So, let me so, this is the effect of looking at this the image of this line, under tau going to minus 1 by tau. Let me look at the effect of that line under tau going to minus 1 by tau plus 1; which means I just have to translate this by one all right. So, what I will do is consider again consider the image of imaginary part of tau is equal toy naught under tau going to minus 1 by tau plus 1.

(Refer Slide Time: 13:00)



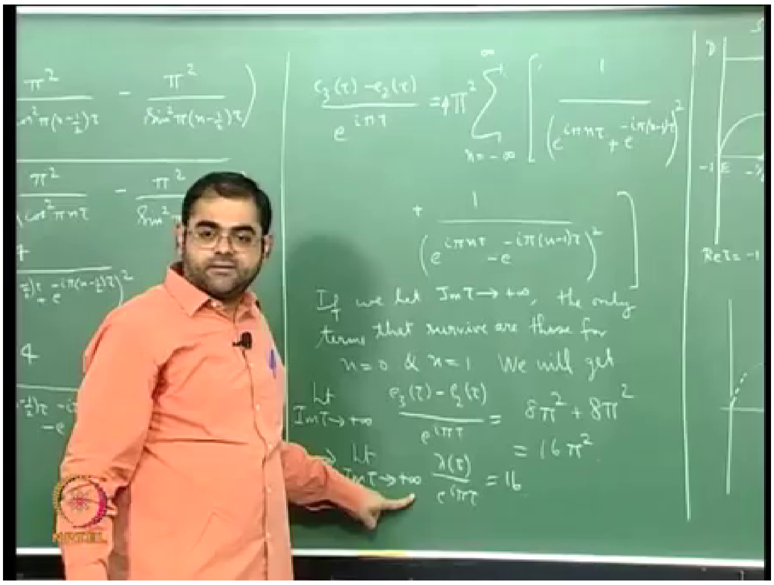
So, that is that is you know tau prime of tau is so, let me call it as tau double prime if you want. So, tau double prime of tau is minus 1 by tau plus 1 all right. Now you will see that if I draw the diagram in the tau double prime plane. So, what will I get? I will get you know what to expect. So, this is 1, this is 0. And well this does not look the same length as this does not look the same length as this is.

So, let me draw it right. So, this is minus 1 all right. And then again you have this semi circled centered at half, and what will be the image of B C D under the mapping tau going to tau double prime, which is minus 1 by tau plus 1. It will be just translate of this circle by one all right. And I will get again the corresponding points B double prime C double prime and D double prime. So, what I will get here is well you can imagine a mirror image of this circle here, and then I will get I will again get a circle here. Being traced in this manner, and I will see I will get that this is well this point here will be B double prime, this point. So, yeah because it is just translation, this will be C double prime and so, this will be C doubled prime, and well and this point here will be D double prime.

So, the point I want you to understand is that if you go from B to C in the tau prime plane I will get this portion of the arc from B prime to C prime. If you go from C to D, I will get this portion of the arc from C prime to D prime. If you go from B to C in the tau double prime plane, I will get this portion of the arc from B prime to C B double prime to C double prime, and if I go from C to D then the double price I get this portion of the arc from C double prime to D double prime, all right.

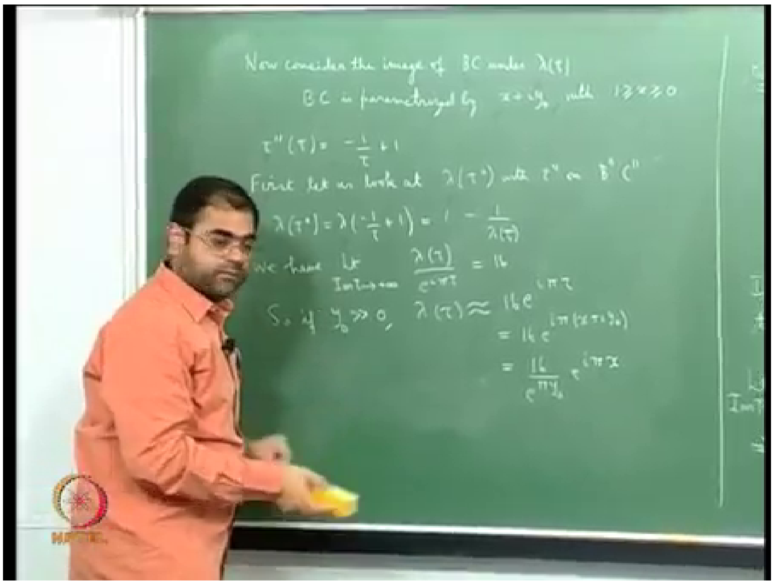
So, I will use this now, all right. I will use that in conjunction with this with this estimate right.

(Refer Slide Time: 16:09)



So now consider the image of let us consider one by one.

(Refer Slide Time: 16:12)



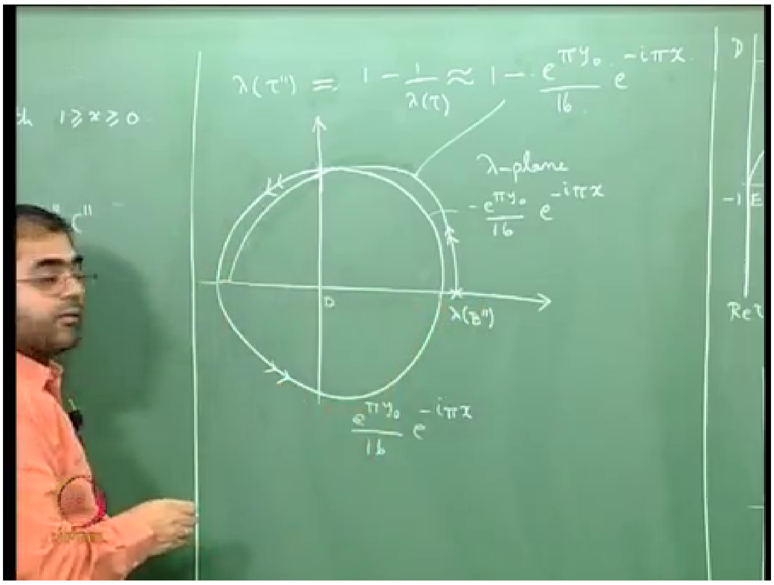
Let us consider the image of B C under lambda of tau. Now consider the image of B C under lambda of tau. B C is parameterized by x plus iy naught with x varying from so, when I met B x is 1. When I met C, it is 0. So, x is decreasing from one to 0. This is the orientation x is starting at one, and it is going to 0. If I if I move from B to C. Tau double prime of tau is what it is tau double prime of tau is minus 1 by tau plus 1. First let us look at lambda of tau double prime with tau double prime on B double prime C double prime.

So, let us look at what is the image under lambda of this piece B double prime C double prime right. So, you see you see what you will get is you know, lambda of tau double prime, but you know tau double prime is minus 1 by tau plus 1. So, it will be lambda of minus 1 by tau plus 1, and you have a functional equation. Lambda of minus 1 by tau plus 1 is what is it 1 minus 1 by lambda of tau, this is 1 minus lambda of tau. This is 1 minus 1 by lambda of tau. Now you see, as we have limit imaginary part of tau tending to plus infinity, lambda of tau by E power i pi tau is equal to 16. And so, but you see our tau is x plus iy naught, and imaginary part of tau is y naught.

So, you assume y naught to be sufficiently large. If you assumey naught to be sufficiently large. Then you can you can approximate lambda tau to 16 E pi i tau. So, if y naught is sufficiently large. Lambda of tau can be approximated to 16 E power i pi tau. And what is this 16 E power i pi tau 16 E power i by tau is just 16 into well, it is E power i pi my tau is x plus iy naught. So, it is E power i pi x plus iy naught right.

So, I am going to get 16 by E power. So, I will get E power minus p iy naught. So, I will get pi y naught in the denominator all right. And then I will get E power i pi x. This is what I will get, right. And this is with what this is with x varying from one to 0, I need to look at 1 by lambda tau. So, let me write out what 1 by lambda 2 is. So, you see it is a little bit involved in the sense that in order to trace you have to keep track of all these small arcs and trace their images carefully. You can then the whole proof is I literally doing this in a care full meticulous way all right.

(Refer Slide Time: 21:03)



So, you see now see because I want to calculate what is lambda of tau, double prime lambda of tau double prime is will be approximated to well, what is it is 1 minus 1 by lambda of tau. I mean this is equal to this and that is approximated to 1 minus well I will get reciprocal of this. So, it is going to be E power pi y naught over 16 into E power minus i pi x. This is what I will get. Now you see now go back now look at the lambda plane. You see if I look at E power pi y naught by 16 E into E power minus i pi x, and let x vary x decrease from 1 to 0. Then here pi x minus pi x is going to you know it is going to increase from minus pi to 0.

So, what I will trace is I will trace the circle centered at the origin. So, this is circle centered at the origin rather large say I will trace the semi-circle. So, what I get is I will get this. This is what this will trace beta without the minus sign only this.

So, the angle changes from minus pi to 0 all right. So, I will get this. So, this is what you will get for E power pi y naught by 16, times E power minus i pi x. This is what you will get. Now if I take minus of that, I will get this circle which is just reflection about the origin is z going to minus z. So, what I will get is I will get you see I will get a circle like this. So, I will get this circle. This is this will be minus E power pi y naught by 16 t power minus i pi x.

So, I will get this semicircle all right. And you see mind you the radius of the circle is pretty large. It is E power pi y naught by 16. And you assumey naught is positive and large enough. So, this is huge circle it is a huge circle. So, this is a huge semicircle in the upper half plane. And centered at 0, but then I have to add one to it. So, it will be slightly displaced all right. So, the net effect is that you know if I. So, I if you if you assume that y is if you assume that y is y naught is pretty large. Then this displaced then shifting this upper semi-circle by one is going to be negligible. So, what you are going to get is you are going to get something like this. You have you are your final image is going to be something like this.

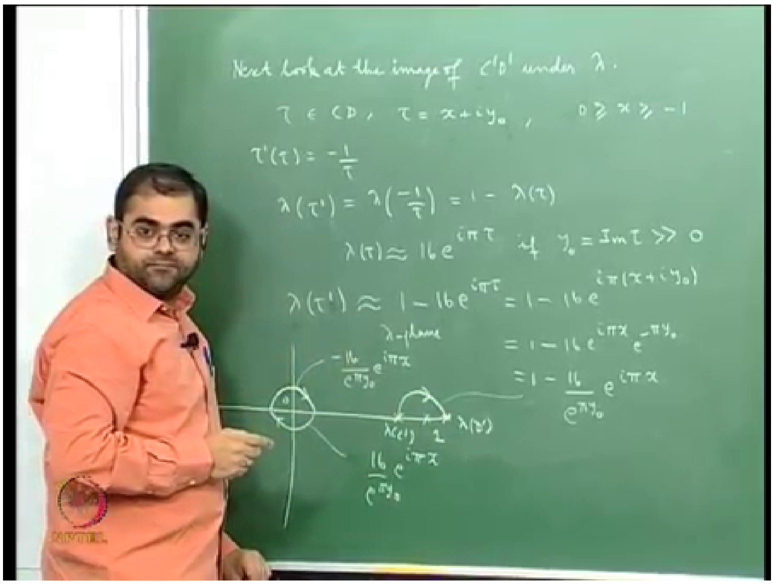
So, you will get something like this. This is so; this is going to be my final image. This is going to be in my final image. So, so you see this is what this is going to be all right. So, the so, the moral of the story is, that if I apply lambda to this to this small segment I mean to this small piece of the semicircular arc, all right. I end up getting a full semicircular arc something that is closed to it, because you know. There is an approximation here. So, I get of course, I get a curve, but I am, but what it says is the curve can be approximated by this semicircular arc all right. And the approximation can be made to any degree of accuracy, you want by simply taking y naught large enough.

So, and mind you therefore, you see B double prime all right. We will go to will start here all right, and then so, I will have to be careful about the orientation. So, if one keeps track of this. I end up with, that is correct. So, you see this will so, this point will be lambda of B double prime. And this point will be lambda of C double prime. This is what I will get. And I will get this large semi-circular arc, which is approximately which will approximate the curve that I will get which is the image of B double prime C double prime under lambda all right.

So, I have been able to get hold of the image of this portion of the curve, but I am thinking of this as in the I am thinking of it as actually lying in the tau plane itself. When I apply directly lambda to it all right, but then so, I need to see next the next thing I will like to do is to see what happens to the image of this portion of the curve C prime to D prime under lambda. I need to find out what is the image of that.

So, you can you can guess you see a this portion of the curve is close to 1, lambda at one is infinity. So, you got a very large semicircle in the upper half plane whereas, this portion of the curve is close to 0. And the lambda value there is 1. So, it is natural if you expect a very small semicircle as the image of this portion of the curve all right.

(Refer Slide Time: 27:38)



So, let us write that; next look at the image of a look at the image of C prime D prime under a lambda.

So, look at the image of C prime D prime under lambda. Notice that C prime D prime under lambda. Notice that C prime D prime is in the; if you want to think of it this is in the tau prime plane where tau prime is minus 1 by tau and tau is varying on C D. So, you see so, tau using C D using in the line segment C D, tau is of the form x plus iy naught. With a x as you go from C to D x is going from is decreasing from 0 to minus 1.

So, minus 1 SNR equal to I mean 0, greater than or equal to x greater than equal to minus 1. This is the parameterization of C D, for a point tau one C D. Then tau prime of tau is well minus 1 by tau. Because tau prime of tau is a minus 1 by tau, and we have seen that this map tau going to minus 1 by tau take C D on to C prime D prime all right.

So, you see lambda of tau prime will be lambda of minus 1 by tau. And again, I use a the functional form formula lambda of minus 1 by tau is 1 minus i guess 1 minus lambda of tau, it is 1 minus lambda of tau. It is 1 minus lambda of tau. And you see and again lambda of, but you know lambda of tau can be approximated to 16 E power i pi tau, if y naught which is the imaginary part of tau is sufficiently large all right.

So, you see if you calculate so, you see lambda of tau prime can be approximated to 1 minus 16 E power i pi tau. And well this is so, this the so, this turns out to be if you write it out this is going to be 1 minus 16 E power i pi tau is x plus iy naught. So, this is going to be 1 minus 16 E power i pi x E power minus pi y naught. So, I will get 1 minus 16 by E power pi y naught into E power i pi x this is what I will get.

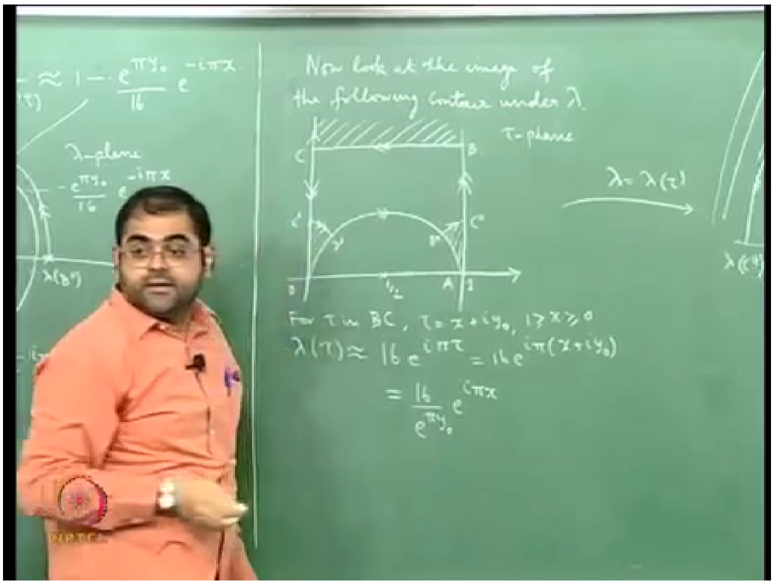
Now, you see trace the trace the image of C prime D price under lambda. So, I would trace this. What will I get? You see in the so, if I if you take the if you take the lambda plane, then 16 by E power pi y naught E power i pi x is going to be a circle centered at the origin with a very small radius. Because you see you see y naught is pretty large this is very small radius, and it centered at the origin. And you see the angle is changing from you see 0 to minus pi.

So, you see what you will get yes you will get a small circle here like this. So, this is this is what this is this is the this is what you trace if you plot 16 by E power pi y naught times E power i pi x. This is what we trace. Now if I if I want to trace minus of that, then I will have to take the reflection of that about the origin. So, what I will do is so I so, essentially what I will trace is a so, I will I will get this. So, that is not a that is not very, very neat, but so, you see I will get this.

So, this will trace for me minus 16 by E power pi y naught into E power i pi x. And this will be this will be centered at 0. And then if you want to trace the image of this, you will have to shift it, you have to add one to it. So, you have to shift the shift this whole arc to the point one. So, you know if you have the point one here all right. Then what I will get is I will get this. And this will be the image of this all right. And what will this point be this point will be the image of C prime. So, this will be this point will be lambda of C prime, and this point will be lambda of D prime. And again of course, I will get a and I will get a smooth arc, but it will it is approximated by this semi-circle, but I in this case. So, in this case I am getting a small semi-circle on the upper half plane centered at one all right.

Now, I think more or less we have come to the end of the story, except that I need to also know because you see I am trying to plot the image of this contour under, under, under lambda. So, in fact, what I have done is you see I have taken the following contour. So, let me and so, instead of taking this full contour, I have I am I will take this contour and I will cut it by this piece here. And then continue along this cut it by that piece there, and then go back and come up.

(Refer Slide Time: 35:02)



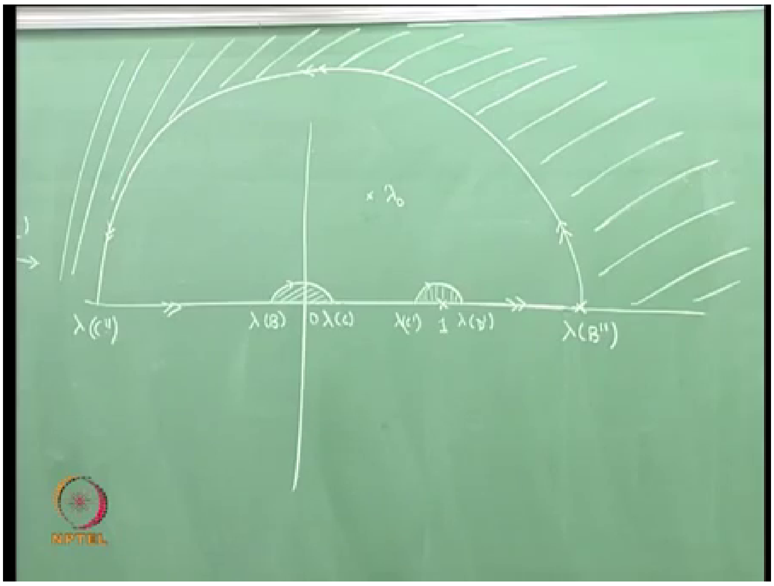
So, you see so, let me write that down. Now look at the image of the following contour under lambda. So, you see this is my contour now. So, let me draw it properly. A bigger diagram will help. So, here is my so, this is my tau plane right, and you see I have this, point which is this line. And then I have this large I have the semi-circle, center at half this is of this is the origin right. And then I have this line segment. And I will call this as A, call this as A. So, let me keep a these notations properly A, then I will take this portion of this segment this portion of the segment is this portion. So, it is B double prime C double prime, but think of it also as lying in the tau plane all right.

So, this is B double prime C double prime. And then this is of course, B this is C. So, this of course, this line this line segment is imaginary part of tau is equal toy naught right.

And then I have. So, you know I go along this arc, then I go along this. So, I avoid this piece. So, I have I avoid going to one right. So, I come like this I come up to B double prime then go along this semi circular the circular arc to C prime C double prime, then I will go to B then I go to C then I come down. And then I take this piece of the arc which is the peace from C prime to prime right.

So, let me draw that. So, C prime here is D prime. So, my you see my contour is this all right. So, you see this is my contour, I start with C then go to C prime D prime, then I go along this then go to B double prime C double prime then go to B then go to C. So, this is my close contour, and I am trying to look at the image of this. Now let me draw the image.

(Refer Slide Time: 37:53)



So, you see so, let us draw what let us look at what we get. So, you see if I started at the point at infinity, I would have got 0 all right. Perhaps the best thing that I should do is may be start with looking at first the image of C prime D prime.

So, you see if I take C prime D prime I have it somewhere here. So, the C prime D prime is approximated by a small semicircular region, which is in the upper half plane, it is a small semicircular arc in the upper half plane centered at one all right. So, what I will get is so, here is so, let me draw let me put one here. So, I will get this. So, this will be lambda C prime, this B lambda D prime. And you see the fact, is you see I am not I am avoiding going to 0, I am avoiding going to 0. Therefore, you, but you know as you go to 0 lambda goes to 1. So, I am avoiding 1 all right.

So, therefore, you see what will happen is if I so, what you should realize is that this whole shaded region here is actually being mapped to this region; which you can which you can understand by making y larger and larger. As you make y larger and larger and larger, you see these circles here they will shrink. So, the fact is that this whole piece here is being mapped to this approximately semicircular region all right. Then you see from, then from D prime to D double prime I have to go along the boundary and you know along the boundary lambda is of course, continuous and you know it is real.

So, from lambda of D prime to lambda of B double prime, I will have to go along the real axis. So, but what is lambda of B double prime, that is a huge you see there is a huge semi-circle, centered approximately it is centered at 1 all right. But it is a huge semi-circle. It starts at lambda B double prime, and comes to lambda C double prime all right. So, what will happen is you see I will get a well if I draw it properly, I will end up getting A, let me just look at the diagram I have drawn. So, that I do not draw something that looks too disproportionate, yeah that is right.

So, you see well, so you know I will I will end up getting or rather a oops, that was too huge. That is why I am trying to draw. Do not mind if it does not look like a semi-circle. Let me let me draw it a little smaller. So, you see I will get something like this. So, this point will be so, this is this diagram you see, it is lambda B double prime and approximately going to lambda C double prime. So, you see so, this is lambda B double prime. And you know it goes on like this. And you see from lambda D prime to lambda B double prime, I am still on the real axis this see this is the image of this semicircular segment by continuity right.

And because you know it is real all right. Then I get this it is a huge semi-circle that goes all the way and here I end up with lambda C double prime. I end up with lambda C double prime, then you see from C double prime, I go all the way up to B all right. You see if I so, of course, you know if I go all the way to the point at infinity, I will again so, I will then I then the image will go to 0 all right. But the point I stop somewhere here, and I need to find out what is the image of B C all right. I need to calculate what is the image of B C, and perhaps that is the only thing that I have not calculated, but that can again be done directly using this estimate. So, let me do that also. So, you see lambda of so, what is lambda of B C for tau in B C? Tau is x plus iy naught and as I mount B C x is decreasing from 1 to 0.

So, you see one greater than or equal to x greater than or equal to 0. If I calculate lambda of tau, I it will be so, lambda of tau because y naught is sufficiently large lambda of tau can be approximated to 16 E power i pi tau. So, it can be approximated to 16 E power i pi tau. So, this will turn out to be 16 E power i pi tau is x plus iy naught. So, what I will get is 16 E power 16 by E power pi y naught into E power i pi x.

So, the result will be, if I if I plot this, I will get a circle a semi-circle centered at the origin all right. It will be centered at the origin. And the x is see x is decreasing from 1 to 0. So, pi x is decreasing from pi to 0 all right. So, I am going to get a small semicircle all right. And it is going to so, it is it is going to be traced like this. And it will be centered at 0. So, the image of B the segment B C will be a will be a small semi-circle here.

So, this will be lambda of B, this will be lambda of C, and I will get this semi-circle. And the fact is I mean the fact that you must understand is well of course, I did forget to say that you see if I take this small region here. What happens to this small region? You see this small region is a neighborhood of 1, but lambda goes to infinity at 1. Therefore, you see as I make this if I make this finite y larger and y naught larger and larger, then this arc will become smaller and smaller. So, the image of this arc will become larger and larger therefore, this shaded region actually corresponds to the exterior of this circle. This is the, this is what this small region is being mapped to. Because you know if I make y naught larger and larger, this this this arc will shrink and as think shrinks this guy will become larger and larger.

So, this shaded region will go to the exterior of this semicircle semicircular region all right. And then also note is that if I take this shaded region, then this shaded region is this is this is a neighborhood of infinity, and at infinity the lambda dies to 0, all right lambda dies to 0. Before you know if I make the segment if I push the segment higher, and higher I am going to get smaller, and smaller semicircular arcs centered at the origin therefore, you know that this shaded region actually goes to this piece. And therefore, you know as I go from B see if I go from C w double prime to B by continuity you see I trace I go from here to here. Then from B to C I get the small semi-circle. Then from C to C prime I get this peace on the real line.

So, actually this gives you the complete image of if this gives you the complete image of the lambda. Now comes the final point. You see give me any value lambda not in the upper half plane, give me any value lambda not in the upper half plane which is then you see I can adjust this y naught to be large enough. So, that lambda naught lies inside this contour the image of this contour. What does it mean? It means that the winding number of lambda the you see the winding number of this the winding number of the image of that contour in the lambda plane about any point in the upper half plane is 1. But you know the winding number is precisely the number of times that value is taken by the function.

Therefore, what it tells you it not it tells you in one stroke that every value lambda not in the upper half plane is taken by lambda and it is taken exactly once. It is taken exactly once. So, this proves that lambda is a one to one holomorphic mapping from a omega to the upper half plane. And of course, you have already proved that you can extend it to the boundary in a continuous manner so that at 0 it takes the value one and at one it takes the value infinity. So, that literally completes the proof of the theorem right.

So, in fact, this proof also tells you that lambda is actually monotonic on the boundary. So, you so, if there is no question of the lambda going somewhere like this, and then and then you know stopping at somewhat point and then going back and then going forth, such things do not happen. The monotonicity is because expect for the points 0 and 1, at all other points lambda is analytic, and if the derivative vanished the mapping will not be conformal and you will never be able to get a semicircular neighborhood in the image.

So, therefore, this argument actually proves that lambda takes every value in the upper half plane precisely once, and lambda on the boundary is monotonic. So, that proves the theorem. Now you can take the you can use the symmetry of argument, and write it out and show that you know, the mirror image of this omega about the imaginary axis will be mapped by lambda onto the lower half plane right.

So, that completes the proof of the theorem. So, at first you know it might look a little confusing, but the point is that one has to be patient enough and do a draw a lot of diagrams, and keep track of these estimates. Then you will be then you can understand how lambda actually maps right. So, I will stop here.