An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves.
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Lecture-39
A Fundamental Region in the Upper Half-Plane for the Elliptic Modular J-Invariant

So, let me recall several things to continue the discussion.

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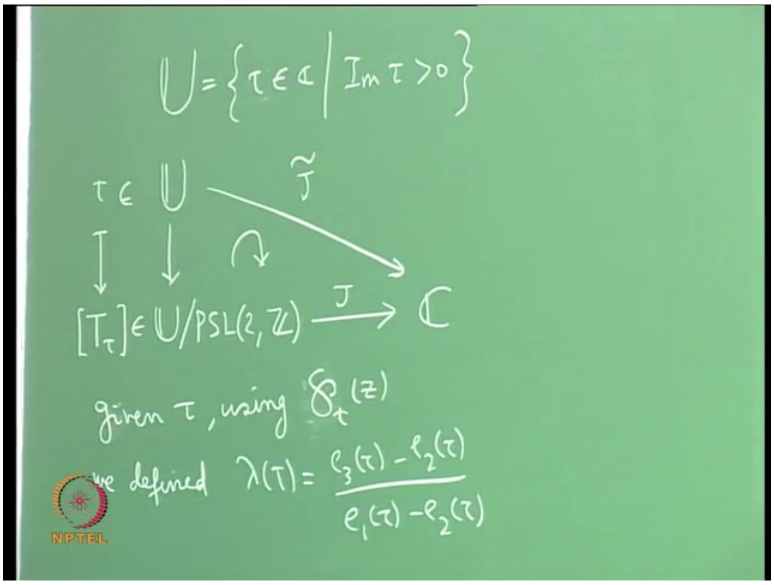

See we have, we have the upper half plane, the set of all complex numbers such that or let me use tau such that imaginary part of tau is positive. And then we know that as we are we are looking at this quotient U to U mod PSL 2 z. And this quotient has the fallowing meaning tau going to the equivalence class of tau model of the action of this group PSL 2 z which is the unimodular group namely, tau going to it is orbit. This is the just the set of orbits of the group unimodular group PSL 2 z on the upper half plane. And it this is just tau going to it is orbit, but then we think of it also as the isomorphism class of the complex torus defined by tau. And our aim is we, want to show that we want to show that this as a Riemann surface is biholomorphic to see, that is isomorphic holomorphically isomorphic to the complex plane.

So, what we did was we have constructed a function J. The so called elliptic modular function which value which is a holomorphic function with values in c. And the way we got J was we got first function J tilde which was defined on the upper half plane which is a which was a holomorphic function defined on the upper half plane. And we proved that J tilde is PSL 2 z invariant. And therefore it goes down to a holomorphic map J.

So, if you if you recall for a tau, we given tau, using the Weierstrass phi function associated to tau. We defined the partially modular function, the function that is modular not under the whole unimodular group, but only under the congruence mod 2 subgroup, which we called as lambda of tau. And this was a e 3 of tau minus e 2 of tau by e 1 of tau minus e 2 of tau. Where e 1, e 2, e 3 are related to the Weierstrass phi. These are the these are the zeroes of a the derivative of the Weierstrass phi function.

And then, using lambda we defined the function J tilde. J tilde was the function that was invariant under the whole unimodular group. So this lambda was invariant only under the congruence mod 2 subgroup. But, using this we cooked up J tilde and J tilde was invariant under whole unimodular group. And what does the definition J tilde it was as follows yeah.

So, let me write that here. So, before I write that on let me say, lambda is holomorphic on the upper half plane. Lambda never takes the values 0 and 1.

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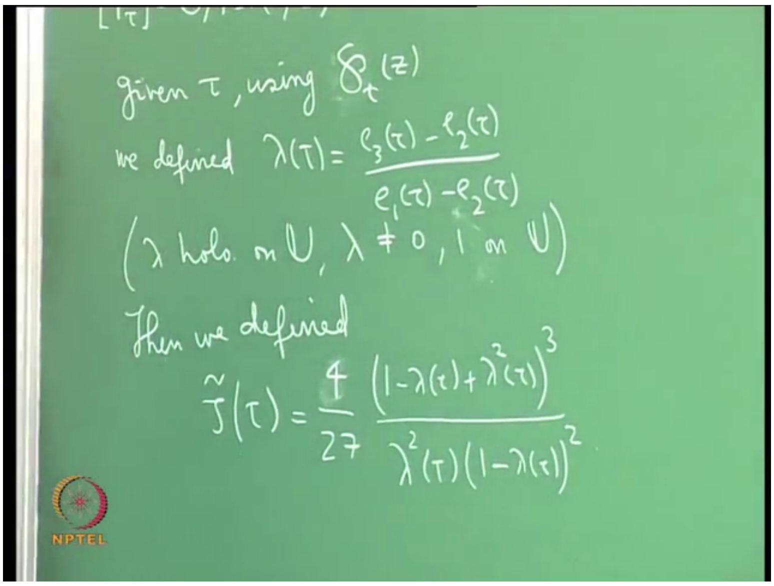

So, lambda holomorphic on the upper half plane, lambda not equal to 0, 1 on the upper half plane.

So, then we defined J tilde of tau to be, well this has given by a formula. So, let me write it out. It was, well it is, 4 by 27, 4 by 27 into 1 minus lambda of tau plus lambda square tau the whole cube. Was it whole cube or was it whole square. So, let me check for a minute. Yes it is whole cube divided by 1 minus by lambda squared tau into 1 minus lambda tau the whole square.

This is how define the function J tilde of tau. And since lambda is never 0 or 1 on U, it turned out that, so this denominator is never going to vanish and this is a quotient of a 2 holomorphic functions on u with the denominator never vanishing. Therefore this is a holomorphic function on U, which is holomorphic on U.

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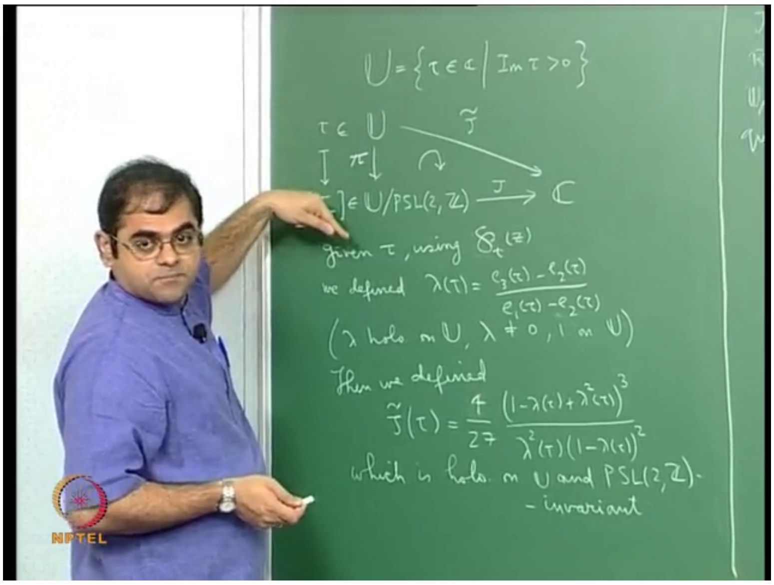

And we proved in the last lecture that this is this function J tilde was PSL 2 z invariant ok.

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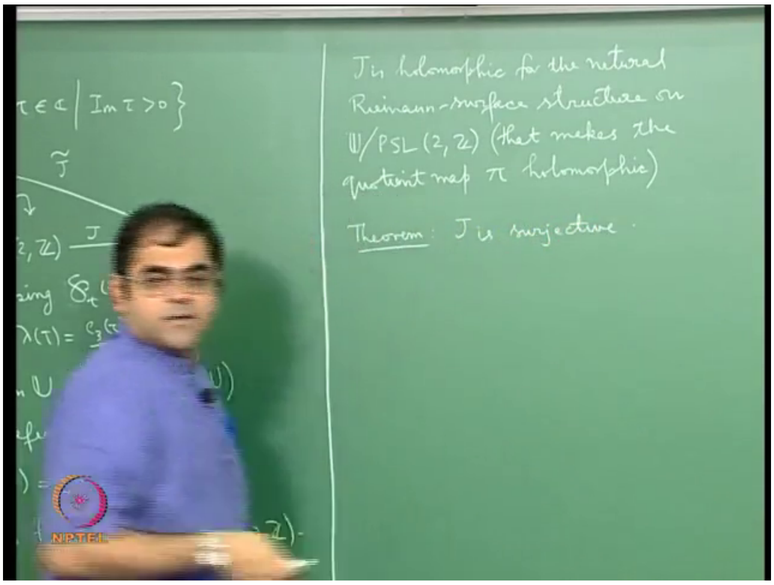

So, the moral of the story is that we have gotten hold of this J tilde. We need to next say that this J tilde is. So, of course, J tilde goes down to a map J because any map from U which is constant on orbits will go down to a map to the set of orb[its] from the set of orbits ok.

So, J tilde goes down to map J. And of course, I want you to realize that J is holomorphic. Because you see we have already proved that U mod PSL 2 z which is, it is already a Riemann surface, such that the map from U to U mod PSL 2 z is a holomorphic map. And therefore, you see that the and if you and if you look at it carefully ok;

If you look at that construction carefully, you can show that J is also holomorphic. So, J is J is holomorphic. So in fact, you see J is holomorphic for the natural Riemann surface structure on U mod PSL 2 z given, making, that makes the quotient map. Well let me call this as pi, let me call this map as pi, that is a quotient map holomorphic.

So, the there is a natural. We have already proved this. On U mod PSL 2 z there is a natural structure of Riemann surface. And the structure of Riemann surface is such that if you consider U also as a Riemann surface, then this map is a holomorphic map. And so, J becomes a holomorphic function right. And our aim is to prove that J is an isomorphism alright. So, there are 2 steps that we have to I mean we have to do it do this in.

The first step is to show that J is surjective. Then the second one is to show the J is injective. The easy part is a surjectivity. The hard part is a injectivity. The injectivity will require us to again go back and look at the mapping properties of a J which depend on the mapping properties of lambda which we have already know. Then there is another thing that will have to study, we will also have to study, the mapping properties of PSL 2 z. In the sense that you will have to find it fundamental region for PSL 2 z in the upper half plane ok.

So, the surjective department is pretty easy. So, let me write that down. So, let me write this here, well.

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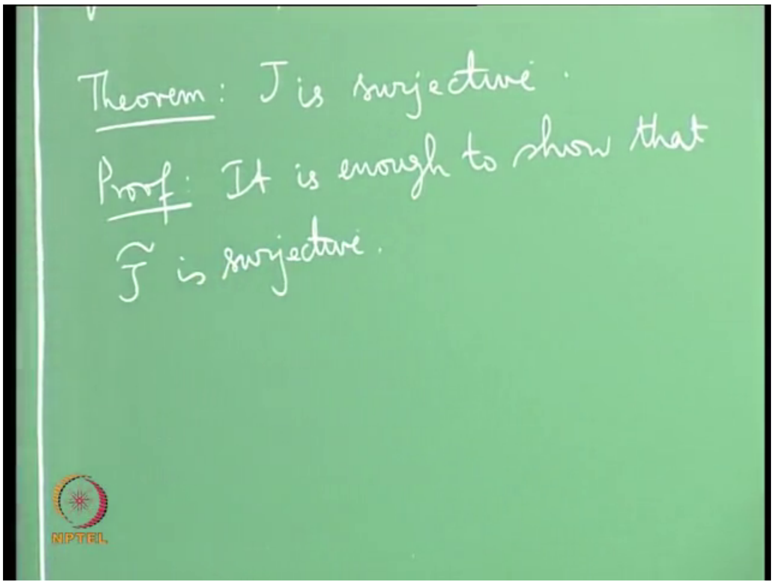

Theorem J is surjective. J is surjective. So, you see lambda; lambda does not take the value 0 and 1. But J takes all values. And you know, to show that J is surjective, it is enough to show the J tilde is surjective, alright.

So, because, this is already surjection. This is a surjective map. It just every point going to it is orbit right. So, proof, it is enough to show that J tilde is surjective. So, I will have to show that J tilde takes every complex value ok.

So, what do I do? I literally look at the formula of J tilde and literally solve for, solve for a value.

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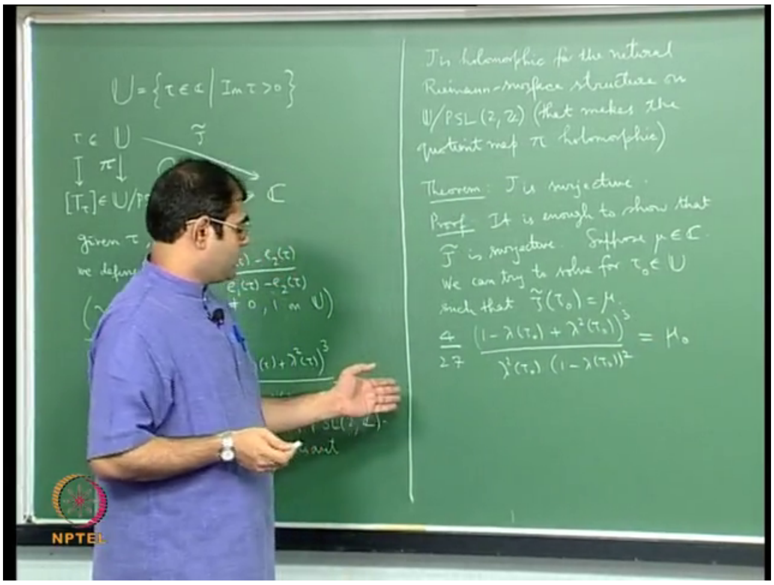

So suppose, mu is a complex number. Suppose mu is a complex number. We can try to solve for tau in the upper half plane. Or let me say tau naught in the upper half plane such that, well J tilde of tau naught is mu.

We can do we can try to solve for this. So, let see what it means. So, you see. So, you will have well, if I write it down 4 by 27 times 1 minus lambda tau naught plus lambda squared tau naught the whole cube divided by lambda square tau naught into 1 minus lambda tau naught the whole squared. This is, you want this to be equal to mu naught.

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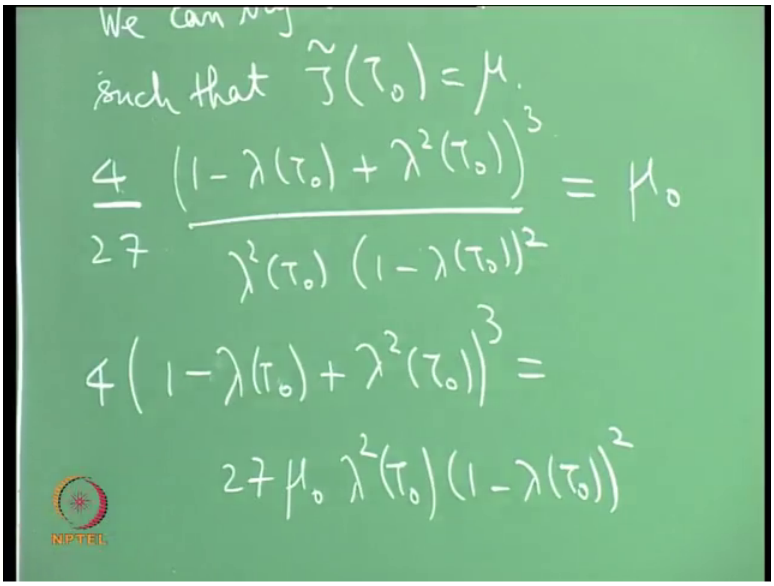

Now, if I cross multiply it out and write it out, what I will get is I will get 4 times 1 minus lambda tau naught plus lambda squared tau naught whole cube is equal to 27 mu naught lambda squared tau naught times 1 minus lambda tau naught the whole squared. This is what I will get, this what I will get.

Now, you think of, you first of all think of lambda tau naught as a variable. Think of lambda tau not as a variable. Then you see, if I think of lambda tau naught as a variable, let us call it z. Then what I have here is a polynomial equation in z. And it is a non trivial polynomial equation because on the left on the left side the highest power of the variable is 2 into 3, 6 on the right and it is coefficient is not 0.

So, and the right side you have lesser powers. And now you know the fundamental theorem of algebra guarantees that you give me a polynomial with complex coefficients in one variable, there is always a root. So therefore, the moral of story is, I can always find a certain, I can always find a certain value of a lambda of tau naught which a when which when I plug into this equation will satisfy this equation ok.

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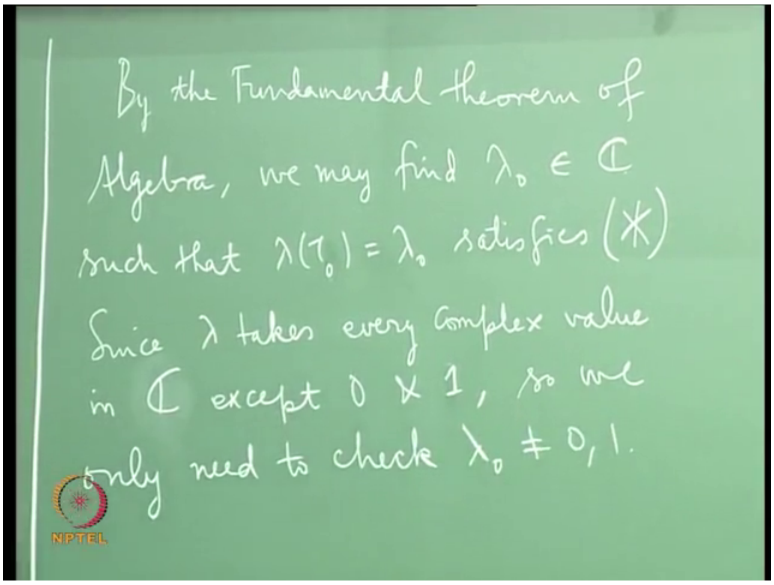

So, let me write that down, by the fundamental theorem of algebra, of algebra namely, the fact that the complex number are algebraically closed. Which is saying that every non trivial complex polynomial with polynomial in 1 variable with complex coefficients certainly has a 0, which is the complex number. We can find, we may find lambda naught belonging to C such that that lambda tau naught is equal to lambda naught satisfies this equation. May be I will label this equation as star ok.

So, I can certainly find here lambda naught such that lambda tau when I instead of lambda tau naught if I put lambda naught here, then it will satisfy this equation. Namely, it will be it will be a solution to this polynomial equation aright. Now you see, now the aim is I want to find a tau naught such that therefore, you see I have to find a tau naught such that lambda tau naught is lambda naught alright.

So, you see. So, what I have done, in order to show that J tilde takes a certain value mu naught, because of the formula for J tilde and fundamental theorem of algebra I have reduced it to problem of trying to find solve for lambda taking a particular value. Now you we make use of fallowing fact namely the mapping properties of lambda that you see lambda takes every complex value on the upper half plane except for 0 and 1 ok.

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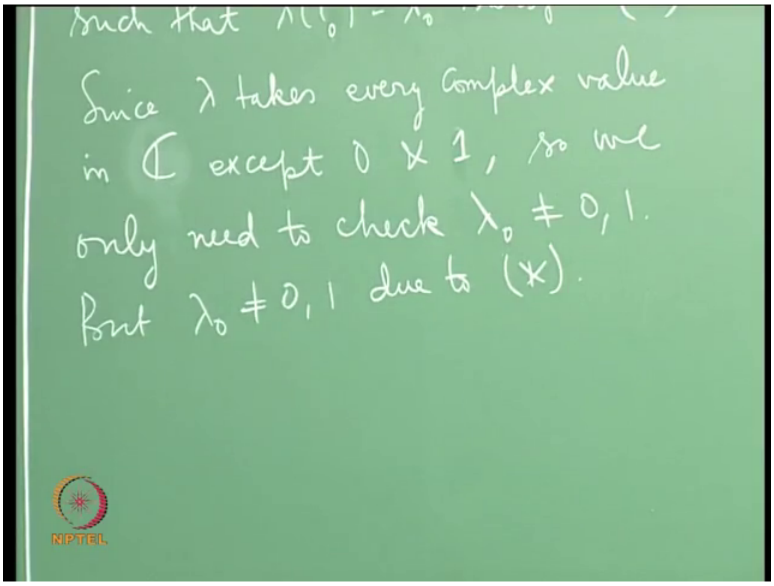

So, since lambda takes every complex value U in U well in fact, I should say lambda takes every complex value in U except 0 and 1, 0 and 1. We have already done this. I need to therefore only check that you see that lambda naught is not 0 or 1 ok.

So, lambda takes every complex value in c x of 0. So, yeah that is right. So the only thing I will have to check is that, I will have to check that this lambda naught is not 0 or 1. So, we only need to check, we only need to check lambda naught cannot be 0 or 1. But you see if I put lambda naught equal to 0 or lambda naught equal to 1, you will get the contradiction from star. Because after all instead of lambda tau naught I have to put lambda naught and if I put lambda naught as 0, I if I put lambda naught as 0, I will get 4 equal to 27, 4 equal to 0 which is observed. And if I have put lambda naught equal to 1 then again I will get 4 equal to 0 ok.

So, I will get 4 equal to 0 it any case which is observed. Therefore, it cannot happen. So, lambda naught is not 0 or 1 and therefore, we are done. But lambda naught is not equal to 0, 1 due to star. So, we have done alright. Therefore, you see the function J is surjective. So, this is a pretty easy thing pretty easy given the fact that you know the mapping properties on lambda, aright.

Now, the, for the rest of the discussion what we will need to do is we need to show that J is injective.

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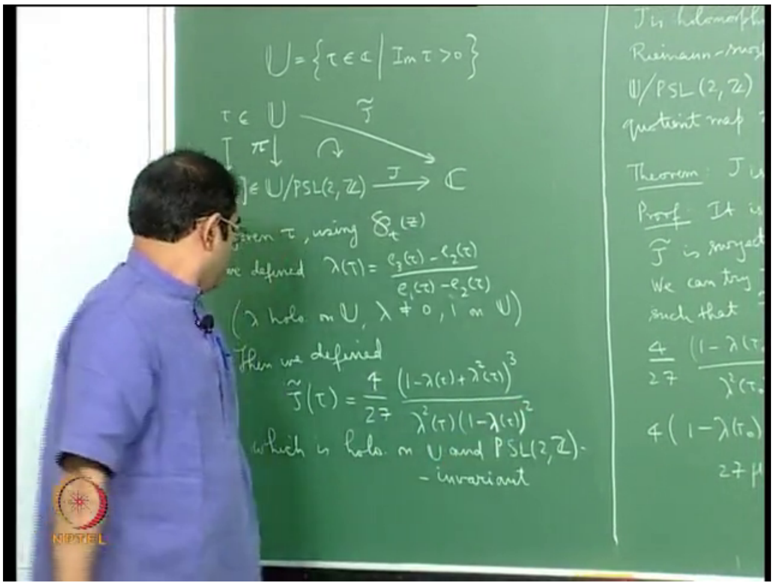

So, for this we will have to again get what is what is called a fundamental region for j and so that will again involve studying maps. So, let me begin that. So you see.

So, what I am going to do now is, I am going to draw big diagram here and yeah. So, let me draw it here yes.

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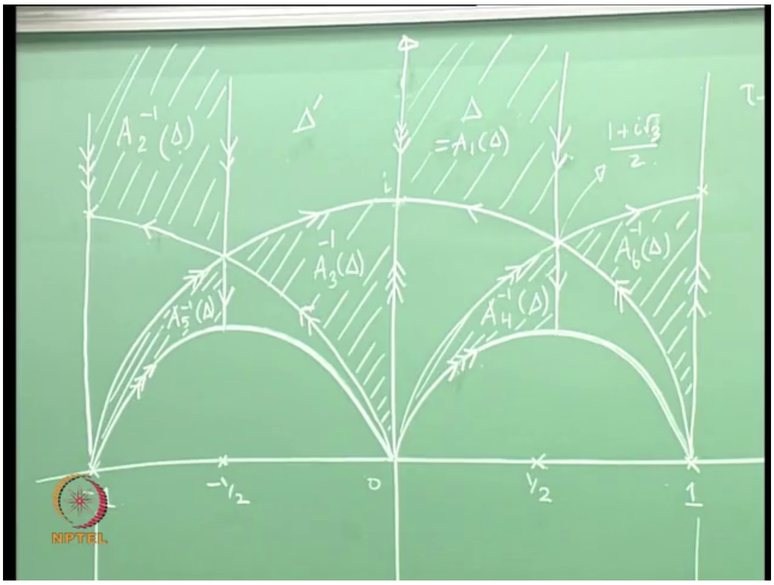

So, this is 1. This is minus 1. This is the vertical line through 1. This is the vertical line through minus 1. And then I have this I have this point which is half and I have this circle. So, this is 1. So, I will draw something here see here.

So, this is 1 plus i. Well this is i. And this is minus 1 plus i well. So, I am going to, So, I will, I will draw the circle centered at half the radius half which we have already seen while studying the mapping place of lambda and I will draw similar one here centered at minus half again radius half. So, this another circle and that what I am going to do is I am going to now draw circles centered at 0, 1 and minus 1, which are radius 1 alright.

So, I am going to draw these circles well here is 1. So, here is 1. Here is a second 1. So, this centered at 0, radius 1. This is one centered at 1 radius 1 which will look like this. There is one centre at minus 1 radius 1 which will look like this. And well I am also going to draw couple of lines. Well I am going to draw this vertical line like this that goes through this and passes through the point half. Another vertical line that passes through minus half right.

So, the first thing that I want to tell you is just to recall what I use here. See this you take the region that is bounded by this the; you know the positive imaginary axis. And this semicircle and then this ray namely you take this region right. And that region was suppose to be mapped by the; that region was proved to be mapped by lambda on to the upper half plane. And this and the corresponding region here which is a reflection of that region by the imaginary axis namely the region bounded by this. And this, namely this region, this region was mapped by lambda on to the lower half plane. And they you we were able to extend the mapping lambda to continuous 1 map to the boundary.

So, that you know the real line is also covered. And therefore, put together; both put together you see we proved that lambda takes all values on the complex plane except for the value 0 and 1. Because lambda went to 1, as you go lambda takes a value 1 at 0 and it takes a value infinity at 1 ok.

So, you will have to consider the point it infinity if you want the value 1 right. And of course, you know this much was enough to study because lambda was having period 2. So, this whole thing is you know spread over an x coordinate of length 2. So, it is enough to study lambda here ok.

So, that something that we already done. Now you see this function J that we have cooked up is slightly more complicated. So, let me tell you that for J, what is going to happen is that the fundamental I mean the region that is going to that J is going to map on to the upper half plane will be this piece bounded by this.

So, this piece will be mapped by J outer the upper half plane and it is reflection delta prime is going to be mapped by J on to the lower half plane. And so, that is what we will have to show first. So, how do we do this? So, we do this by considering several Mobius transformations. See if you remember; how did we show that J tilde was PSL 2 z invariant because we showed that J tilde was invariant under yes. Well bunch of Mobius transformations in fact 6 of them including identity which gave as a complete set of complete set of unimodular elements in z mod 2 that is mod 2.

So, that is how we verified it. And these in fact, in fact studying lambda on these helped us to understand what happens if an arbitrary unimodular element acts on lambda. So, let me write out those let me write recall those transformations.

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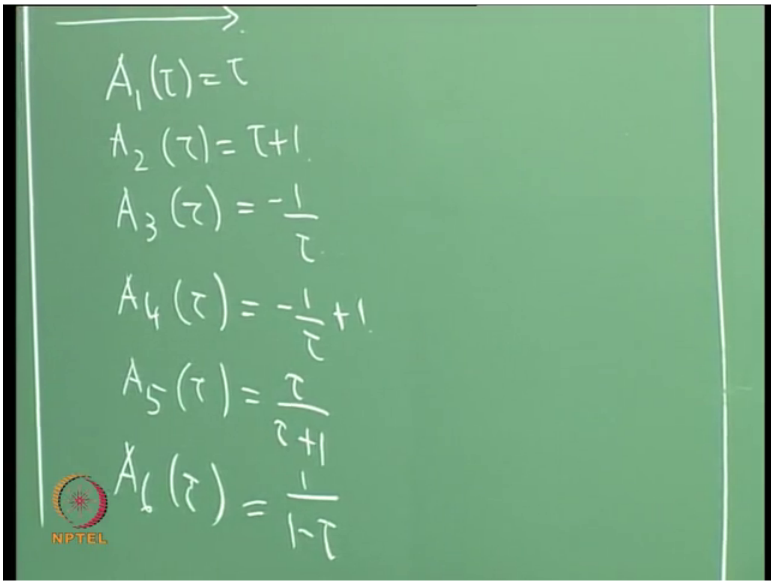

So, they are as follows. Here they are may be the numbering is, well, I be consistent with what I wrote earlier. So, this is the identity map the identity Mobius transformation. Then you have A 2 of tau which is a translation by 1. Then you have A 3 of tau which is a minus 1 by tau. Then you have A 4 of tau which is minus 1 by tau plus 1. A 5 of tau is tau by tau plus 1. And A 6 of tau use well 1 by 1 minus tau. So, these were the 6 Mobius transformations which if you if you consider them as elements of PSL 2 z and read them mod 2 then you will get all the 6 elements of PSL 2 z mod 2.

So, and in fact, lambda when you apply this to lambda; each of these to lambda, then lambda satisfies a certain functional equation. That is something that we proved and that was used to show that that was used in the proof of showing the J tilde is a you know is not effected by any element of PSL 2 z, PSL 2 z invariant right.

So, you see now what I want to tell you is. So, let me, let me write this down. So, I will draw some, I will draw some arrows and I will draw some, I will do some shading to tell you what happens. So, this delta is. So, what I want to say is, see if I call this thing as delta, then this delta is of course, well this is, that is A 1 of delta because after all A 1 is identity alright. And then this one, this see this region here is A 2 inverse of delta which is pretty obvious which pretty obvious because see A 2 is translation by 1, A 2 inverse is translation by minus 1.

So, this region is got by this region just by translating by minus 1. So, it is a 2 inverse of delta. So, I will shade that also. And right then, the nice thing is, this region here, this region here turns out to be A 3 of delta. So, this region here is A 3 of delta and mind you this is well A 3 is a same as A 3 inverse A 3 is A 3 inverse alright.

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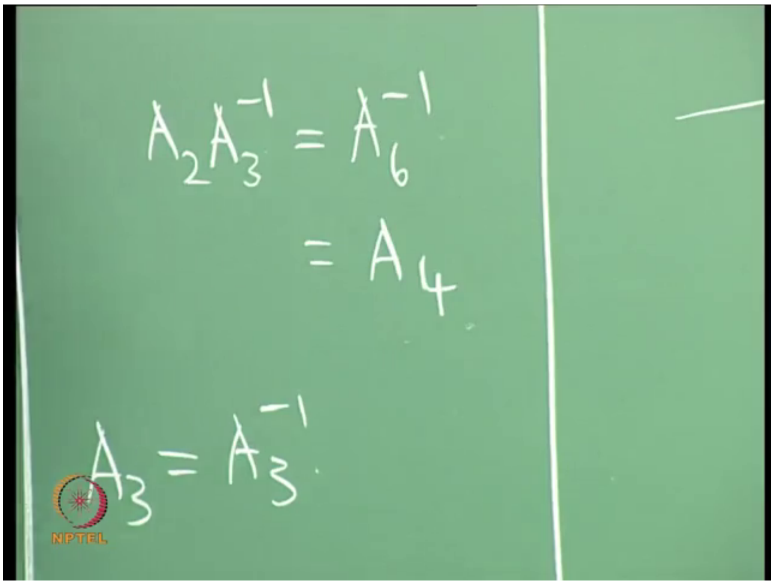

So, well you can call this as A 3 delta or you can also k call it as A 3 inverse delta alright. But I am trying to write everything in terms of inverse. So, let me call this A 3 inverse delta right. And that is this region right. Then, this region here, this region here is translate of A 3 by A 2. Literally translate by translation by 1 ok.

So, and this turns out to be A 6. It this turns out to be A 4 delta which is same as A 6, A 6 inverse of delta. So, that is right. So, this is A 6 inverse of delta right. And what is happening here is well and let me write this from certain results here.

So, you apply A 3 inverse, then you apply translation by 1 which is A 2 what you get is A 6 inverse and that is a same as A 4. So, this is something that you can check. And then this guy here I mean this we this piece here, this piece here turns out to be this is A 4 inverse delta that is this piece.

This is A 4 inverse delta. And of course, you know A 4 inverse delta is A 6 delta because A 6 inverse is A 4. And then there is this piece and this is A 5 inverse delta which is this piece. And A 5 inverse turns out to be of course it is A 2 inverse A 6.

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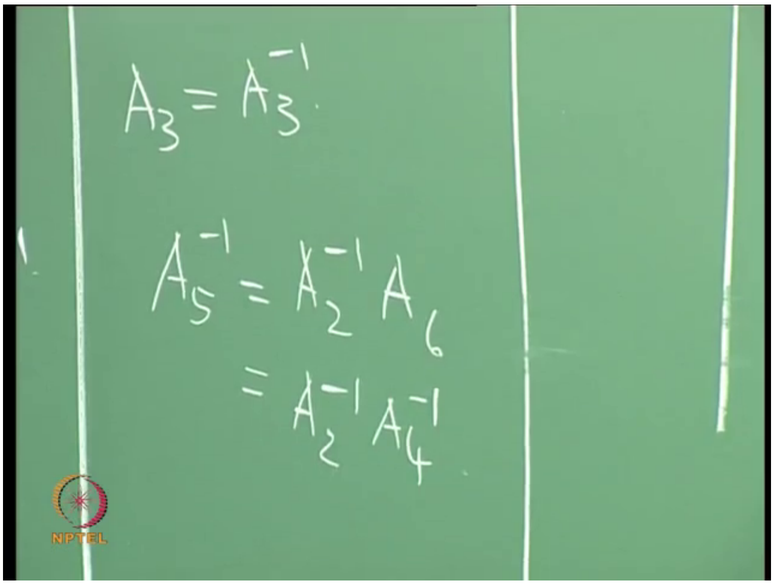

Because you see it is just A 4 inverse is A 6 and then this is applying A 2 inverse translating by minus 1 alright.

This is also the same as A 2 inverse A 6 is A 4 inverse alright. So, the moral of the story is that, this is how the regions are mapped into each other by A 1 inverse of course, A 1 inverse is just A 1, A 2 inverse, A 3 inverse, A 4 inverse, A 5 inverse, A 6 inverse. Of course, you can renumber them if you want a certain order. But that is not the point. The point is they are all mapped by holomorphically on to one another with the boundaries. And if you want to also I am going; I will, let me explain a couple of these how do you how do you check this for a couple of regions and then you can do it on your own for all the region alright.

So, but of course, when we say we should also, because these are all conformal maps, you will have to give orientations to the boundary. So let me do that. So, the orientations are as follows. So, if I take, I will use a triple arrow orientation for, I will use single arrows, double arrows and triple arrows. So, here is and this is not to be confused with on any of these.

So, let me just put it here and put that triangle here and similarly a triangle here. So, here is a triple arrow. And then use the single arrow and there is a double arrow and what do the; what do they correspond to. Well of course, when I go to here to here, the orientations are not going to change. So, you see this is continue going to continue to be a triple arrow.

This is going continue to be a double arrow and this is this is going to continue like this right. So this going to continue at this. And the question is; how does it go from here to here. So, this goes to this. So, this goes to this and this goes to this. So, this arc goes to this arc from infinity to i, goes from 0 to i. And the double arrow use this one.

So, this goes to this right. And in the same way, you can draw orientations for all of them. So, let me draw it for this one. So, here. So, this arc, it will go to, this goes to this. And well the other one is this. And let me draw something here for this one. So, here it is this. And it is these 3 and well it is these 2.

So, this is how the regions are. So, there of course, you know this is just translation. So, you know that from this I can put a 3 arrow heads here and I can put double arrow head here and of course, an arrow head like this. So, this is how these regions are mapped and well the.

So, you see, now you see if you look at. So, what I wanted now tell you is that the unshaded region is mapped also to. So, you see there are 6 shaded regions, where a 3 here and 3 here. And all these 6 shaded regions are mapped by the inverses of all these guys. The region delta is the; is mapped to all the 6 regions including itself by the inverses of all these maps alright. And the fact is that if you take the region delta prime that will be a map to the others to the 6 unshaded regions by all these maps themselves ok.

That is the claim. And what is the result of this (Refer Time: 34:20) claim. The result of this claim is, you see you. So, in particular if you take delta and delta prime, then using these 6 elements or their inverses, I can map the regions composing of delta and delta prime to all the to this region as well as this region. And then you see that is mapped by lambda on to the whole complex plane.

So, the upshot of this is that you can see that this delta will be mapped on to half plane by J tilde. So, that is the point alright. So, let me explain how to get this, how to get this things. So, let me explain a couple of them alright. So, let me look at let us look at this guy.

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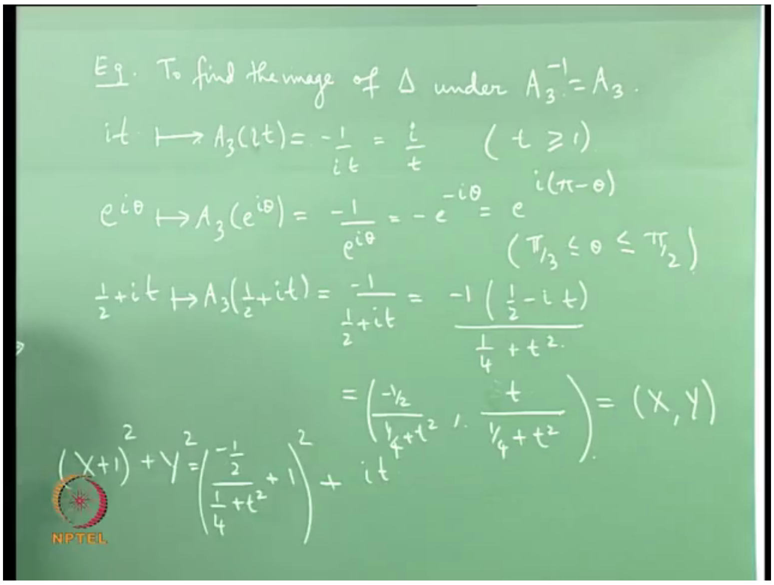

So, to find example, to find image of delta under A 3. Suppose, I suppose I will work this out and you can do it for the others the others fine.

So, I am just looking at the images under Mobius transformations and you know Mobius transformations will preserve boundaries alright. Therefore, what I will have to do is, check what happens to each of the boundary curves. So, in this case if I take this boundary curve that is parameterized by I, it is parameterized by i times t where t is greater than or equal to 1 ok.

So, if you take if you take i t, i t goes to A 3 of i t. And A 3 of i t is well, it is minus 1 i t and this is going to be i by t. So, and you know t for me is greater than or equal to 1. So, as if t is 1, i simply goes to i and if t is infinity, then A 3 of i t, A 3 of infinity will go to 0 alright. So, the moral of the story is that, this line which coming from infinity to i. So, this point is I, this line which is coming from infinity to I, is mapped on to this line which is coming from this line segment from 0 to i ok.

So, that is the reason this triple arrow head corresponds to this triple arrow head and the orientation is from infinity to i is same as 0 to i under the image. So, next look at, look at this, look at this boundary curve. This boundary curve is just the unit circle alright. So, it is parameterized by e power i theta ok.

So, if I take e power i theta, e power i theta will go to A 3 of e power i theta and that is going to be minus 1 by e power i theta and this is minus e power minus i theta and this is e since minus 1 is e power i pi. So, this is e power i into pi minus theta.

So, you see which is just reflection. It is just the reflection about the imaginary axis. So, if theta varies. So, this point is actually minus 1 is actually 1 plus i root 3 by 2. This is exactly 60 degrees and this is a and this points corresponds to a complex cube root of unity, the 1 and the upper half plane ok.

So, this. So, this point is actually, let me write this, 1 plus i root 3, i root 3 by 2 right. And so, theta is varying. So, as theta varies from 60 to 90, pi minus theta will vary from 120 to 90. So, this the image of this curve, this arc of the unit circle will be precise to this arc of the unit circle. That is a reason why I have put a single arrow head from here to here and that corresponds to single arrow head from here to here.

Then I will have to look at this boundary curve. So, that boundary curve is. So, so this is here the parameter is pi by 3 less than or equal to theta less than or equal to pi by 2. That is the that is this portion of the arc right which is map to this portion of the arc. Now I will have to next look at this boundary curve which is the line real part of z equal to half or real part of tau equal to half. Because we consider the variable; think of this is a tau plane cannot the z plane. So, this is a tau plane ok.

So, well what is the parameterization for this line? It is half plus i t. That is the parameterization for this line. And where will it go to. It will go to well A 3 of half plus i t. That is going to be, write it out. It is going to be minus 1 by half plus i t. That is let us multiply and divide by the conjugate complex number. So, that I get a real denominator.

So, I end up with. So, which is which is as a point in a with coordinates. It is minus half by 1 by 4 plus t squared comma i t by 4 plus t squared. And well that suppose to correspond to you know this image of this line should be this arc.

So, what is this? This is a circle centered at minus 1 radius 1. So, to show that it is indeed that, you use you show that that parametric representations satisfies the equation of the circle. So, what you do is at if you calculate, if you call this as capital X comma Y, then do you check whether it satisfies equation of the unit circle centre at minus 1 comma 0. So, you calculate X plus 1 the whole squared plus Y squared.

You will see that X plus 1 the whole squared plus Y squared turns out to be in fact 1. Let us write it out. It is minus half by 1 by 4 plus t squared plus 1 the whole squared plus i t sorry I should write it coordinates, I should remove this i.

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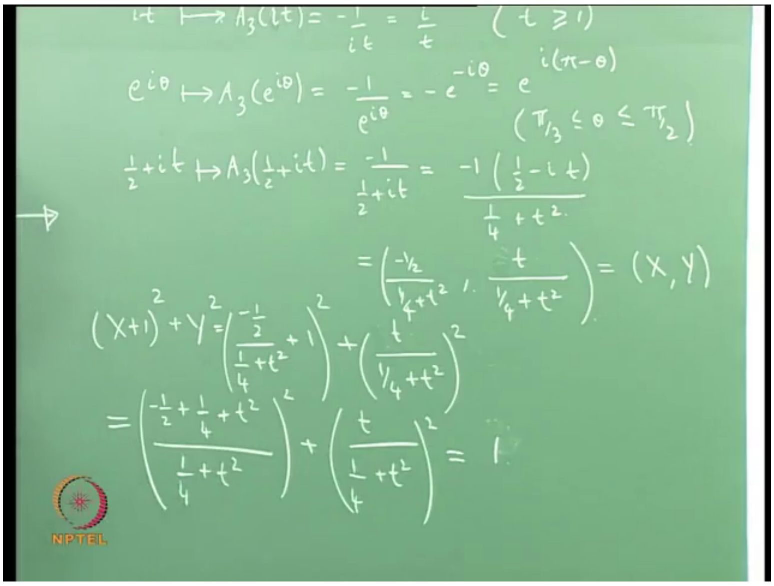

So, it is i, it is t by 1 by 4 plus t squared the whole squared and if I write it I will get, I will get 1. So, what is this is. So, let me write it here. This is going to be well the numerator I am going to get. Here I am going to get, you know minus half plus 1 by 4 plus t squared whole squared plus t by 1 by 4 plus t squared the whole squared and ah.

This is of obviously going to give me t squared minus 1 by 4 the whole square identify simplify this, I will get 1 I will simply get 1. Because it is will t squared it will be t squared minus 1 by 4. The whole squared and then there is a t squared there if I add it I will get a t square plus 1 by 4 the whole square that will cancel. So, I will get 1. So, the moral of the story is that, the image of, the image of this line is certainly going to lie on this circle. On the circle and of course, the point at infinity I will get by putting t equal to infinity. If I put t equal to infinity here I will get, I will get 0. Both entries will be 0.

So, I will get this point. And if I put t equal to for this point corresponds to t equal to root 3 by 2. So, if I put root 3 by 2 you will see that I will get this point which is minus 1 plus i root 3 by 2. Therefore, this line segment from infinity to 1 plus i root 3 by 2 is mapped on to the arc of the unit circle centered at minus 1 comma 0, from 0 to minus 1 plus i root 3 by 2. Then that is a reason why I put the arrow, the double arrow for this and at and the double arrow for this. And now because everything is conformal, then the region enclosed by this is going to be map to the region enclosed by this.

Where in principle, the region inside can be mapped a to the region inside or region outside, but you can you can test it at any point and you can see that it has to be mapped to the region inside. So, that completes the proof of the fact that this region is mapped by A 3 inverse which is a same as A 3 on to this region. Now what you can do is, you can makes similar computations and show that if you apply I mean A 4 inverse, then A 5 inverse and A 6 inverse you get all these regions as stated.

And then, as a further exercise, what you can do is you can take delta prime and you can show that delta prime is map can be mapped to all the unshaded regions. There are 6 unshaded regions 1, 2, 3, 4, 5 and 6. Or we never always consider this region inside the semicircle. We this is this is always left out. It is everything in we are considering our is above that, is above these 2 semicircles.

So, you can check that as well. And so let me state what it is that we need to we need to prove namely. So, here is a claim.

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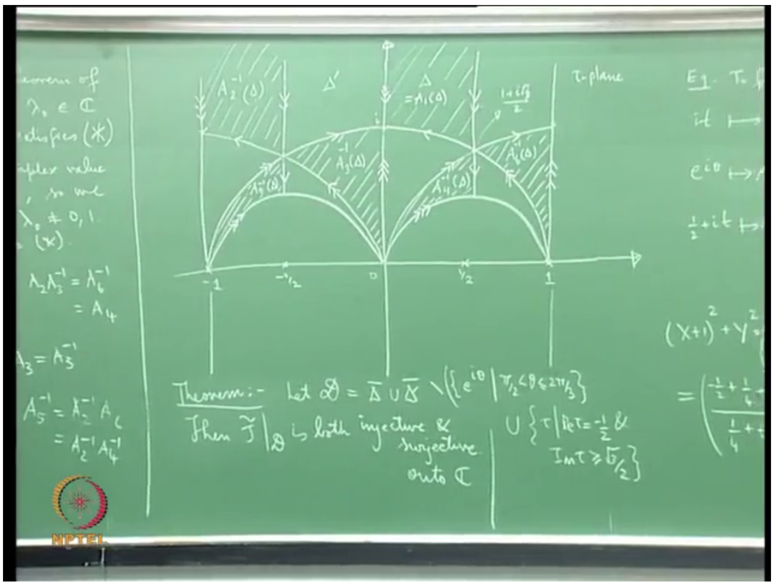

So, here is a. So, here is a, here is a theorem ah. For which for which we will have we will have to further work. So, the theorem is that you see if you take delta and delta prime and then you take in for the boundary you take of course, a boundary will contain this and then this and then this and then this. But you leave out these 2, these 2 you leave out. So, you take delta and delta prime and you take the only this segment. You take this segment and also of course, you take the imaginary axis.

So, basically you take this whole, this whole region and add to it a part of the boundary namely this arc of the unit circle centered at the origin and this vertical line. The claim is you restrict J tilde to this set. Then J tilde is both injective as well as surjective on that set.

So, let d be delta bar union delta prime bar. And then from that throw away the arc well, e power i theta varying from. So, through this arc which is a theta varying from i by 2 to 2 pi by 3. Throw this away. And also throw away, throw away this line. Because actually you see this line to that line it is translation by 1 and you know translation by 1 is a unimodular element and J, values of J are going to be the same.

Therefore all the values of J it corresponds here or the same as values of J on the corresponding points here therefore, throw away this vertical line as well and that is the set of all are tau such that real part of tau is a minus half. That this whole line and then I will take imaginary part of tau greater than or equal to root 3 by 2. So, I will leave out, I will leave out this, I will just take out this whole thing ok.

So, and imaginary part of tau is greater than or equal to 3 by 2 right. So, I am just throwing away this piece, I am throwing away this piece and then I am taking everything else. And therefore, when is set delta bar. Therefore, these imaginary axis, this portion of the imaginary axis is included. i is also there. I have not thrown out i, because I have put theta greater than pi by 2 ok.

So, let d be this. This is not a, it is actually, it is not closed. Then J tilde restricted to d is both injective and surjective. So, this is a statement that one has to prove, this is a statement one has to prove. And so, this see this script d this region is every special region. Of course, such a region is called a fundamental region for J tilde and what will prove is we will prove that even for PSL 2 z it is a fundamental region.

I mean what we mean by that is in the whole upper plane, this region consists of exactly 1 representative of each PSL 2 z orbit. We will prove that as well. So, if you put both together you will get the fact that the function J is actually a bijective holomorphic map ok.

So, we have to prove this and we will, which is saying that this script d is a fundamental region for J tilde. We also have to prove that script d is a fundamental region for PSL 2 z. In principle, it is not a fundamental it is not a region, it is actually, I mean it is not open part of the boundary is omitted and it is omitted because those values are already taken at other parts of boundary. So, that is the point. So, this is what we have to do and we will do this in the coming lectures. So, I will stop here.