An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 04
A Riemann Surface Structure on a Cylinder

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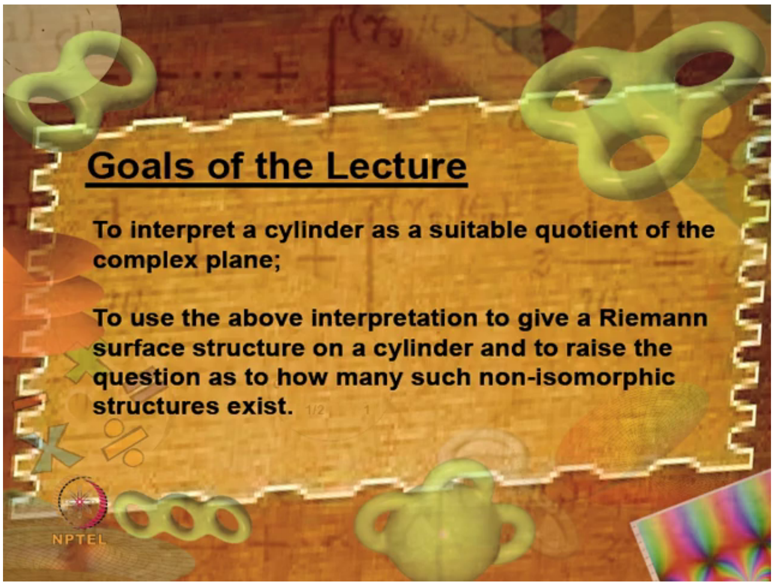

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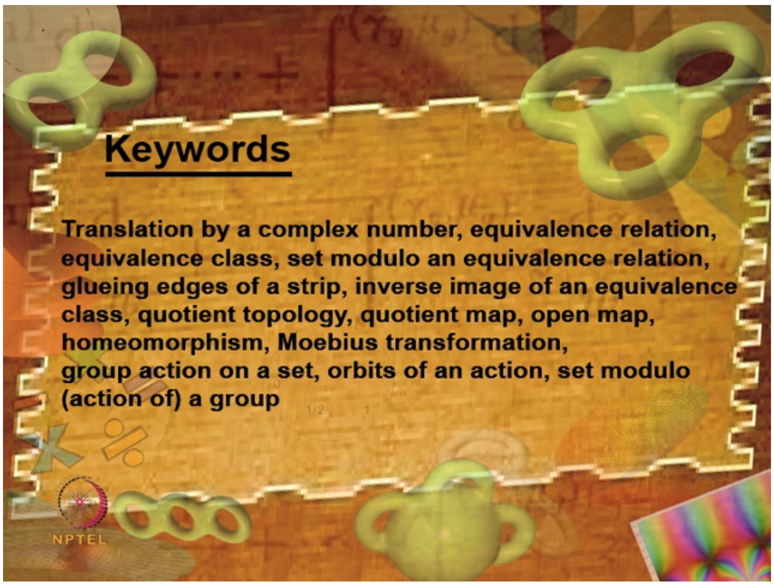

Welcome to lecture four in this series on a Riemann Surfaces and Algebraic Curves. So, what we will try to do in this lecture is to try to put Riemann surface structures on the cylinder and the torus.

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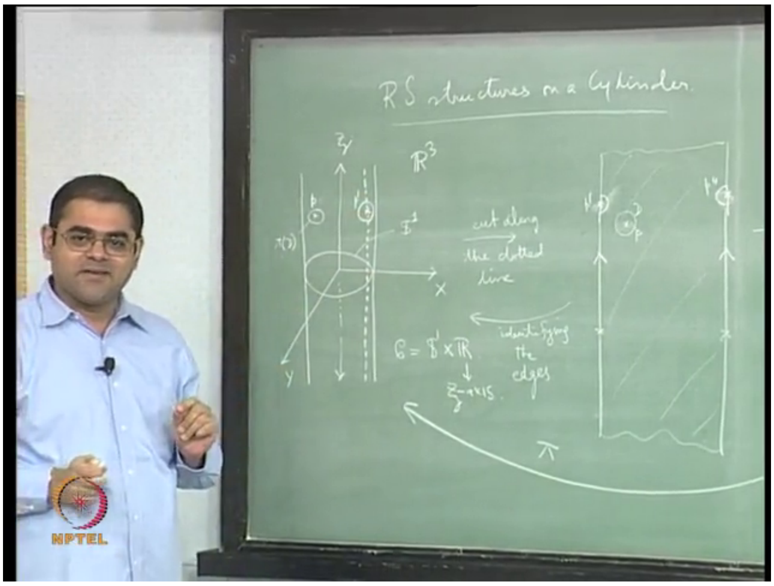

So, let us begin with the cylinder. So, let me write that down Riemann surface structures on a cylinder. So, you see we visualize the cylinder in three space as follows. So, these are the axis and we have here the unit circle which is usually denoted by S 1, the one sphere. And then we have the cylinder with this is base and with axis parallel to the z-axis. So, this is your cylinder here. And let me call it as a script C, you can see that this is just S 1 cross R where this S 1 is the unit circle and R is of course, R refers to the z-axis. So, here is my cylinder and basically I want to turn this into a Riemann surface.

So, what is our aim, our aim is given any point on the cylinder I will have to produce a small disc like neighborhood and coordinate chart on that neighborhood which identifies it with a piece of the complex plane, in fact, with a small disc like a disc on the complex plane. And then I will have to give you a collection of such charts which gives you an atlas. And of course, then the Riemann's surface structure that I am going to talk about is going to be a Riemann surface specified by the maximal atlas which contains that atlas. So, well so somehow I will have to connect this to the plane and in this case it is very easy to do that in an intuitive way.

So, what I do is that well I let us draw a dotted line on the cylinder a parallel to the z-axis. And assume that let me just cut it up cut along the dot dotted line. And when I do that what I will get is basically I will get a strip like this I will get a strip, of course, it is going to look like a vertical strip which is going to go to infinity in both directions vertically. And of course, this length is going to be the equal to the length of the a unit circle. And well how do I undo this operation, I undo this operation well by actually identifying the edges.

So, this strip has two edges and I put arrows to tell you that you have to identify this edge with that edge namely just you stick this edge to that edge and you will get back your cylinder well. So, here is my strip it is still not yet not quite the plane, but you can make this into the plane in the following way.

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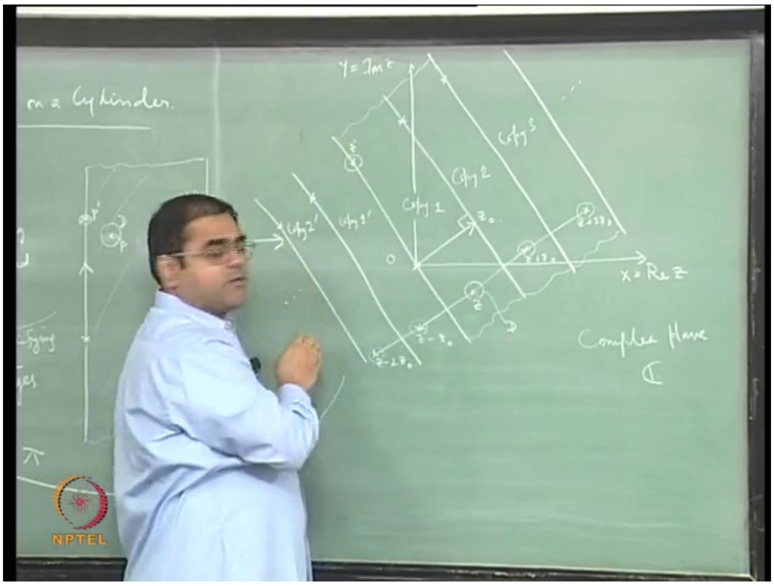

What you do is well at least for the purposes of a the present lecture at least for the moment let me erase this dividing line because I may need the whole of this blackboard.. So, you see what you do is you just well take the strip and repeat it infinitely that is put several infinitely many copies of this strip on both sides; and if you continue it you get the plane. So, I will do it like this. So, what I do is well there is no particular way to do it in the sense that I can think of the strip to be like this. So, here is my strip and let me call this as a copy one of the strip. I put another copy of it on this side. Here is copy two, and can I put one more, here is copy three and so on. And I do it also on in this direction. So, I put one more copy here if you want I will call it copy one prime and then get another copy two prime and so on. When I do this, what I get is really the complex plane. So, you see here is my complex plane C.

And well how do I go from back from here to here? To do that all I have to do is well choose a point here on one of these lines one of these lines that form the edge of this edges one of the edges of the strip call that as the origin. And then draw a draw the axis there the usual axis like this. So, this is the real axis, and this is the imaginary axis; incidentally you should not confuse that z with the z here because this was a z in R 3 and you should not you should forget this z when I am talking about that. Well maybe I will I will I will put a z like this, so that you do not get confused with this z and that z.

Well, and then you know I draw this perpendicular, and I take this vector, so this is a complex number z naught. And you can see that a translation by z naught, the translation map by z naught which is take any complex number z and translate it by z naught. So, this is z plus z naught translate it once more I am going to get z plus 2 z naught and translate it well in the in the other direction. So, this is z minus z naught translated it once more I will get well e z minus 2 z naught and so on. So, this is the translation by z naught.

If I do it so many times then you see that this translation by z naught will map this strip exactly onto this strip, translation by 2 z naught will map this strip onto that. And translation by z naught by minus z naught will map this report this translation by minus 2 z naught will map this report this. So, this translation by multiples integer multiples of z naught is precisely the operation that identifies all these strips together to give you back this strip, so that is the whole point of trying to get a complex structure on this. So, we will have to know formalizes.

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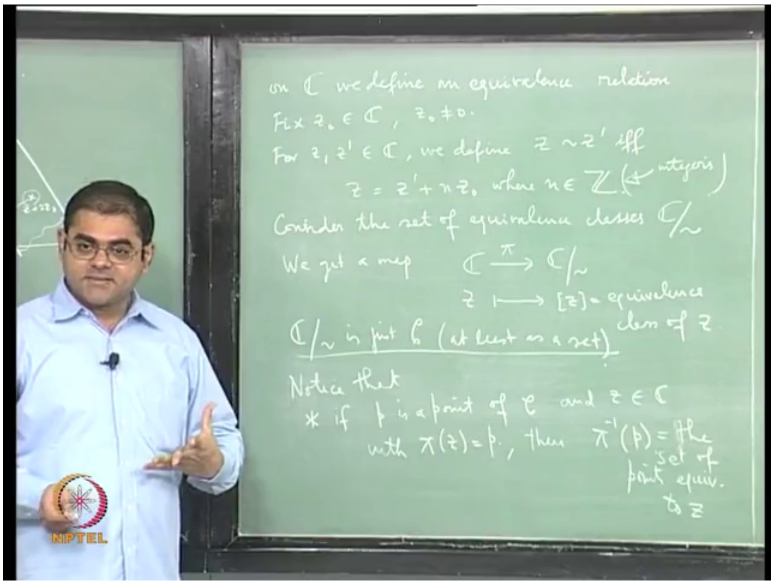

So, what you do you do the following thing? We take the complex plane, we define an equivalence relation, and we define an equivalence relation. What you do is you fix a z naught a complex number z naught and make sure that z naught is not 0. So, you think of z naught like this. And what you do is for z and z prime complex numbers we define z is equivalent if and only if z is equal to z prime plus n times z naught where n is an integer. So, this script Z denotes the set of integers. So, you can easily check that this is an equivalence relation. And therefore, you can look at the set of equivalence classes.

So, what you do is consider the set of equivalence classes which I will write as C mod tilde. So, this is the usual notation set modulo and equivalence relation. And you also get a map C to from the complex plane to C mod tilde and I will call this map as pi is a natural map its takes any complex number z to its equivalence class, this is the equivalence class of z under this equivalence relation.

So, if you think about it if I take a z naught to be this vector here, this complex number here then c mod tilde the set of equivalence classes its precisely my cylinder because you see what is going to happen is that all the copies of the strip are going to get identified with one copy. So, I am going to get this, but the identification is still not complete because on this there are still given a point here, it has to be still identified with the corresponding point here.

And given a point here it has to still be identified with the point here and that is because they are still translates by z naught. So, I still have to do essentially gluing this edge with this edge, it is only then that I that I get this set of equivalence classes. So, I therefore, get the cylinder. So, C mod tilde is just C at least as a set that is because it can also be seen in another way if it makes it makes it more clearer for you take any point in a c mod tilde that is an equivalence class.

Now, what I want you to understand is that if you take a strip like this any point inside if you take its equivalence classes it will be full of points which are translates at this point. And if this is a an interior point of a strip you will get exactly one point. So, for all the in interior points of a strip, they are the unique representatives for their equivalence classes in that strip, so that is what happen to interior points. But if there is a point on the strip which is a boundary point, then in one strip, you get two representatives, namely the point on one edge and the corresponding point on the other edge.

And so if you want to still get an equivalence, you still have to identify these two and when you do that you get the cylinder. So, you must understand that this cylinder is exactly c mod tilde at least as a set and that is the whole point. The point is that you are able to realize a cylinder as a set of equivalence classes for a nice equivalence relation on the set of complex numbers, and this is what will help you to give a Riemann surface structure on the cylinder.

So, let me make a couple of remarks the first thing is well you see notice that number one if p is a point of cylinder if p is a point of the cylinder and e z is a complex number with a pi of z equal to p. So, you see I am identifying the set of equivalence classes at least as a set with the cylinder. So, well, if you give me a point p on the cylinder, now that point p is going to correspond to a point on the strip. Either it can correspond to a point the interior in which case you will get only one point or it may correspond to two points if it is a boundary point, I will get two points. So, I will get p prime and p double prime I get two points. And well if I look at this point in these many copies that I have written down then saying that the image of z is p it is same as saying that that is one of those points.

So, z is a point which goes to p the way I have drawn it this z has to go to this p because this see this is an interior point of the strip, so it has to actually go to this. But it could this e z could have well been on one of the edges, and that is the situation I am looking at I am I am I am taking a point of the cylinder you think of it as an equivalence class and you take a representative of the equivalence class, well take this. Then what is pi inverse of p, pi inverse of p if you look at it, it is going to be just the equivalence class, the set of points equivalent to z. So, because it is what is this map, it takes every point to its equivalence class. So, if you take the inverse image of an equivalence class, you will get all the points in the equivalence class that is exactly what you will get. So, pi inverse of p will be the set of all those points are equivalent to z.

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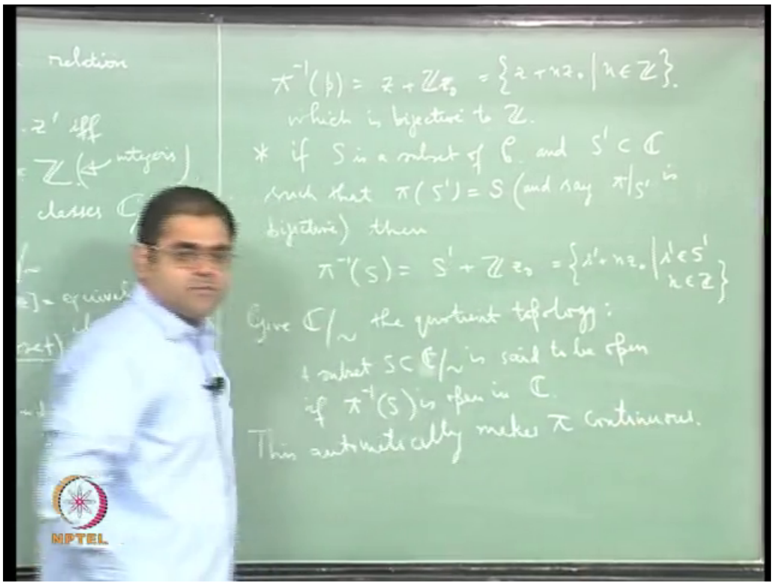

And you can see that that is nothing but pi inverse of p will be just z plus integer multiples of the points z naught. Because the set of points equivalent to z is all translates by integer multiples of z naught which is what I have drawn here, these are all the translates. So, the inverse image of a single point will be a set this set of points they are all p translates of z, so that is what you will get. And what this notation means is the set of all z plus n z naught where n is an integer. And you can see that this is bijective to z, it is just a copy of z because I will have to just map this z plus n z naught to n. So, it is a copy of z, you take any point the inverse image is just a copy of z.

Well, I can generalize this a little bit more if S is a subset of the cylinder and S prime is a subset of a complex plane such that pi of S prime is equal to S, and say pi restricted to S prime is bijective. So, here what I did was I took a point on the cylinder and I took a representative. Here what I am doing is I am taking a subset of points on cylinder, and I am taking a subset of representatives. And then the same kind of argument will tell you that pi inverse of S will be nothing but S prime plus all translates of S prime. So, it will be S prime plus again z time to z naught, this is going to be the set of all S prime plus n z naught, where S prime belongs to capital S prime and n belongs to z, this is what I am going to get.

I am just trying to make you understand what this map is I am just trying to make you understand what this map is. So, it is very clear that you know now if I choose on the strip a very small a disc if I choose a very small disc here, so the radius of disc is extremely small. And suppose I call this disc as say D, then if I take the image of this D in the cylinder what I am going to get is I am going to get a small disc like neighborhood surrounding the point p. So, it is going to be just, so what I will get here is so well let me rub the circles off and draw it like this, so here is my D.

So, if I take a disc like this and I take its image I am still going to get a disc here a disc like neighborhood. And if I take the inverse image of this, what I am going to get this I am going to get all possible translates of this disc by my integer multiples of z naught. So, this is what I am going to get. And since I have chosen the disc sufficiently small, the map pi mind you the map pi is this is a map pi, this is exactly the map pi. What this map pi is going to do is its going to map all these translates of z to the point p, and all these all these discs to this disc here.

So, well I can call let me call this as pi D, I call this as pi D. And I can use this to now give a coordinate at the point D. And why is that that is because of the fact that well I just now said that this cylinder is just the set of equivalence classes as a set it is just a set of a equivalence classes as a set. But then the cylinder has more structure in fact, it is a topological space and it is a nice surface in r 3. And you can do calculus on it if you want there is a notion of differentiable functions and so on and so forth, whereas the set c mod tilde does not seem to have anything.

So, to begin with at least you can put a topology on c mod tilde on the set c mod tilde, so that the identification of c mod tilde with this cylinder is a real identification even as topological space. And the way you do that is a standard technique in topology which is called a giving the quotient topology. So, let me come to that give C mod tilde set of equivalence classes, the quotient topology, what is this quotient topology. The quotient topology is a set here is open if and only if it is inverse image under pi the set given by the inverse image under pi which is a subset of the complex plane that is open. So, and set in the complex plane of course, you can recall is said to be open if it is a union of a discs. So, S is a subset is said to be open, if I inverse of S is open in C. So, this is called the quotient topology and this is a technique that can be used whenever you have a surjective map from a topological space to any set.

If x is whenever you have a topological space and you have surjective map to any set then to this set you can give a topology by namely that quotient topology, namely you call a subset to the target open if and only if the inverse image under this map is an open set in your source topological space. So, that is what I am doing here. Now, the advantage of this definition is that it automatically makes pi continues, because well this is actually begging the question. You see if normally if you are given two topological spaces and if I give you a map when do we say that that it is continuous only when the inverse image of an open set is open, but now I am demanding the inverse image of an open set to be open. So, this automatically makes pi continues.

So, this automatically makes pi continuous, this automatically makes pi continuous. And in fact what it does is that it actually identifies the set of equivalence classes with the cylinder even topologically. Now, the identification is not just as I said, but it is an identification also in the topological sense. That is because you see if I now take a point z and if I take a disc small disc D surrounding z then its image here will be a small disc a disc like neighborhood pi D surrounding the point p. And if you now take this set, this is open because pi inverse of pi D is D the union of D and it is translates that is a union of open sets. So, it is open, so that makes pi D open.

So, what you have actually proved is that pi is an open map that is it takes open sets to open sets that is another beautiful property of this map. So, the moral of the story is that you not only recover the cylinder as the set of equivalence classes, you actually recover it as this topological space structure on the set of equivalence classes that makes pi a continuous map by the quotient of all.

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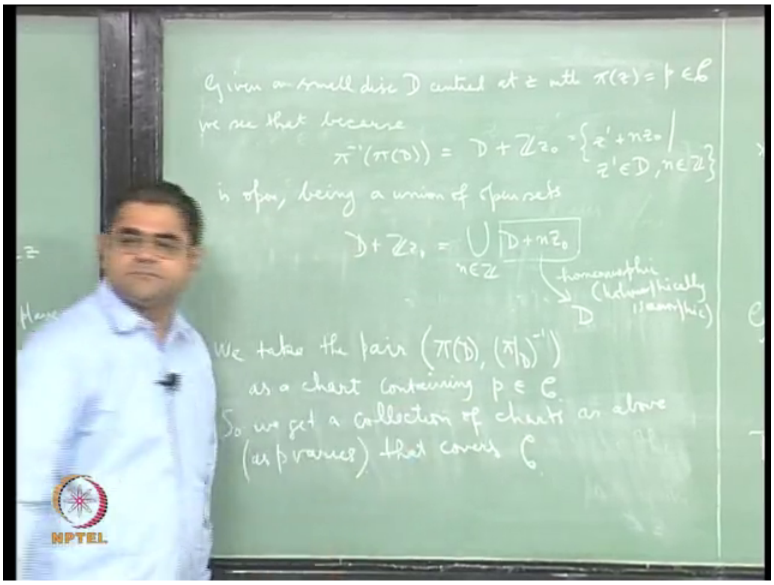

So, let me write the following down given a small disc D centered at z with pi of z is equal to p on the cylinder, we see that because pi inverse of pi of D is just D and all of its translates, this is a set of all z plus z prime plus n z naught. Where z prime is an element of D and n is an integer, because pi inverse of pi of D is a union is open being a union of open sets.

So, each D plus n z naught is an open set, it is just a translate of D. And union of all these D plus n z naught as a n varies is precisely what this set is so in fact, I can write this as D plus z times z naught is equal to union n belonging to z D plus n times z naught. And each D plus n times z naught is homeomorphic to D or even holomorphic even holomorphically isomorphic D because translation is of course, a holomorphic isomorphism translation is a holomorphic map and its injective and the and there is an inverse map for translation.

So, well so each of these sets is like D and there is union of these open sets. So, it is open. So, if you start with a small disc D here, I am going to get its image here is an open set. And this is what is going to help me to give a chart at the point p. Namely, what I do is the following what is it that I want an open set surrounding the point p and I want a homeomorphism of that open set with an open set open subset of the complex plane.

So, what will I do I will take this open set pi of D and the map I will take is pi inverse and that pi inverse will be not just any pi inverse I will just take this D itself I will just if

I restrict pi to this D mind you it is bijective. That is because you see I have made this disc very small. So, two points in two different points in this disc cannot go to I mean they cannot go to the same point here if this disc where that made large enough, so that it extended beyond the boundaries. So, for example, instead of this disc suppose I took a huge disc then I will get several points which will go to the same point there, the map will fail to be injective. So, that is the reason to choose the disc to be small enough, so that that is what is going to help me to give a complex coordinate at this point.

So, let me do this. We take the pair pi of D comma pi restricted to D inverse as a chart containing p. So, what I have finally been able to do is for every point here, I have been able to give you a chart. And I can do this for every point because my point was arbitrary. The only thing that remains to say that this is a Riemann's surface is to say that all these charts are compatible that is the only condition we will have to check. And once we check that then this collection of charts is going to give you a Riemann surface structure. And of course, when we talk about that Riemann surface structure, we of course think we also keep in mind the maximal chart which contains this chart.

So, let me write that down. So, we get a collection of charts as above as p varies that covers your cylinder. Now, to say that this collection of charts is an atlas, I will have to just verify the compatibility condition. So, let me do that that is also pretty easy you can even do it diagrammatically. Of course, all these things can be written down a little bit more formally, but nevertheless it is not very difficult to write things down more formally.

Student: (Refer Time: 32:44) there will be some boundary points, for that boundary points, how do you give this correspondence.

You mean a neighborhood.

Student: Neighborhood.

No, there is no problem. If the point you select was a point on the dotted line, then you are going to get two representatives for that point. So, here well the point is you are going to get a point and inverse image will look like z prime; if you want to call it. And you know what you will get is all of it is translates, you will get all these points. And if you take a disc like neighborhood small enough disc like neighborhood surrounding this then again the image of that small enough disc like neighborhood in for example, on this strip you will get, so let me rub this off and just put a very small cross here let me rub this off.

So, the inverse image of that neighborhood I mean the under this map is going to be well you are going to find a piece here. So, let me call this as a p prime here. So, I am going to get a piece here, and I am going to get another piece here, but then when you glue it you are going to get if my point is p prime, I am still going to these two things are going to glue I am going to still get a nice disc. So, in this way you are actually covering every point even the boundary points. So, it will work even if you had chosen a point on the dotted line where you have cut it.

And of course, here I have put in some arbitrariness because you know the way I pasted this disc on the complex plane, well I could have even kept it vertical or I could have kept it horizontal. But just to give a general case I have put it at an angle and I have drawn the axis arbitrarily so that I get a vector. The only restriction the way I have done it is that this vector has to have length equal to the length of the unit circle nothing more. And even that restriction you can remove. If you remove that restriction and you take any z naught which is nonzero, you are going to get a cylinder anyway. The only thing is that it is not going to be the circle the perpendicular section of the cylinder is going to be a circle its radius is not going to be 1, it is going to be something else, but nevertheless it is the same as a cylinder up to a scaling. So, it is the same surface. So, on that also you will still get a Riemann surface structure.

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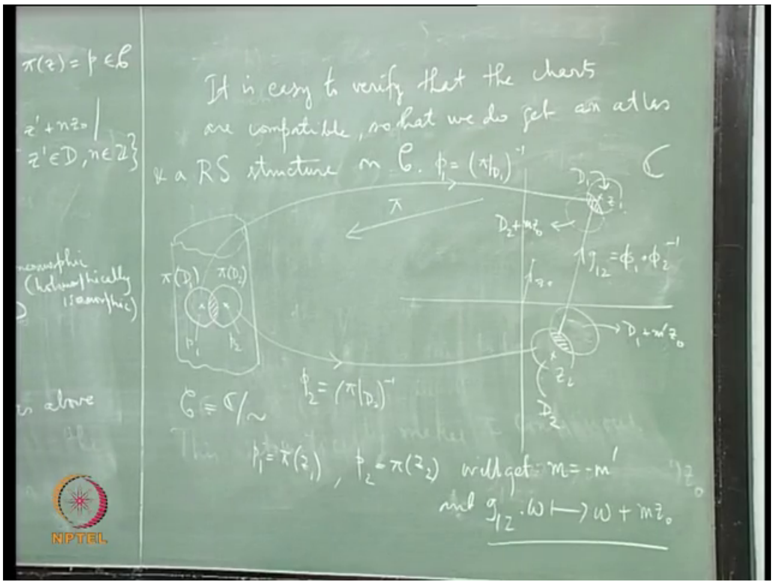

So, it is easy to verify that the charts are compatible; so that we do get a Riemann surface we do get an atlas and a Riemann surface structure on the cylinder. So, let me do that a little bit in a diagrammatic way. So, you see so here is my situation. So, here is my cylinder. So, here is my cylinder. And well I have a point let me call this as p 1 and it is surrounded by a small disc like neighborhood D 1. Let me draw it a little bigger, so that it is easier to label these things. So, here is my point p 1, and it is surrounded by a disc like neighborhood pi of D 1, the way I have done it.

So, it means that what I have what I have done is well here is my complex plane. I have chosen a point z 1, which is being mapped by this; this map is just the map pi, and this is a cylinder which is identified with the set of equivalence classes. This z 1 is going to go to p 1, so p 1 is pi of z 1. And of course, this pi of D one is just the image of a disc D one small enough disc D 1 which is surrounding z 1. And then what is this compatibility I will have to say that whenever two charts intersect then I will have to say that the transition function is holomorphic.

So, take another such chart. So, that is going to be centered at some other point p 2 and I am this is going to be just pi of D 2 where well p 2 is going to be the image of a point z 2 here. So, p 2 pi of z 2, and D 2 is going to be again a small enough disc centered at z 2. And what I will have to say is that I will have to say that transition functions are holomorphic. So, what is it that I do well let us let me write down the transition function.

So, I have, so the transition function from here to here is what is this function this is well if you want let me call it as phi 1, phi 1 is just pi restricted to D 1 and inverse that is a transition function, because that is the way we have defined it.

Always a chart consists of a pair u comma phi, where u is an open set and phi from u to an open subset of the complex plane is a homeomorphism. So, my open set is pi D it is open because I have already proved why it is open because of the quotient topology and pi restricted to D is a homomorphism. So, pi D inverse is also homeomorphism. So, it is a pi D pi restricted to D inverse is the homeomorphism from pi D to d. So, pi restricted to D one inverse is I call it if I call it as phi 1, it is a homeomorphism from D pi D 1 to D 1.

Similarly, I will get another homeomorphism like this, this homeomorphism is going to be just phi 2 if I want to call it as phi 2 it is pi restricted to D 2 inverse. And what is the conditions I will have to verify take this intersection which is pi D 1 intersection pi D 2. And this will correspond to well a piece of the disc here and it will also correspond to a piece of a disk here. And I will have to say that the transition function which is the transition function is now going to go from here to here which is go by phi 2 inverse and then follow it up by phi 1 this is my so called g 1 2 my transition function.

And I will have to say that this is just a holomorphic map. Well, if you complete the picture properly you can see that if I take all translates of D 1, which will be the inverse image of pi D 1 under pi. And if I look at all translates of D 2, in fact, what will happen is that I will get there will be a translate of this D 2, which will look which will which will essentially be like this. And then this D two this intersection comes from a translate of D one which look like this. And these two will be translates of each other and this translation will be translation by the vector this will be the vector z naught.

So, what you will get here will be some D 2 plus some m times z naught that is what this disc will be. And what you will get so this is my D 1 and let me write it properly and this is my D 2. And this disc will be D 1 plus some m prime z naught because you see if you think about it these two discs you go back to this picture, these two discs side by side are going to give you two discs here. And then when I take the inverse image, I am going to get a pair of discs along with your intersections and they are all translates. So, it is actually going to look like this. The only thing is that you see I have to translate this to this to get this image and therefore, literally my z naught should be in this direction if this image is correct.

So, the moral of the story is that this g 1 2 is nothing but translation by you can write that down what will happen is that you see this disc is D 1 plus m times z naught if I translate it by minus m primes z naught I should get D 1. And this disc if I translate it by m z naught, I should get this D 2 plus m z naught. So, what will tell you is that this m is equal to minus m prime you will get m equal to minus m prime and g 1 2 to be the map that sends omega two omega if you want plus m z.

This g 1 2 will just be a translation by integer multiple of z naught, you will see that D 2 will go to D 2 plus m z naught and D 1 will go to D 1 plus m z naught, but m is minus m prime. So, it will go D 1 will go to D 1 plus ms D 1 plus m prime z naught will go to D 1 plus m prime z naught minus m prime z naught. So, I will get back D 1. So, you will see that a g 1 2 is just translations by a suitable integer multiple of a z naught and this is certainly a holomorphic map it is just a translation.

So, the moral of the story is if you write it down you will see that this is just this is the transition function is just translation by a multiple of integer multiple of z naught and that is holomorphic. So, that gives you the compatibility condition between two charts. We will have to of course, I am assuming that these discs are very small and then this is true you can convince yourself you can write it down. So, having done this you have we have been able to give a Riemann surface structure on the cylinder. Now, that is there is one more aspect of this that one can look at and that is the so called group theoretic interpretation of this, which I will try to now explain.

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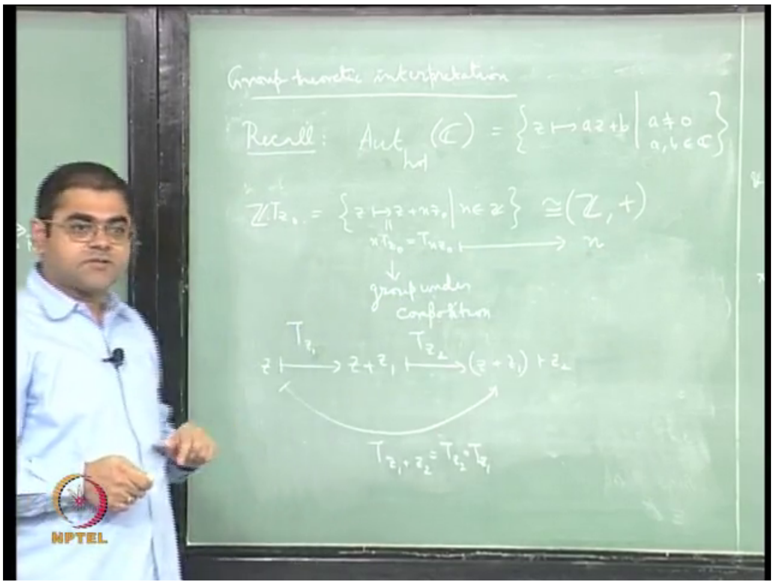

So, let me give you also a group theoretic interpretation. And this interpretation is important because what is going to happen in is that the whole idea is that as we will see that may we will prove that all Riemann surfaces can be gotten from well known Riemann surfaces like the plane or the Riemann sphere or the unit disc by going modulo a group of automorphisms. So, the idea is you can get every Riemann surface as a quotient by a group of automorphisms. And this philosophy in general is called the uniformization theorem, the general uniformization theorem. And this is the technique which allows you to translate questions on Riemann surface to questions on the complex plane or something as simple as the unit disc or this Riemann sphere.

So, let me give you this with theoretic interpretation. So, what we do is that we first recall that the automorphisms or the complex plane set of holomorphic automorphisms of the complex plane that is given by the set of maps of the form z going to a z plus b where a is nonzero; and of course, a and b are complex numbers. So, these are all the possible holomorphic automorphisms for the complex plane. I think you would have proved this in a first course in complex analysis, but if you have not you can still do it. And well these are in particular they are mobius transformations.

Now, what I am going to do is that I fixed this vector z naught and I was looking at translations by z naught or rather translations by integer multiples of z naught. So, what I am going to do is instead of looking at z naught, I look at the translation by z naught.

And mind your translation by z naught is also an element here a translation is an element here namely I take a equal to 1, then it becomes translation by b. So, what I do is that I define this group z times z naught if I want rather let me put T z naught which is what this is a set of all maps e z going to e z plus n z naught, where n is an integer. Of course, mind you again z naught is not zero and T z naught is this map and well this map will then be n times T z naught. And you can check that this is also equal to translation by of course, it is equal to translation of n times z translation by n times z. If T sub lambda denotes translation by lambda.

So, here is my group. And you can see that this isomorphic to the group of integers and addition after all two translations when you so here you see this is a group under composition. In fact, all these a holomorphic automorphisms that itself is a group and the operation is composition take any two holomorphic automorphisms compose them, you get another holomorphic automorphisms. And but the only problem is that that is a and of course, any element here a holomorphic automorphisms here has an inverse, the inverse of a holomorphic automorphisms is also a holomorphic automorphisms. And you can also write that inverse down explicitly using this formula and you will see that the inverse is again a map of this type. So, it is actually a group under composition of mappings.

And here how do a two elements combine they combined the compositions. So, in particular if I take two translations they combined by composition. And that corresponds to adding integers you see that is because you see if I take an a complex number z and first apply T by T z naught or let forget z naught let me put z 1 then what I will get is I will get z plus z 1. Then if I apply T z 2, I am going to get a z plus z 1 plus z 2 and you can see that this composition is just T z 1 plus z it is T z 1 the two composition T z naught first apply T z 1 then apply T z 2 you get this.

And you can see that since I am getting z 1 plus z 2 it is a commutative group. So, these whether I apply first T z 1 and then T z 2 are whether I first apply T z 2 and then T z 1 it is going to give me a complete group. And in fact, you can get this isomorphism by sending the T n z naught to n. So, this T n z naught going to n it will be an isomorphism of groups, namely it will preserve the additions here; and the addition here will correspond to addition of integers. So, the moral of the story is I am looking at the translations as a group a subgroup of automorphisms of the complex plane.

And you know whenever a group acts on a set, you can talk about the orbits of the group. The orbits of the group are just given any point of the set, you look at all those points that you get by applying elements of the group. But that in this case just translates to the following, you take any point and you just take translates of the point by a integer multiples of z naught. And therefore, the orbit of a point z is just going to be just the equivalence class of the point z. So, this tells you that you can think of the quotient a set of equivalence classes that is c mod tilde as the complex plane modulo this group.

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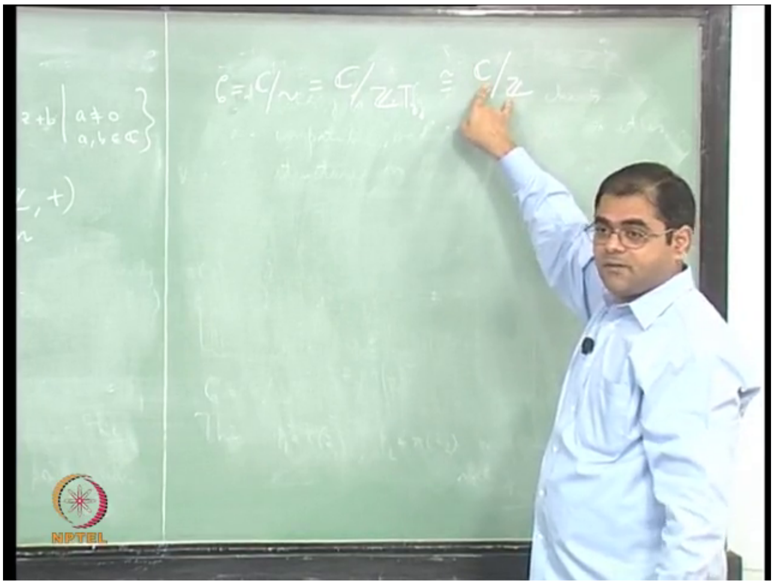

So, you see the cylinder which has been identified with the set of equivalence classes is also just the complex plane modulo this group z T z, it is quotient by a group. And well if you take this point of view then it explains that this is also you know isomorphic to C mod z because after all z times T z naught identified with z by this map. So, what it tells you is that your cylinder is C mod z basically its C mod z complex number is mod z.

And you if you really look at it in a very natural way that this should have a group structure after all it is a group by a subgroup. If you think of z as sitting inside C as a subgroup then this is a group, and that should make you expect that there is going to be some group structure on the cylinder. And that is because after all its S one cross r you take an element of S one both of them are groups R is group under addition, S 1 is a group under multiplication and this is just a product group. And therefore, this group structure actually is natural to expect.

So, all I am trying to tell you is that your group structure also comes into the picture, if you think of it as a quotient of C by a group of automorphisms and this viewpoint is going to be extremely important in the lectures that follow.

So, I will stop here.

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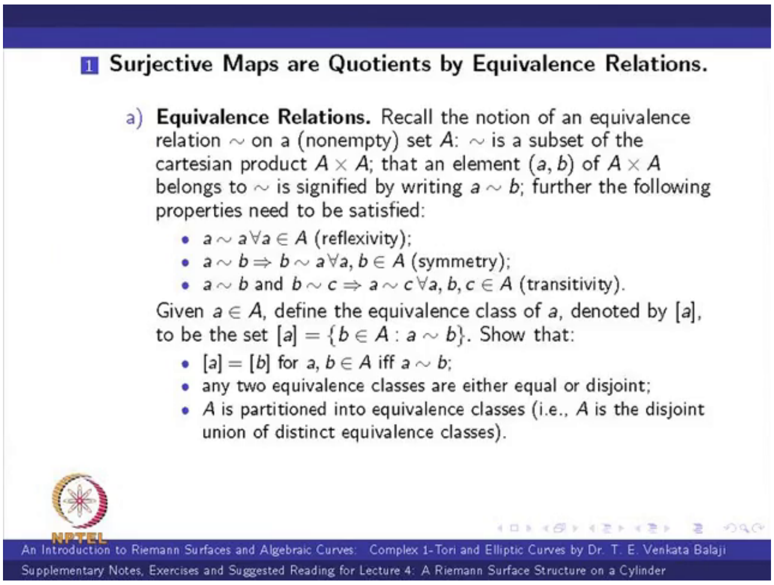

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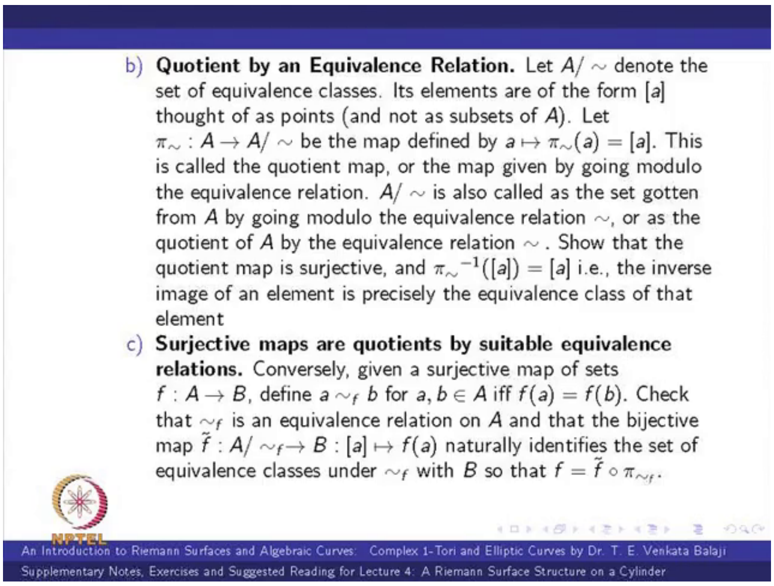

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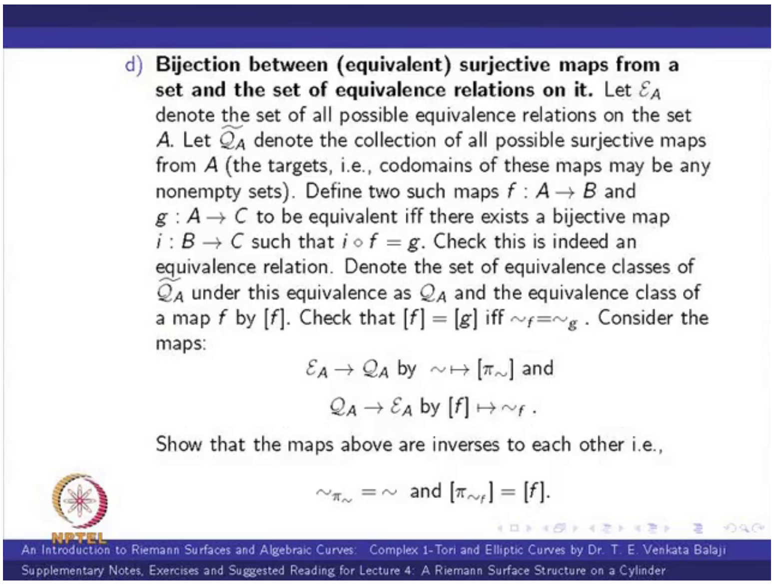

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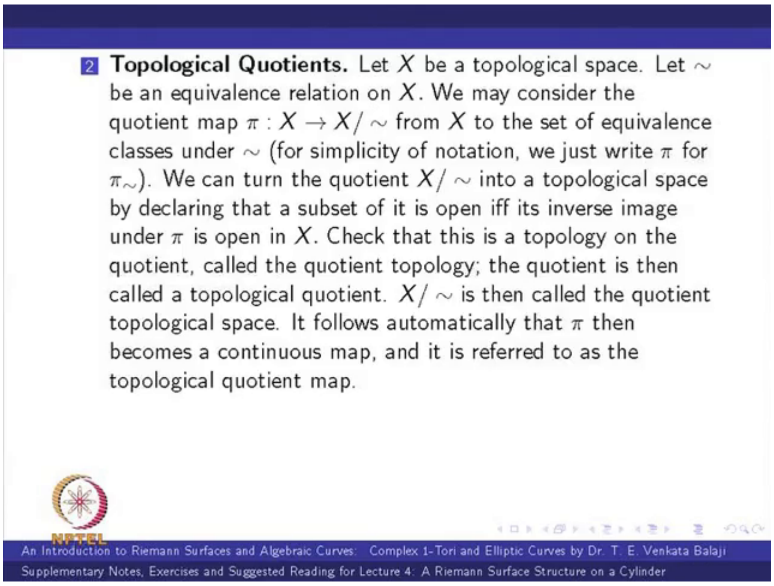

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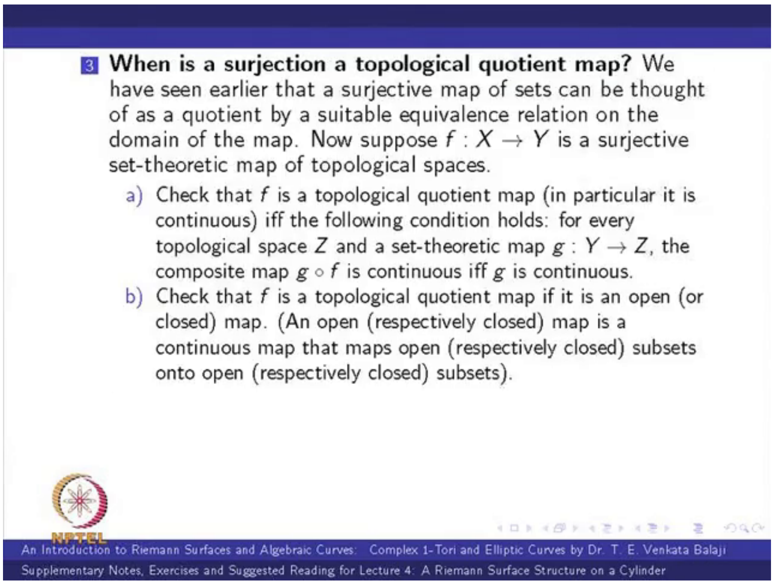

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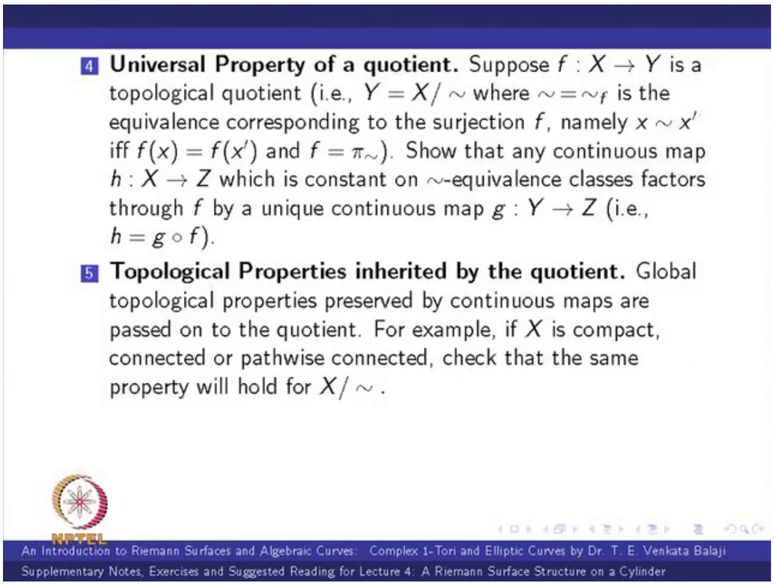

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6 Separation Properties of the quotient. Global separation properties like being T_i $(0 \leq i \leq 4)$ including Hausdorffness $(= T_2)$, regularity, normality, paracompactness, second countability, metrizability (see the slides at the end of Lecture 3) need not pass on to the quotient. For a pathological example, just look at the equivalence relation given on the real line by declaring two real numbers equivalent iff their difference is a rational number.

The separation properties of the quotient are linked to the nature of the subset $\sim$ of $X \times X$ (product topological space) and of the quotient map and its fibres (which are the equivalence classes), rather than those of X itself.

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Check the following:

a) $X/\sim$ is T_1 iff every equivalence class is closed when considered as a subset of X.
b) If $X/\sim$ is Hausdorff $(= T_2)$, then $\sim$ is a closed subset of $X \times X$.
c) If the quotient map is open, then second countability of X passes on to $X/\sim$.
d) If the quotient map is open and $\sim$ is a closed subset of $X \times X$, then $X/\sim$ is Hausdorff. Local compactness of X passes on to $X/\sim$. In this case if X is moreover second countable, then it follows that $X/\sim$ is further paracompact, regular, hence T_3, normal, hence T_4 and hence metrizable.
e) If the quotient map is closed and its fibres (the equivalence classes) are compact as subsets of X, then the properties of local-compactness, Hausdorffness and second countability pass over to the quotient. It follows that if all these properties are possessed by X, then not only is X paracompact, regular, T_3, normal, T_4, metrizable, but all these also hold for the quotient $X/\sim$.

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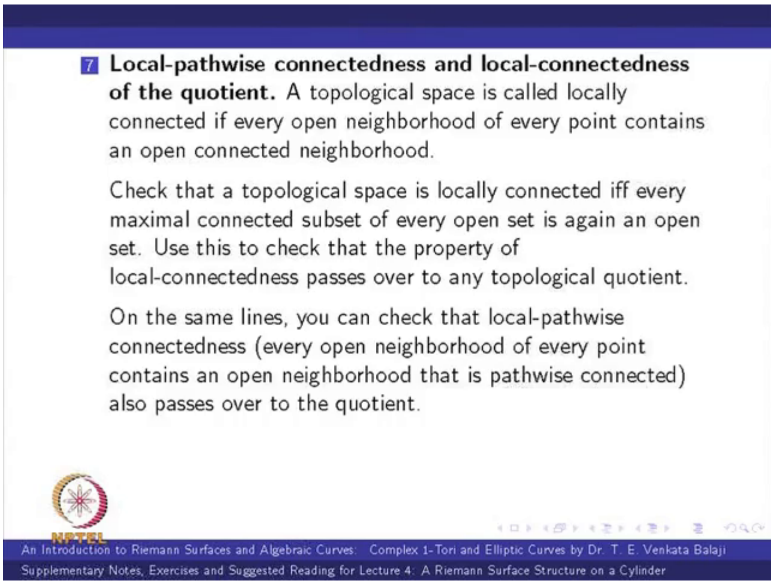

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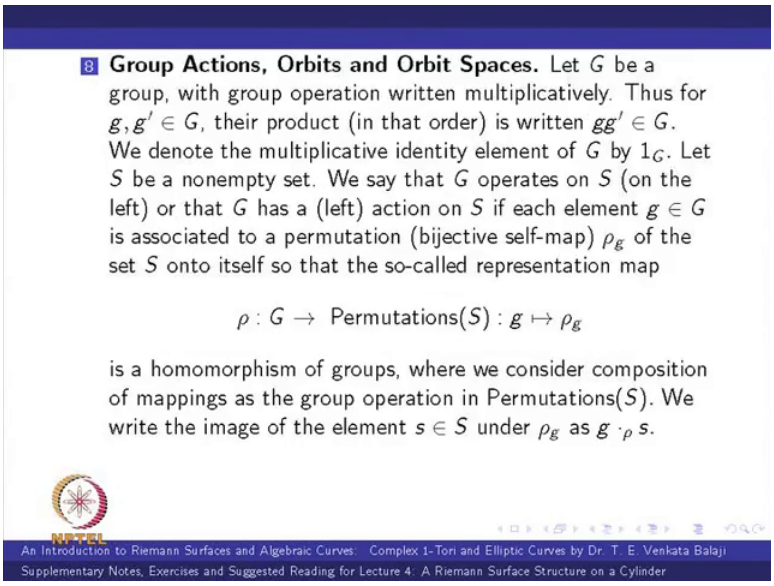

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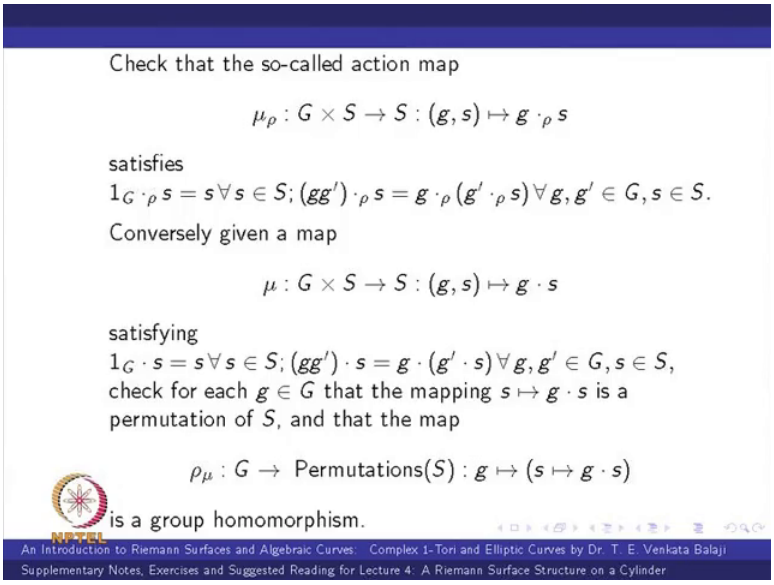

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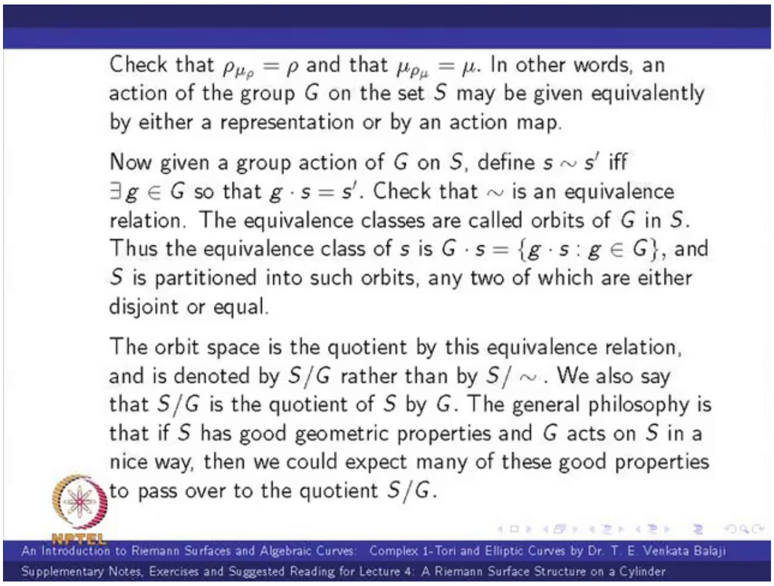

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Moebius transformations, Automorphisms of $\mathbb{P}^1$ and the Projective Special Linear Group. Recall that a Moebius (or) Bilinear (or) Linear Fractional transformation is a map of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where z is a complex variable, and a, b, c, d are complex numbers satifying $ad - bc \neq 0$.

Let Moeb($\mathbb{C}$) denote the set of all Moebius transformations. Note that one has to be careful while writing down elements of this set: the Moebius transformation $z \mapsto \frac{az+b}{cz+d}$ is the same as the one defined by $z \mapsto \frac{a\lambda z+b\lambda}{c\lambda z+d\lambda}$ for any nonzero complex number λ.

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We want to indicate the relationship between Moebius transformations and holomorphic isomorphisms of the Riemann sphere $\mathbb{P}^1$ onto itself (also called holomorphic automorphisms of the Riemann sphere). The set of all such automorphisms is denoted $\mathrm{AUT}_{hol}(\mathbb{P}^1)$ and one can check that it is a group under the operation of composition of mappings.

We further want to indicate the relationship between Moebius transformations and the elements of the so-called Projective Special Linear Group of order 2 with complex entries, denoted PSL($2, \mathbb{C}$). This is the quotient of the so-called Special Linear Group of order 2 with complex entries, denoted SL($2, \mathbb{C}$), namely the group of determinant 1 complex 2-by-2 matrices under matrix multiplication. The quotient is to be taken by the (normal) subgroup $\{\pm I_2\}$ where I_2 is the 2-by-2 identity matrix.

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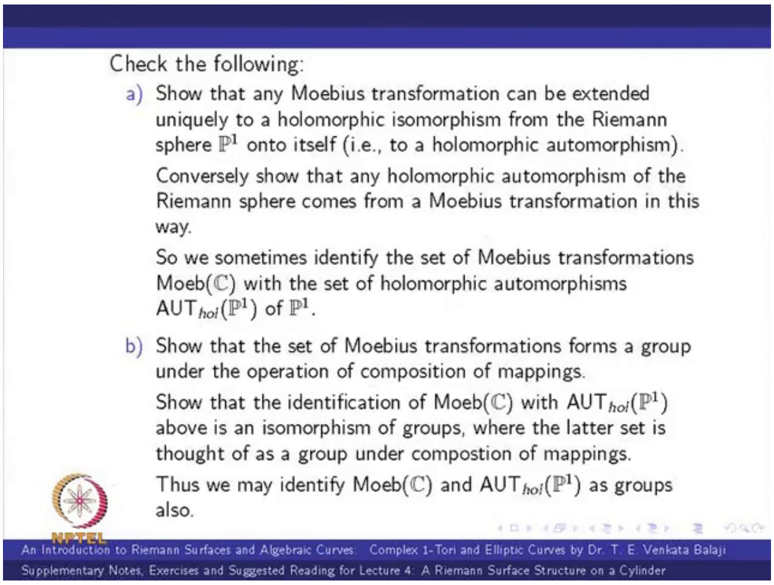

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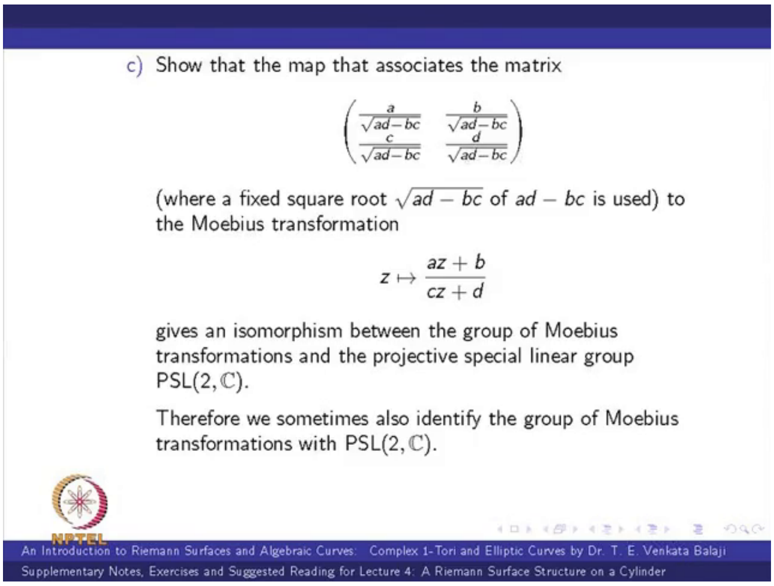

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d) Show that the set $\mathrm{AUT}_{hol}(\mathbb{C})$ of holomorphic isomorphisms of the complex plane onto itself (i.e., holomorphic automorphisms of the complex plane) is given by Moebius transformations of the form $z \mapsto az + b$.

Show that this subset is a subgroup of $\mathrm{AUT}_{hol}(\mathbb{P}^1)$ which under the identification of the latter with $\mathrm{PSL}(2, \mathbb{C})$ corresponds to the subgroup represented by upper-triangular 2-by-2 matrices (those for which the entry below the diagonal is zero).

e) Show that translation by a complex number $z_0 \neq 0$, namely the map $T_{z_0} : z \mapsto z + z_0$, is a Moebius transformation. Write down the inverse transformation. Find representatives in $\mathrm{SL}(2, \mathbb{C})$ for the elements these transformations correspond to in $\mathrm{PSL}(2, \mathbb{C})$. Show that the group of complex numbers under addition is naturally identified with an abelian (i.e., commutative) subgroup of $\mathrm{PSL}(2, \mathbb{C})$ under the map $z_0 \mapsto T_{z_0}$.

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10 **The Group of Translations as a $\mathbb{Z}$-module.** A commutative (abelian) group M with operation written as addition "+" is called a module over the ring of integers $\mathbb{Z}$, or a $\mathbb{Z}$-module for short, if there is a map

$$\mathbb{Z} \times M \to M : (n, x) \mapsto nx$$

satisfying:

- $1x = x$;
- $(n + m)x = nx + mx$;
- $n(x + y) = nx + ny$;
- $(nm)x = n(mx)$;

for any $n, m \in \mathbb{Z}$ and any $x, y \in M$.

For example, multiplication by an integer on the left makes $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ into $\mathbb{Z}$-modules.

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Show that the subgroup G_{z_0} of Moeb($\mathbb{C}$) generated by a single translation T_{z_0} is a $\mathbb{Z}$-module with nT_{z_0} defined as T_{nz_0}.

Thus nT_{z_0} is the same as T_{z_0} composed with itself n times for any positive integer n and is the same as $T_{-z_0} = T_{z_0}^{-1}$ composed with itself $-n$ times for any negative integer n.

Show further that the map

$$G_{z_0} \to \mathbb{Z} : nT_{z_0} \mapsto n$$

is bijective and respects the $\mathbb{Z}$-module structure thereby giving an isomorphism of $\mathbb{Z}$-modules.

We thus say that the subgroup generated by a translation is isomorphic to $\mathbb{Z}$ (as $\mathbb{Z}$-modules).

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11. **Cylinder as a Quotient of the Complex Plane by the Subgroup generated by a Translation.** Show that the group of Moebius transformations acts on the complex plane in the natural way: if g is a Moebius transformation and z is a complex number, then $g \cdot z$ is just $g(z)$.

Show that the quotient of the complex plane by the subgroup of Moebius transformations generated by a single translation is a Riemann surface which as a topological space is homeomorphic to a cylinder.

The main technical point of this exercise is to verify that the quotient is Hausdorff, locally-compact and second countable; the Hausdorffness being concluded by checking that the equivalence relation defined by the group action is a closed subset of the product $\mathbb{C} \times \mathbb{C}$ and that the quotient map is an open map.

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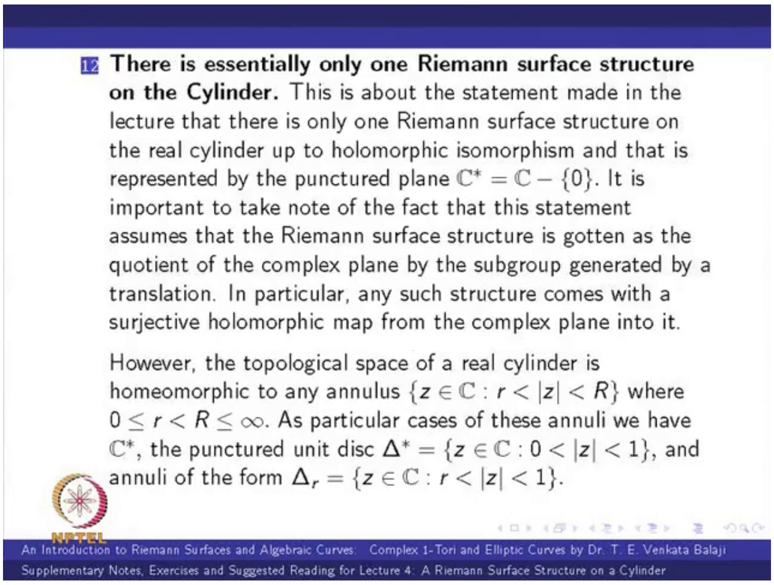

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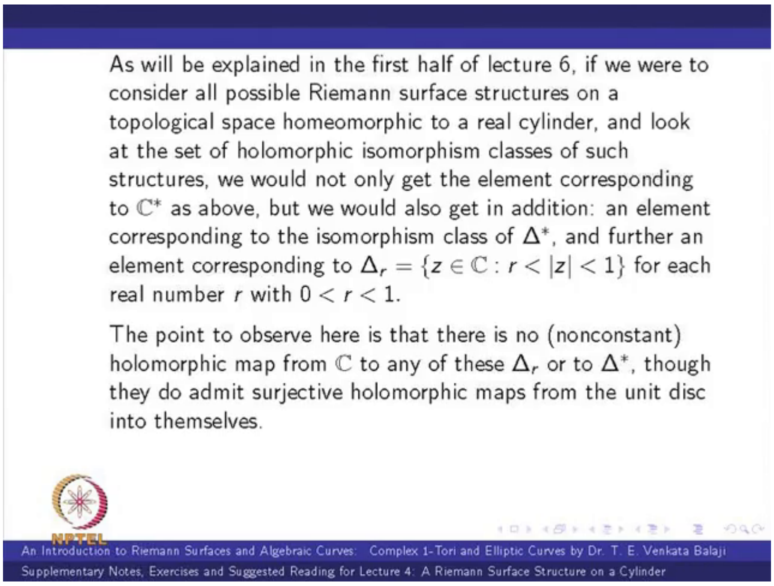

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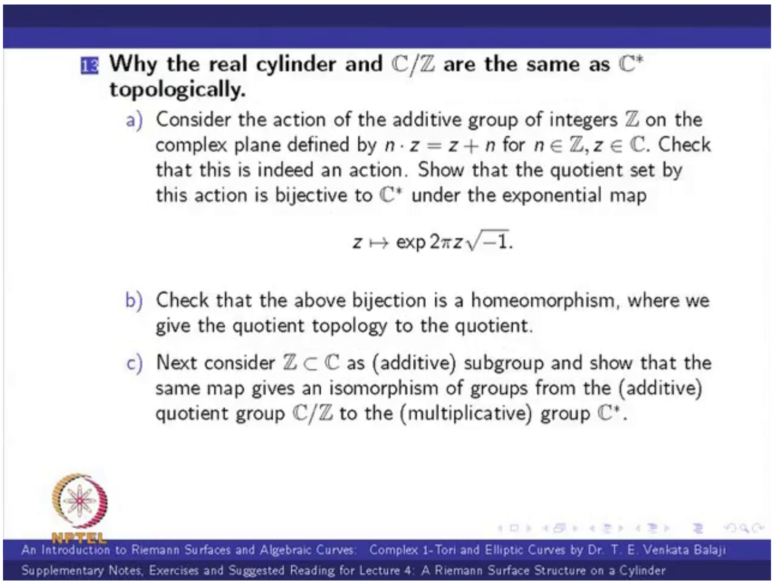

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