And Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 – dimensional Tori and Elliptic Curves
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Lecture – 38
The J - Invariant of a Complex Torus (or) of an Algebraic Elliptic Curve

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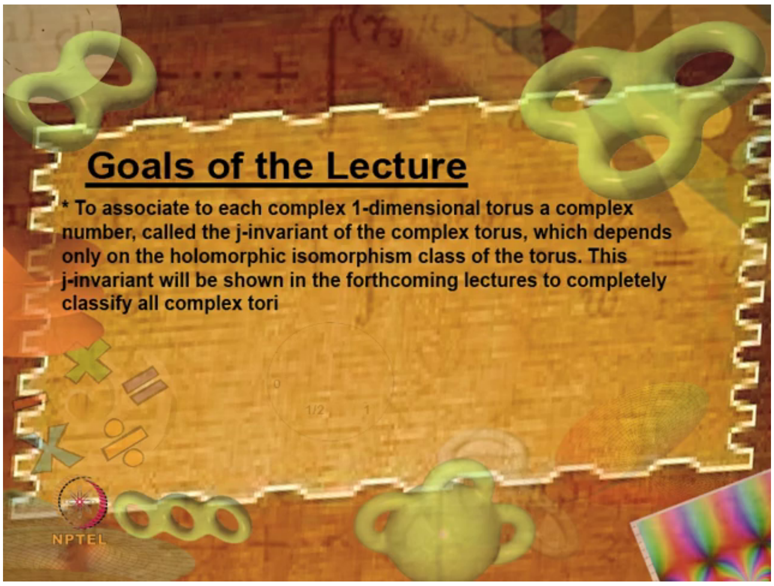

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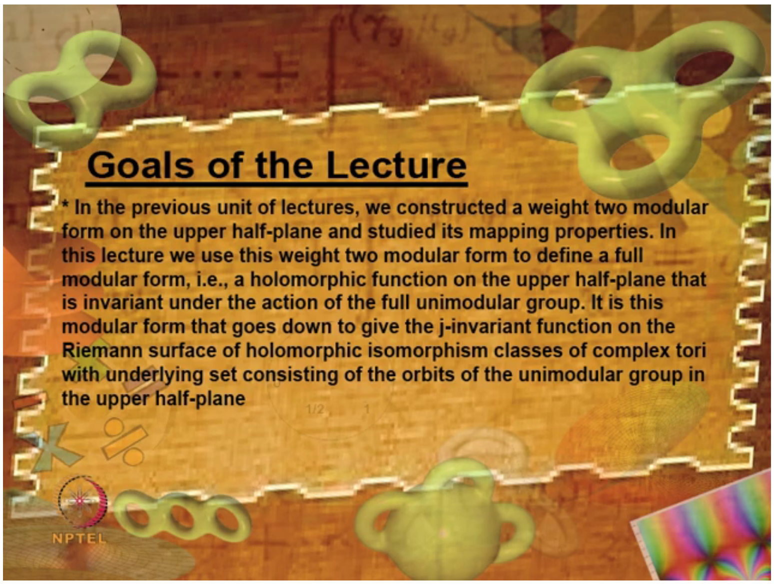

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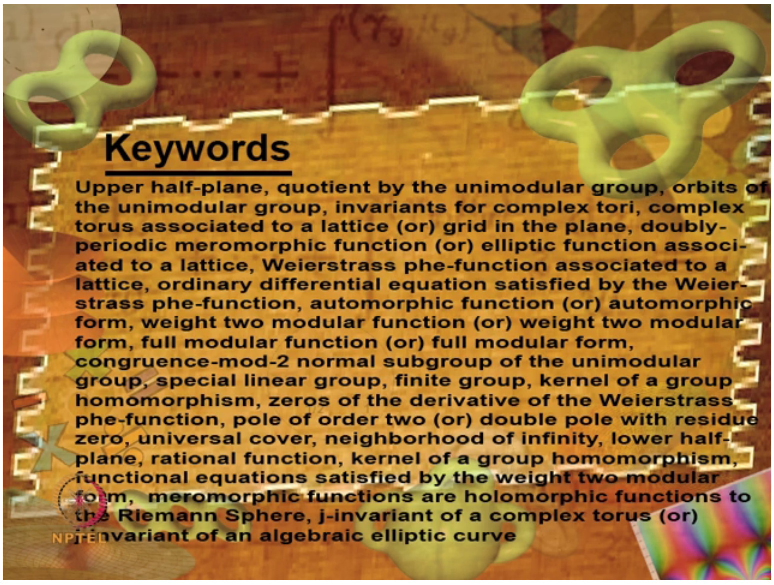

So, let me try to again give a you know brief revision of what you have been doing because since the things we are trying to prove are deep results and you cannot prove them in a single lecture. What we are trying to do is we are trying to take the upper half plane modulo the unimodular group and we are trying to show that the natural Riemann surface structure on that is nothing but the complex plane with the natural Riemann surface structure on it.

So, this is a rather deep result and, so it will help if I at this point of time try to give a summary of what we have done and what we are going to do.

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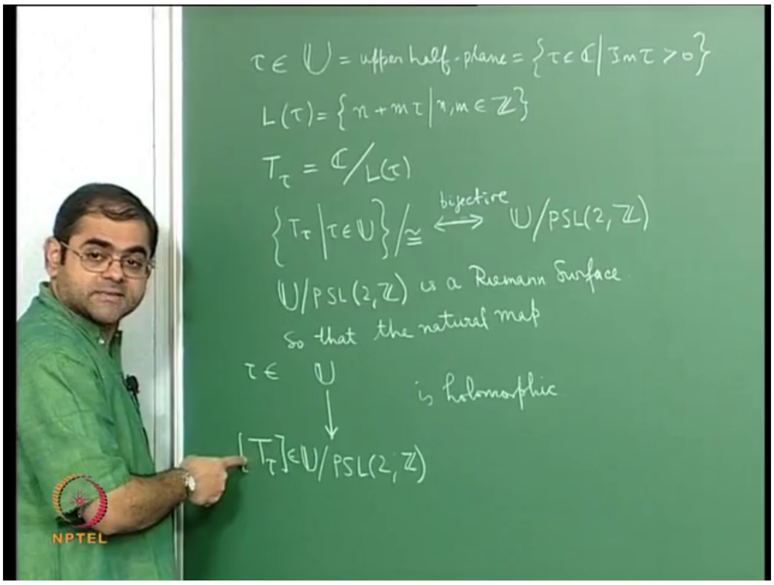

So, you see if you recall we had U the upper half plane, namely the set of all tau in C such that in mathematic part of tau it is positive. And then if you start with tau in upper half plane then you have the corresponding lattice L tau which is generated by which is which consists of integer multiples of I mean integer linear combinations of 1 and tau that is it is the Z sub module of C generated by 1 and tau.

So, it is of the form n plus m tau where n and m are integers and then you know that we get the torus associated to this lattice T sub tau this is just complex numbers modulo L tau and this is you know that this is the quotient of C by L tau. And you know that this has universal cover C and the covering map is just the natural map from C 2 this quotient and you know of course, that the fundamental group of this can be identified with L tau and it can also be thought of as the Mobius transformations that I have given by translations by elements of L tau.

And then you also know that you see the if you look at the various tori T sub as tau change tau varies over U if you take the isomorphism classes holomorphic isomorphic classes then this is bijective to U the upper half plane modulo PSL 2 Z. Namely, tau the tori corresponding to tau 1 and tau 2 are holomorphically isomorphic which means isomorphic as Riemann surfaces if and only if tau 1 and tau 2 are in the same orbit for the action of the unimodular group PSL 2 Z on the upper half plane. And then of course, we have also seen that U mod PSL 2 Z is a Riemann surface is naturally is a Riemann surface it is naturally a Riemann surface, so that the natural map the natural map U 2 which is just the quotient map is holomorphic.

So, this is a natural quotient map which sense any tau in the upper half plane to the isomorphism class of the complex torus defined by tau which the isomorphism class I denote it with a square bracket. And of course, we are trying to show that this Riemann surface U mod PSL 2 Z is isomorphic to C. So, we have to find a holomorphic map which is defined on this and which is whose image is all of the complex numbers and which is also injective. So, you know you have to find the bijective holomorphic map from this to C and that will give you automatically, that will be automatically a bijective biholomorphic map.

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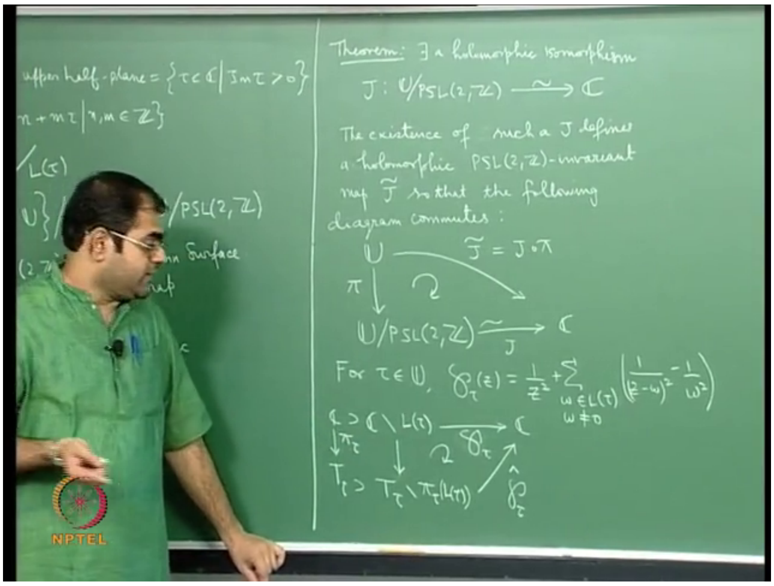

So, the claim is claim is, so let me write this here. So, what we are trying to show is well. So, can save space and draw the line here the claim or theorem is that there exist a holomorphic isomorphism from U mod PSL 2 Z to the complex numbers this is the theorem this is what we are trying to show. So, theorem is you want to show that this Riemann surface is none other than C. And so, what does it mean? It means, given such a see the existence of such a J defines a map holomorphic PSL 2 Z invariant map J tilde so that the following diagrams commutes.

So, you see you have this quotient from U to U mod PSL 2 Z, U mod PSL 2 Z is just the set of orbits of PSL 2 Z in U and of course, each orbit corresponds to the a holomorphic isomorphism class of tori alright. And see suppose you have this J here which is a holomorphic isomorphism on to C then what happens is that you get this J tilde above which is just this followed by this. So, you see well if I call this, if you want to call this as projection pi then J tilde is just first apply pi then apply J that is what it means to say that this diagram completes and that is why I put this circular arrow here.

So, the point is we will have to first find a map J tilde we have to find a map J tilde and this J tilde has to be constant on the orbits of PSL 2 Z. So, that it goes down to define the map J. So, in other words what you are trying to do is we are trying to find a function on the a holomorphic function on the upper half plane which is whose image is all of C whose image is all of C and which is invariant for the action of PSL 2 Z. And such I have told you that you know functions which are invariant under the action of the unimodular group or a subgroup of the unimodular group are called modular functions in general if they are holomorphic or meromorphic functions which are invariant under a certain group of Mobius transformations are called automorphic functions and those automorphic functions which are invariant under the unimodular group or a sub group of the unimodular group are called a modular functions.

So, the moral of the story is that if you want to get hold of J you will have to construct a modular function on U. So, you have to construct a holomorphic function on U which is invariant for the action of the group PSL 2 unimodular group PSL 2 Z. So, you know this, so in fact, what is this J it associates to every orbit of PSL 2 Z a unique complex number. So, in other words, but you know orbits of PSL 2 Z are precisely isomorphism classes of a complex tori.

So, you are trying to find a complex number which associated to any complex torus such that the complex number depends only on the isomorphism class of that complex torus. So, whenever you find a quantity which is dependent only on the isomorphism class that that quantity is called an invariant because it depends only on the isomorphism class it is called an invariant. So, what you are trying to do is you are trying to find the invariants for complex tori.

So, constructing this modular function J get trying to get hold of a modular function J tilde is actually it actually translates to finding invariants for complex tori. So, you see if I start with a, if I start with the point tau in of the upper half plane J tilde of tau has to be something that depends only on the isomorphism class of the complex storage defined by tau. So, in other words it has to depend on the geometry of T tau as tau changes.

So, that is the reason one hopes that you know one looks at T tau and tries to you know construct some function of tau which will behave in this way. So, that lead to us try to look at functions the possible holomorphic functions on T tau, but then you know you know that T tau is compact therefore, there are no global holomorphic functions. So, the only thing that you can expect, are meromorphic functions and then you we argued that the simplest such meromorphic functions is given by the Weierstrass p function.

So, for tau in the upper half plane we have the meromorphic function the Weierstrass phi function phi tau of Z we defined the and this is well it is given by a series 1 by Z square plus summation over omega in the lattice omega not equal to 0 of 1 by Z minus omega the whole square minus 1 by omega square we defined this Weierstrass phi function and this function, this phi function, so how was this phi function it was. So, from the complex from the complex plane minus minus the lattice 2 the complex plane this phi function this phi function turned out to be a holomorphic function with a double pole at each point of the lattice with sum of residues 0. And the singular part at a point omega of the lattice consisted of only 1 by Z minus omega the whole square alright.

And then this function, you see this function goes down to the torus minus you see the image of this whole lattice in the torus will be a single point on the torus. So, let me call that as pi sub tau of L of tau that is the single point and you get you get a meromorphic function on the torus minus of single point and these are simplest meromorphic function that you can, maybe I can call this as phi tau hat and well this is the. So, what is a, this map is just the natural projection. So, this is a sitting inside C and this is sitting inside T tau and this is the natural map this is the natural quotient map pi sub tau which has which is which is a universal cover and.

So, the phi function is the simplest kind of function you can get on the torus and then of course, there is another way of writing it since every point of that the lattice is a pole I can extend this function to have the value infinity there and I can define, but then I have to change the target to the Riemann’s sphere C n in infinity I have to add the point at infinity.

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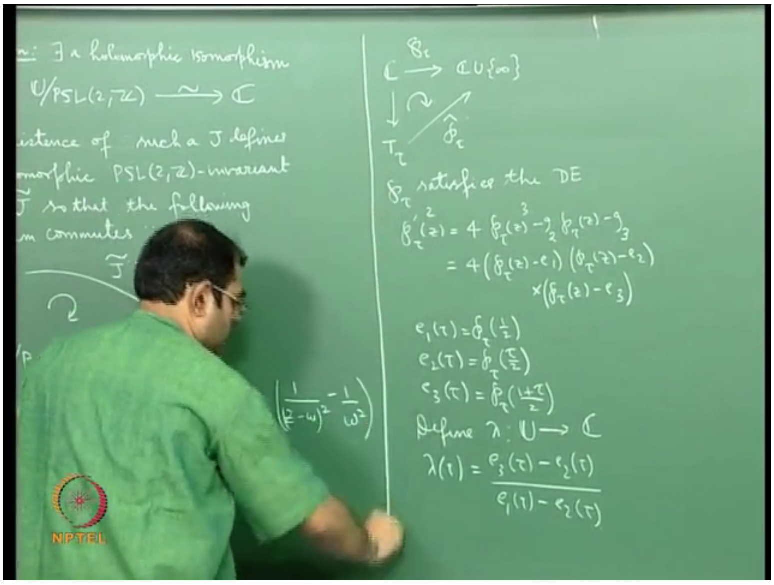

So, another way of writing this is well I can also write it as you know I can also write it like this C 2 C on a infinity this is the if this is a Riemann sphere which the natural structure of Riemann surface and then I can write this as phi sub tau here and there is a holomorphic function and that goes down to the torus and that gives you well phi sub tau hat. So, I can also write it like this.

So, the point is either if you write, if you want to write the meromorphic function as a holomorphic function you throughout the poles or if you want to write it as a holomorphic function you write it as holomorphic function into the Riemann's sphere you include the point at a infinity. So, in other words this defining the value of the function to be infinity at a pole continues to make it meromorphic map continues to make it a holomorphic map in the neighborhood of that pole provided you take the complex structure the natural Riemann surface structure on the Riemann's sphere.

So, well, then what we did was. So, the aim is of course, you know to cook up for tau as tau varies a certain complex number. So, it is a variation of phi sub tau as tau varies that is a important and, what we did was well we found that this Weierstrass phi function satisfies a natural differential equation. So, we have, so maybe I will. So, let me write it here phi tau satisfies the differential equation well it was I guess phi prime tau square of Z is equal to 4 e tau of Z q minus g 2 p tau of Z minus g 3 where g 2 and g 3 where certain functions of tau they were certain summations over this lattice. And then we factorize this as 4 times we factorize the polynomial on the right as into 3 linear factors we wrote it as phi tau of e Z minus e 1 into phi tau of e Z minus e 2 into well into phi tau of e Z minus e 3.

We factorize it like this and then we and then we found the 0s of phi prime and the 0s turned out to be the 0s in the fundamental parallelogram define by tau namely the parallelogram that consist of vertices 0 1 1 plus tau and tau. We found that essentially we have three 0s, three distinct 0s and those 0s were at half tau by 2 and 1 plus tau by 2 and we set e 1 of tau to be the value of p tau at half e 2 of tau to be the value of phi tau at tau by 2 and e 3 of tau the value of phi tau at 1 plus tau by 2 this is what we did and. So, then we constructed a partially modular function a modular function lambda of weight 2 namely it is not modular for the fill modular group, but it is modular only for the congruence mod 2 sub group.

So, we defined lambda, lambda the following so the lambda was defined from the upper half plane into the context numbers. So, lambda of tau was just e 3 tau minus e 2 of tau by e 1 tau minus e 2 of tau and well maybe I can draw a line here, I can draw one more here and we were studying the we notice that lambda is a holomorphic function and it never takes the values 0 and 1.

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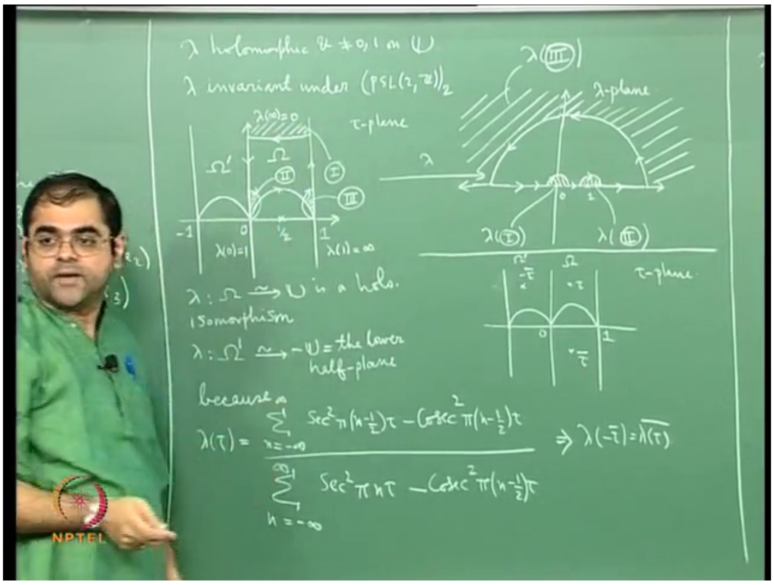

So, lambda holomorphic and not equal to 0 1 on the upper half plane and in fact, we proved that lambda is invariant under the congruence mod 2 subgroup of PSL 2 Z namely PSL 2 Z subscript 2, this consist of all those transformations in PSL 2 Z which have representative matrices which when you read the coefficients mod 2 you get identity matrix 2 by 2 identity matrix.

So, of course, our aim is this is only a normal sub group and we want in fact, what we are looking for here is a function which is in varying around to the whole modular group. So, how to extend this function so for that we had to study the mapping properties of lambda alright and what were the mapping properties in fact, we proved that, let me draw let me draw a small diagram here. So, this is e. So, this is the complex, this is the tau plane and this is real axis this is the imaginary axis this is the origin and this is the line passing vertical line passing through 1 this is the vertical line passing through minus 1. So, this is real part of tau equal to 1, this is real part of tau equal to minus 1, this is real part of tau equal to 0.

And then what we did was we drew we drew a region like this we took a region like this omega alright and this region was the interior of it is a interior which is bounded by this half line this half line and this semi circle which is centered at half and radius half and of course, what we proved was lambda, lambda from omega to U is a holomorphic isomorphism, isomorphism. So, lambda maps this region omega on to U and let me recollect a few more facts in fact, what we did was well you know if you draw a segment like this if you draw a line segment like this. And then if you draw a circle like a circle like this with center on you know the imaginary axis and well another symmetric circle here semicircle rather, and well if you took you know this contour namely the contour that I start along this line then I come down like this then I go by this arc then I go by this arc then I go by this arc of the circle and I go back.

Then how does lambda map this on to the lambda plane. So, what we proved was that you see well if I take this as lambda then in the lambda plane what happens. So, this is the lambda plane what happened was well the fact is that lambda at infinity takes the value 0. So, and lambda at 0 takes the value 1 lambda 0 is 1. So, let me write it somewhere here and lambda at 1 takes the value infinity, lambda 1 is infinity the point at infinity and see if you take the image of this contour what you would get is you see. So, the there is a point 0 and then there is a point 1. So, you will end up with a small semicircular region here semicircular arc here nearly semicircular arc and then, you will get the semicircular arc then you will get a segment on the real axis then you will get another semicircular arc centered at 1 and then it goes on like this at up to some point and then you get a large semicircular arc like this.

So, let me draw this properly. So, the image of the maybe I will draw this a little bigger. So, that I can do some shading if I want it want to do. So, I will draw it a little bit bigger. So, you see it is like this. So, the image of that contour the image of this contour here this contour is exactly that roughly and this shaded region which you can think of as a neighborhood of infinity of the point at infinity as you keep increasing the height of this segment that is mapped on to a neighborhood of 0 so in fact, So, if I call this as region as 1 then this the interior of this semicircular roughly is the image of 1. So, this is lambda of the region 1.

And if I take this shaded region which is it is neighborhood of 0 you know lambda at 0 is 1. So, this goes to this shaded region. So, this is lambda of 2, where 2 is this shaded region here. So, you see I the sheet the region is the region inside omega. So, it is actually the intersection of this semicircle with omega alright. And then there is another there is another region here which if you call it as 3 then this is a neighborhood of one, but lambda goes infinity at 1. So, this is precisely what is going to be the exterior of this. So, you see, I will have to shade.

So, this is this is the lambda of 3. So, this is how the mapping lambda behaves and in fact, therefore, as you as you keep increasing the height of these this segment these shaded regions becomes smaller and smaller and therefore, these two semicircles semicircular regions shrink completely and this circle becomes large enough to cover the whole upper half plane. And in this way lambda takes all values in the upper half plane and in fact, we can see that it takes each value once because the winding number of this curve for any point in the upper half plane if I take this segment large enough is 1.

So, this tells you that lambda maps omega 1 1 on to you and it is a holomorphic therefore, it is also holomorphic. So, it is a holomorphic isomorphism, but in fact, what you also get is that, you get lambda see if you also take the; well if you take the reflection of omega by the measuring axis, you get another you get another region that is omega prime. So, let me do the following thing let me call this as. So, this was called 2 right. So, maybe I will this as 2 here let me label it here.

So, that is also this region omega prime. So, that is this region omega prime which is just the reflection of the region omega by the imaginary axis and the fact is lambda takes omega prime isomorphically on to minus U and by minus U I mean the lower half plane, so I maybe I will call it as let it be as it is the lower half plane half plane. And you see one way to understand this is as follows if you have not seen it is pretty easy you make use of the fact becomes lambda of tau if you remember we you know to get these kind of picture we had to make estimates of see in fact, how did we get lambda of infinity is 0, lambda of 0 is 1, lambda of 1 is infinity we got all these things by trying to uh look at you know a kind of Fourier series of lambda all right and in fact, trying to well I think I accidently erased this, let me write it properly.

So in fact, we expanded these two terms in terms of you know sins and cosines and for that we made you use of these definition. And the definitions of phi function and in fact, what we got is well if you remember what we got was a following it was, it was sigma n equal to minus infinity to infinity I think it was secant square pi secant squared pi n minus half tau minus cosecant squared pi n minus half tau divided by well, summation another summation of the same type sigma from minus infinity to infinity this was I think secant square pi secant squared by n tau minus cosecant square the same term as in the numerator. So, this was the expression we got for lambda of tau and the way you have to understand it is that the series in the numerator and in the denominator they I mean they converge uniformly on compact subsets and in fact, so this was the fundamental importance to study the behavior of lambda at infinity.

And now you can see that you know if you have well if you have see, if you take suppose this is the tau plane I am drawing let me draw another diagram and you know well this is 1 if I take. So, this is omega if I take a tau in omega then you see its reflection will be tau bar reflection about the real axis and the reflection of that about the origin will be the reflection of tau about the imaginary axis. So, this will be minus tau bar and that is what is going to lie in the minus tau bar that is what is going to lie in the other region omega prime and well. So, I need to draw a line like this so that you do not confuse these two pictures and therefore, you see you see if I calculate lambda of minus. If I replace tau by minus tau bar here first of all if I replace tau by minus tau bar you will see that replacing tau by minus 2 does not do any harm because there is these are all squares of the corresponding trigonometric functions and then replacing tau by bar you can pull the bar out because all these functions have real quotients in their series expansions.

So, the moral of the story is this formula actually tells you that lambda of minus tau bar is actually lambda tau whole bar. So, therefore, it is clear that you know if lambda takes omega holomorphically isomorphically on to U, then it is going to take omega prime holomorphically isomorphically onto the lower half plane minus U. So, you get this from that. And then we also had a good boundary behavior of this of lambda.

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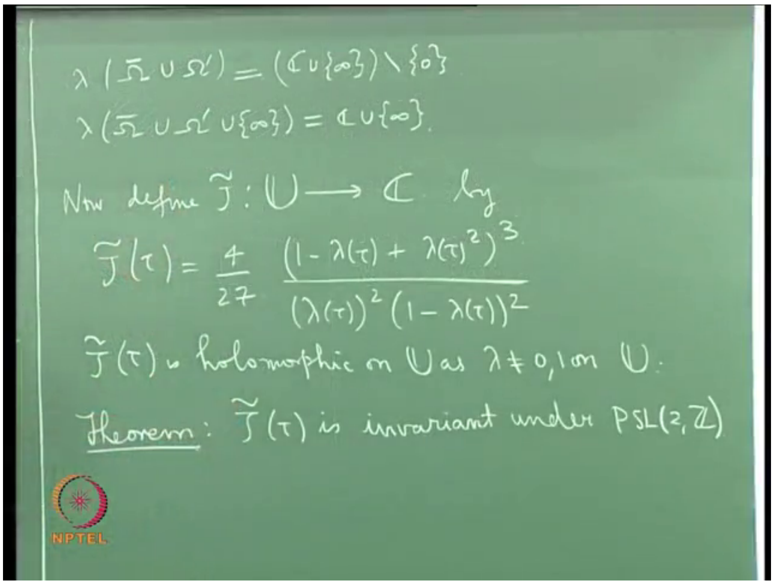

In fact, what we found was that lambda of lambda from if you include the boundary, if you include the boundary then you the results is well I will give the whole complex plane except for.

So, let me write it like this lambda of omega bar union omega prime that turns out to be a mapped to the whole Riemann's sphere minus 0 this will be C union infinity minus the origin this is what you will have because you see the point at infinity as you move from the point at infinity to 0 lambda values move from 0 to 1. And then as you move from 0 to 1 lambda values move from 1 to infinity. So, you see you trace from 0 to 1 to infinity the point at infinity and then you come back and then come back all the way back to 0 as you move here from 1 to infinity then lambda values move back from infinity to 0.

So, what happens is the whole you get the whole real line the only thing that you do not get is the, I get the value to get the value 0 I need the point at infinity, but otherwise I get every other value alright. So, and in fact, lambda of this so in fact, let me put equal to here and then lambda of well if I include the point at infinity, then I will get the whole I get C union of infinity and lambda is the function lambda is continuous and monotonic on the boundary of omega.

So, well now all the whole point of doing all this was to use lambda and it is mapping properties to cook up a function on the upper half plane which is modular that is which is invariant for the whole unimodular group. Namely, a functions such as J tilde, I will define now I will tell you have to define J tilde using this lambda that was that is, that was the purpose of getting hold of this lambda and this the mapping properties of lambda will also be used later on as you will see.

So, you see now define J tilde from the upper half plane took the complex numbers by well J tilde of tau is 4 by 27. So, let me write it down properly its 1 minus into 1 minus lambda of tau plus lambda of tau square whole cube divided by lambda of tau the whole square into 1 minus lambda of tau the whole square. So, it is a rather crazy looking I should say rather simple looking rational function of lambda of tau. It is a polynomial in lambda of tau in the numerator of degree 6 and in the denominator it is a polynomial of degree 4, it is a rational function of lambda of tau.

And the fact that this is doing the job also is that that we are able to get a rational function here is also something that makes you to believe that there is some algebra there is some algebraic geometry going on here. And in fact, as I told you all these complex tori are actually algebraic curves they are given by 0s of the single polynomial in to variables and they are called elliptic curves these are cubic curves and the key to that is this differential equation we will come back to that later and that is why this whole topic is usually called as a moduli of elliptic curves.

That is also the reason why functions on a torus are called as elliptic functions, a functions on a torus are just functions on the C which are invariant under the lattice and the functions invariant under a lattice are basically doubly periodic functions they are periodic with respect to 1 as well as tau these and they are called elliptic functions. So, the fact is actually there are elliptic curves these are actually elliptic curves and we will see that in later lecture. But the point is that this, the fact is that you are getting an algebraic function namely a quotient of polynomials in lambda is suggestive of the fact that there is some algebra going on here.

So, this is. So, we define J tilde in this form mind you J tilde is very well defined because on the upper half plane lambda never takes the value 0 or 1. So, this denominator is never going to vanish and you have a quotient of holomorphic functions the denominator non vanishing, therefore, this is holomorphic on the upper half plane. So, J tilde tau is holomorphic on U as lambda is not equal to 0 1, 0 or 1 on the upper half plane. So, this function has no similarities its holomorphic.

And the first thing I want to say is that well of course, the there are 2 claims the first claim is that I mean the most important claim to begin with is that this is the function we are looking for namely this function is invariant under the whole modular group unimodular group. So, let me write that down theorem J tilde of tau is invariant under PSL 2 Z.

So, this is of course, this was the, so what I want meant to tell you is that all the story. So, far was towards this end to get hold of this function which is you know invariant under the under the whole unimodular group. Now, how do you prove that this is invariant under the whole unimodular group? One does it again cleverly because where because we can use the so called functional equations of lambda which express how lambda behaves under certain transformations.

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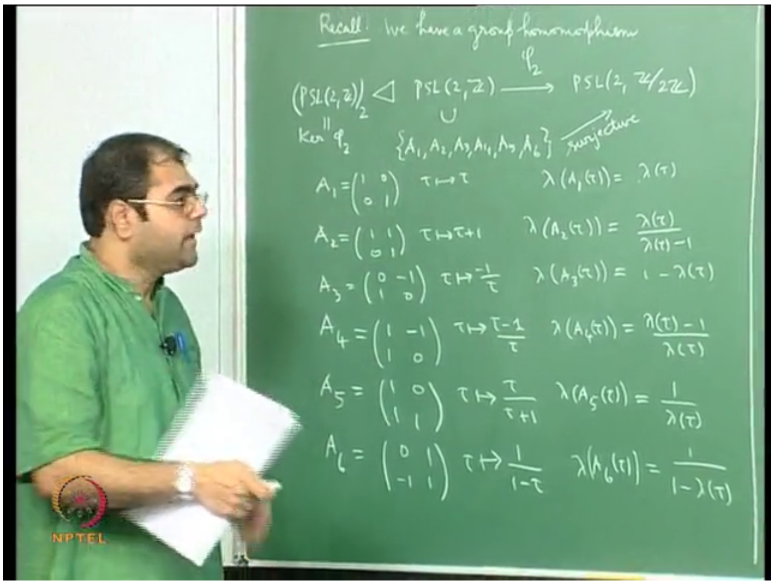

So, let me recall that recall we have a group homomorphism, we have a group homomorphism from PSL 2 Z to PSL 2 Z mod 2 Z which we call phi 2. This is just take a matrix, take a matrix representative after all its represented by a matrix which is determinant 1 and with the integer increase and you read all the entries a mod 2. Namely with values in this smallest field Z mod 2 Z which consist of only you know 0 and 1. And well this phi 2 is a group homomorphism and the kernel of phi 2 is precisely all those all those elements of PSL 2 Z which consist of the congruence mod 2 sub group because it is all those elements which when red mod 2 are give the identity matrix.

So, here you have a normal sub group which is given by PSL 2 Z sub 2 which is actually the kernel of the homomorphism phi 2 and then if you remember that we wrote down 6 specific transformations. So, let me go back to this is from one of the earlier lectures let me write down those transformations, here they are.

So, you know, we take these, so we take these transformations here namely given by A 1, A 2, A 3, A 4, A 5, A 6 and this is certain set of transformations here, and A 1 is A 1 corresponds to the this is the identity matrix this is 1 0 0 1 and this corresponds to the Mobius transformations tau going to tau A 2 is well, is the matrix 1 1 0 1 and that corresponds the Mobius transformation tau going to tau plus 1 translation by 1. A 3 is the PSL 2 Z element given by 0 minus 1 1 0 and this is the transformation Mobius transformation tau going to minus 1 by tau which is its own inverse. Then we have A 4 which is 1 minus 1 1 0 which is tau going to you know tau minus 1 by tau which is also 1 minus 1 by tau alright. And we have A 5 which is 1 0 1 1 that corresponds to tau going to tau by tau plus 1 and A 6 is the element 0 1 minus 1 1 which corresponds to tau going to 1 by 1 minus tau.

So, we wrote down these 6 Mobius transformations and their corresponding matrices and we found that phi 2 of if you take the images of the 6 here, that gives you all the elements in here phi of, let me write that here. So, phi this is surjective, if you take the map phi 2 and restrict it to this set and then it is surjective. So, this in other words if you read all these matrices mod 2 then you get all the 6 matrices this consists of only 6 matrices you get all the 6 matrices. So, we found that lambda satisfies certain functional equations you know this was we did this trying to understand what happens if a lambda what is a effect of a general element of PSL 2 Z on lambda and we found that in order to find the effect of a general element of PSL 2 Z on lambda, it is enough to look at only the effect of these guys on lambda. And what is the effect of these guys on lambda, you get the 6 corresponding functional equations satisfied by lambda and what are those equations well let me write that down. So, the corresponding equations of lambda are lambda of A 2 of tau is just lambda tau.

So, let me write it of course, let me write let me first write A 1 of course, I am not going to get anything because A 1 this is the identity lambda A 1 tau is lambda tau and then lambda of A 2 of tau is well we got lambda tau by lambda tau minus 1. Then lambda of A 3 of tau which is lambda of minus 1 by tau and that turns out to be 1 minus lambda of tau and lambda of A 4 of tau is well 1 minus 1 by, its lambda tau minus 1 by lambda tau and lambda of A 5 of tau is that that is lambda of tau by tau plus 1 that is its 1 by lambda tau and lambda of A 6 of tau is well lambda of A 6 of tau is 1 by 1 minus lambda tau. So, we got these functional equations for lambda.

Now, now the point is that I quickly tell you how it is so easy now to verify that J tilde is invariant under PSL 2 Z.

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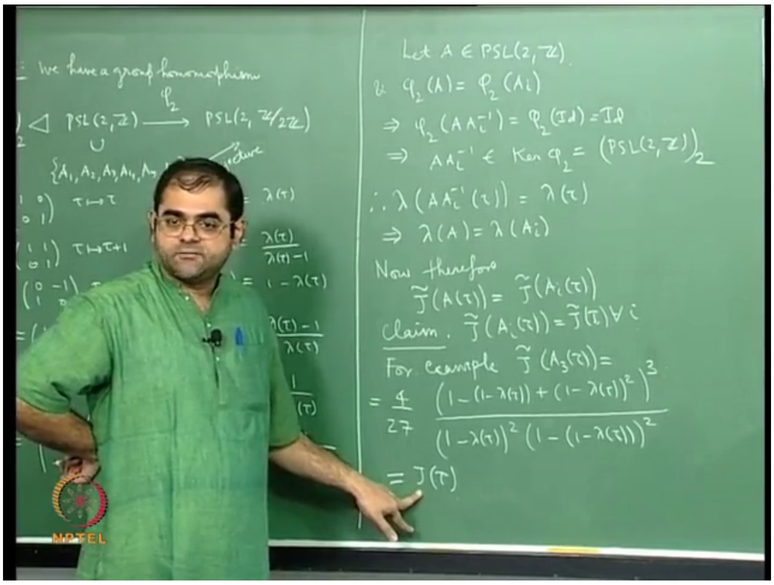

So, well you know let a be an element of PSL 2 Z, let a be an element of PSL 2 Z and look at phi 2 of A. Phi 2 of A has to be here and that has to be phi 2 of 1 of this guys because the image of these 6, give the 6 distinct images the 6 distinct 6 distinct images as you see which are 6 distinct elements here. So, phi 2 of A is let us say phi 2 of A i for unique i all right.

So, what does it mean, it means see phi, you know, this means because phi 2 is the homomorphism this means that phi 2 of like say A i inverse is phi 2 of identity which is identity and this tells you that A A i inverse is in the kernel of phi 2 which is the congruence mod 2 sub group. But then you know lambda is lambda is invariant under the congruence mod 2 sub group therefore, you know lambda of A A i inverse tau has to be lambda of tau this has to happen this is because lambda is invariant under a elements of the congruence mod 2 sub group which is the kernel of homomorphism phi.

Now, so this tells you that you know if I call this. So, this will actually tell you that lambda of A is the same as lambda of A i because you know if I call this A i inverse tau as some tau prime then you will me that lambda of A of tau prime is lambda of A i of tau prime. So, you will get this. Now if I calculate J tilde of A of tau ok, then you see this is this is just if you look at the definition of J tilde I will have to just where ever I have lambda of tau I have to put lambda of A of tau alright, but then lambda of A of tau is same as lambda of A i of tau and therefore, I will get essentially what I will get is I will get J tilde of A i of tau.

So, you see now, therefore, this is the same as J tilde of A i of tau now the claim is all these J tilde of A i of tau is that are only 6 of them in fact, there were only 5 of them because A 1 is this identity, they are all equal to J of tau. Claim J tilde of A i of tau is simply J tilde of tau for all i. How do you verify this? You for example, you verify it for let us say, let us verify it for A 3 for example, J tilde of if I calculate A 3 of tau will be it will 4 by 27 into you see in this formula I will have to put lambda of A 3 of tau, but you know lambda of A 3 of tau is 1 minus lambda tau. So, it amounts to taking that formula and wherever I get lambda of tau I have to put I minus lambda of tau.

So, what I will get is well I will get the following I will get 1 minus 1 minus lambda of tau plus, well 1 minus lambda tau the whole square the whole cube divided by 1 minus lambda of tau the whole square into the other one is 1 minus 1 minus lambda of tau the whole square. And you can readily see that the denominator is the same in the numerator you will again get if you expand it you will simply get J of tau.

So, similarly you can verify that this is true for all the other cases it is just a matter of direct writing down and this will easily in this way we easily see the J tilde is actually invariant under the unimodular group.

So, with that we come to the conclusion of this lecture. Now, what I have to do in the succeeding lectures is to show that J tilde actually is it go since it is you know invariant under the unimodular group it goes down to a function J on you mod PSL 2 Z on the Riemann surface U mod PSL 2 Z it gives you a holomorphic function on U mod PSL 2 Z, I will have to tell you that this function is surjective on the complex numbers and it is also injective and if I do that then I have then I have through. I would have proved that the Riemann surface structure on U mod PSL 2 Z is exactly the complex numbers up to isomorphism. So, this A of tau is called, it is called the J invariant of the elliptic curve T tau or the complex torus T tau, it is called the J invariant classically.

So, I will stop here.