An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimensional Tori and Elliptic Curves

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Lecture – 40
The Fundamental Region in the Upper Half-Plane for the Unimodular Group

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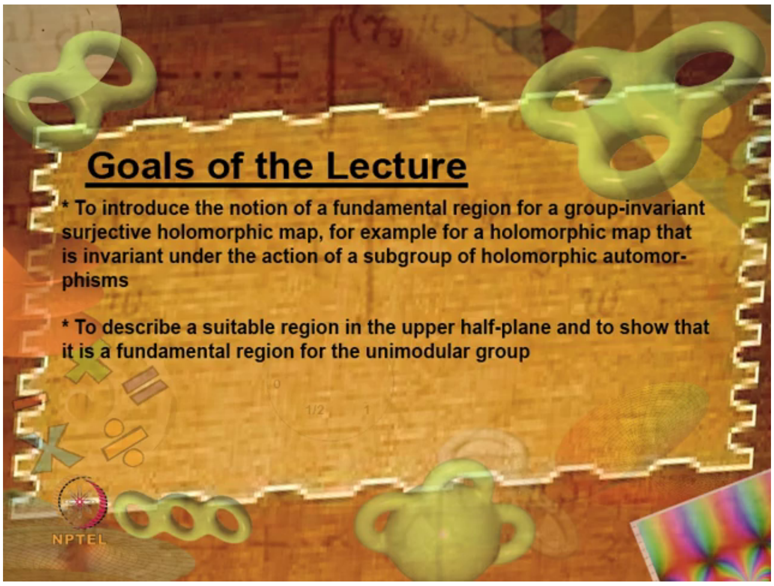

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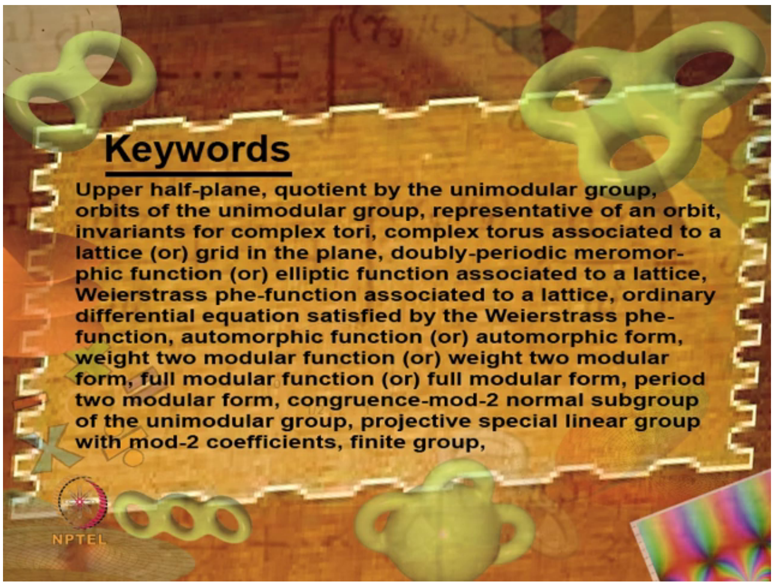

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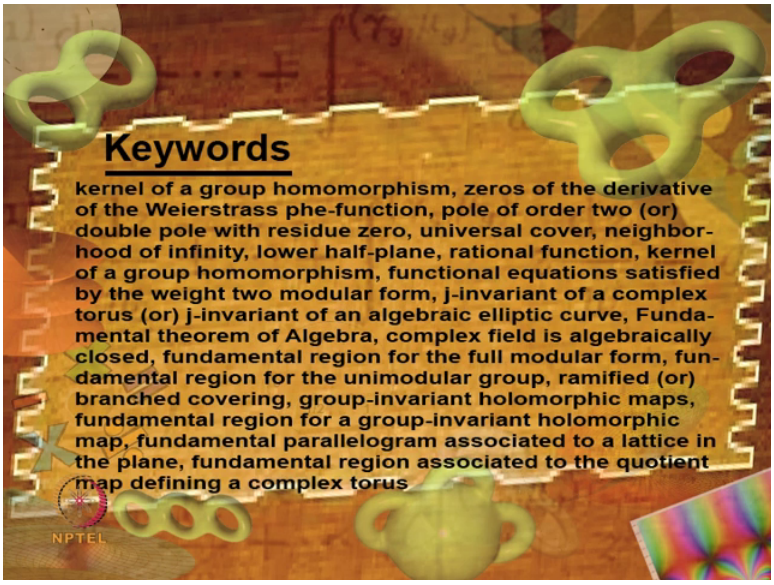

So, let me begin by trying to place in perspective where we are at this point of our discussion. So, we have the following of a half plane and these the upper half plane the on the complex plane and then we have the quotient of the upper half plane by the unimodular group.

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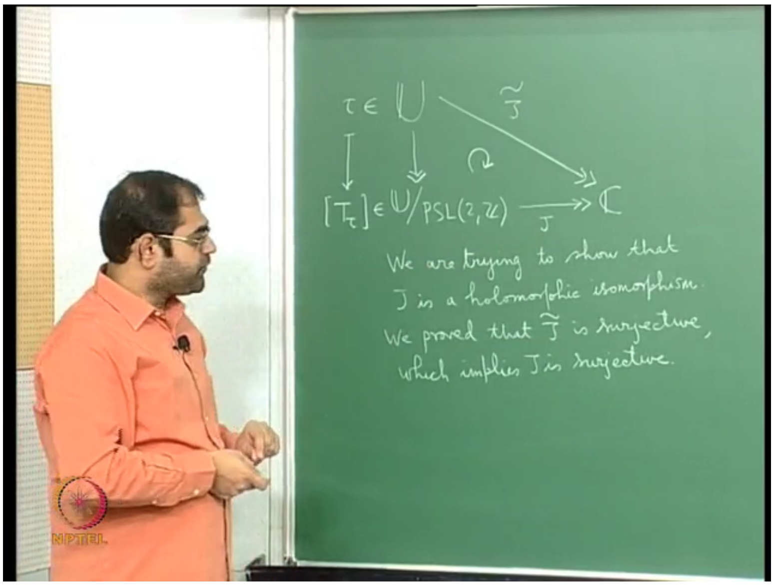

And we have defined a function J tilde which is holomorphic and takes complex values and we are verified that this functions J tilde this holomorphic function J tilde goes down to this quotient which we know is a Riemann surface and it goes down to a function J.

So, this diagram commutes in other words J tilde is invariant for the action of the unimodular group.

So, we are trying to show, we are trying to show that J is a holomorphic isomorphism in other words we are trying to show that the natural Riemann surface structure on U mod PSL 2 Z. Namely the set of orbits of the unimodular group on the upper half plane is the same as the usual complex plane with the natural Riemann surface structure on it and of course, you know that the interpretation for this is this is also the set of holomorphic isomorphism complex tori.

So, you think of if you give me if you take a tau in the upper half plane then you take the corresponding complex torus define by tau and you take its isomorphism class that corresponds to unique orbit half PSL 2 Z and this is the map that takes tau to its orbit can also be thought of as a map that takes tau to the holomorphic isomorphism class of the complex torus defined by tau and of course, the whole point is to say that the on the set of holomorphic isomorphism classes of complex tori that is the classification is achieved of the isomorphism by single invariant and this is call the J invariant.

So, you see we have to show the J of course, J is holomorphic that can be seen by looking at how this how the Riemann surface structure on U mod PSL 2 Z is got. In fact, we showed that PSL 2 Z as properly discontinuously on U and how this map is a ramified covering and if you analyze that construction you can show that because J tilde is holomorphic J is also holomorphic.

So, the other thing that one has to show is the J is both injective as well as surjective because you know bijective holomorphic map is an isomorphism which means its inverse is also a holomorphic map. So, I have to prove J is injective and surjective as I told you the easier part is to show that J is surjective and we did that last time by proving the J tilde as surjective. We have proved that J tilde is surjective which implies J is surjective that is because is already surjective map this is just it is just the quotient map.

So, sending every point to its orbit this is just the space of orbits. So, usually we put a double arrow head to denote a surjective map and I am saying that since J tilde is surjective, so J is also surjective. And of course, in the proof of J tilde is surjective we essentially made usage of fundamental theorem of algebra and also namely that the complex numbers are algebraically close that is every complex, any every polynomial in one variable it complex quotients has all complex roots, all its roots in complex numbers. And of course, we also used the fact that the function lambda the partially modular function lambda which is not invariant enter the whole PSL 2 Z, but invariant under the congruence mod two subgroup of PSL 2 Z that function does not take the values 0 1 1 we use that also in the proof of the fact the J tilde is surjective.

Now, the next thing that one has to do is that one has to show is the J is injective. So, for this I stated the following thing last time. So, let me expand on it.

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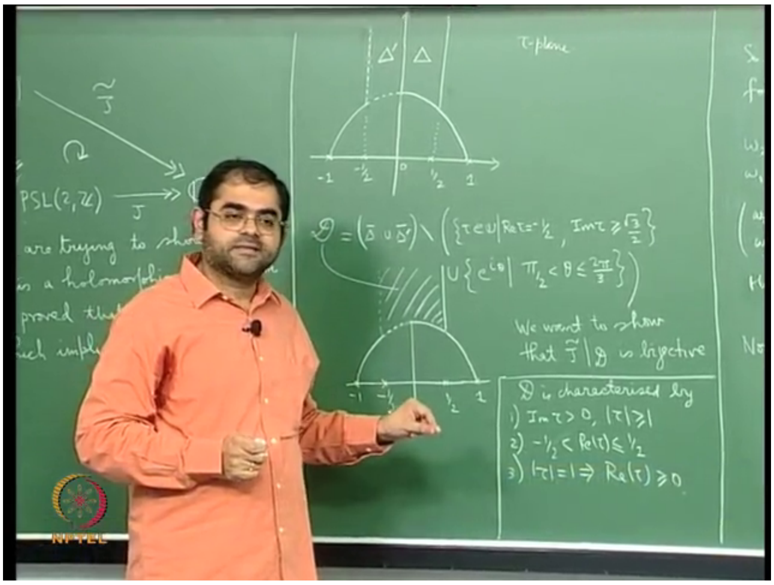

So, we have you see we have so that is this region in the upper half plane. So, this is the tau plane the complex numbers being thought of as the Z plane being thought of as a tau plane and well you take. So, you take this point which is minus half this is 0, this is half and this is one this is the point minus 1 and that draw circle centre at 0 radius 1. So, I am going to get I will only worry about the upper semicircle and then I draw this line vertical line passing through a minus half and another vertical line passing through half.

So, I get something like this. So, this is the line here and well and this another one here and the fact is that we called we call this, we call this, this open region I mean this open side which is bounded by T my this portion of the imaginary axis this portion of the arc of the unit circle and this vertical line we call that is delta. And we call its mirror image by the imagine by the imaginary axis as delta prime and then we define the domain D to be delta it is just the closure of delta prime union a delta and delta prime to union of the closures, but then you just take out you just take out this line away and then take out this arc away.

So, this is this, it is this set minus take out all the. So, this line is the set of all tau is the real part of the tau is half. So, what you do is you take out set of all tau such that tau in U such that real part of tau is half and of course, the imaginary part of tau is greater than equal to root 3 by 2.

So, this point will correspond to a complex cube root of unity. So, minus 1 plus I root 3 by 2. So, the Y coordinate is root 3 by 2. So, real part of tau is half imaginary part of imaginary part of tau is greater than or equal to root 3 by 2. So, that is to delete this whole line. So, in particular I also deleted this point and then I also want to delete this arc of the unit circle, but I would like to retain this point i. So, I also from, so I also take away from this union I am taking away this and I am also taking away this union the other one is this arc which is the set of all e power i theta with theta varying from theta greater than pi by 2 and lesser than or equal to 2 pi by 3 so, but of course, one point is common, but anyway let me write that pi by 2 is less than theta is less than or equal to 2 pi by 3. So, that I wants to taking away this arc.

So, finally, what I get is well this should be real part of tau is minus half correct I want to take off this line thanks. So, is real part of tau is equal to minus half right it is this line it is this, it is this ray that I want to take off thanks. So, you see this is what I want to. So, I take this out. So, see the net effect is you know after I take it out, if I take the region along with the boundary see what I will get is. So, let me kind of you know make this a dotted arc so that and let me also make this dotted so that you know it just to tell you that that I thrown out this, this ray and also this arc.

So, this whole thing is D. So, if you, if I draw it I mean let me draw it once more the region that I am worried about is a following may be I draw it, well may be I draw it here no matter. So, it is the following region. So, this is well this is half, this is 1, this is minus half and this is minus 1 and here is line and region I am worried about is then this (Refer Time: 12:04) is dotted then I start from here then it go here this is not here and this is my D, this is what D is, this is your this is the region D. This is a region D and, what I just trying to tell last time is that to prove the injectivity of J is more difficult.

So, and I said that the first thing is to show injectivity of J tilde in fact, bijectivity of J tilde when restricted to D, so in fact, I said I said the following thing I said we want to prove. So, let me write it somewhere here we want to show that J tilde restricted to D is bijective among other things, but before if you see further let me make a few remarks about what are called as fundamental regions for mappings and for groups.

So, let me make a definition.

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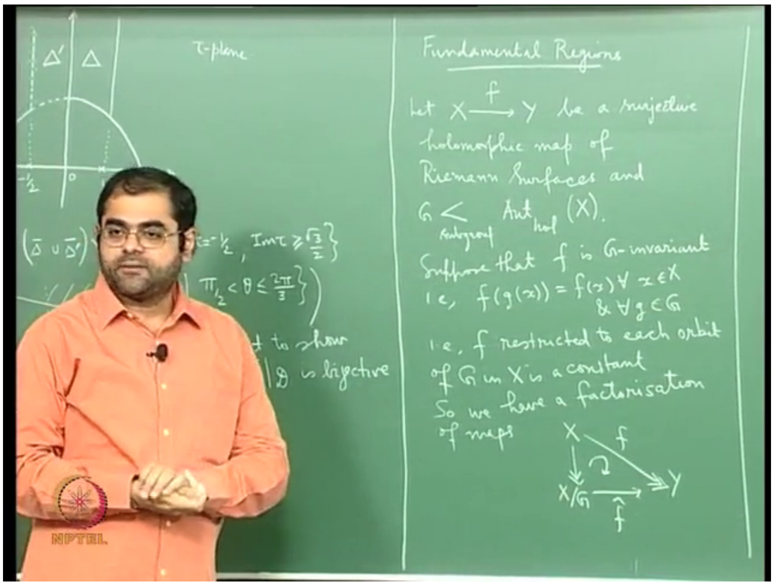

So, you see fundamental regions. So, let me explain what fundamental regions are. So, at least in the context that is relevant to us. So, you see suppose X, suppose you have X to Y you have map f is a holomorphic map be a holomorphic map of Riemann surfaces suppose is a holomorphic map of Riemann surfaces and G a subgroup of the holomorphic automorphisms of the Riemann surface X. So, X is a Riemann surface, Y is a Riemann surface f is a holomorphic mapping and suppose I have this group G which is set in subgroup of holomorphic automorphisms of the source Riemann surface and let us assume that f is surjective. So, let me put that. So, I let me put is a surjective.

Suppose that f is G invariant suppose you have a G invariant map, so that is you see f of g of x is equal to f of x for all x in X and for all for every g in G. So, this is saying that f is G invariant is a same as saying that f is constant on G orbits, it is a same as saying f is constant on G orbits f restricted to every orbit of G is constant. So, that is f is f restricted to each orbit of a G in X is a constant, suppose you have a G invariant map alright

Now, of course, you know if you have a G invariant map like this it means that this map factors through the set of orbits. So, we have a factorization of maps. So, you have X well X to f is from X to Y and you have got X to X mod G, X mod G is the set of orbits of G in X and this is a natural map I put a double arrow because it is just suggestion on to set of orbits and well you have also f is surjective. So, the fact that f is constant on orbits means that f goes down to a map like this. So, maybe I can call this as f hat if you want. And of course, in good situations X mod G could be a Riemann surface, we know already that you know.

For example, if G is acting properly discontinuously on X then X to X mod G also Riemann surface I mean then X mod G is a Riemann surface. So, that X to X mod G is natural projection is holomorphic and there in that case f hat will become also a holomorphic map, but in general f if X mod G does not have if actually of a Riemann surface that is if G does not act nicely then this is you just think of it as a set. I am just saying that saying that f is G invariant is the same as saying that f it goes down to a map from set theoretic map at least from X mod G fine.

So, given such a map what is it we are interested in we will call by a fundamental region for that map f what we mean is an open it is a region in a X, namely it consist of an open subset of X. So, open connected subset a domain an open connected subset of a X together with the part of the boundary. So, it is not open the interior is open, but you have added part of the boundary such that f restricted to that set is actually surjective and you see the translates of that set by G cover all of X.

So, you see, so in other words you see. So, let me write this down.

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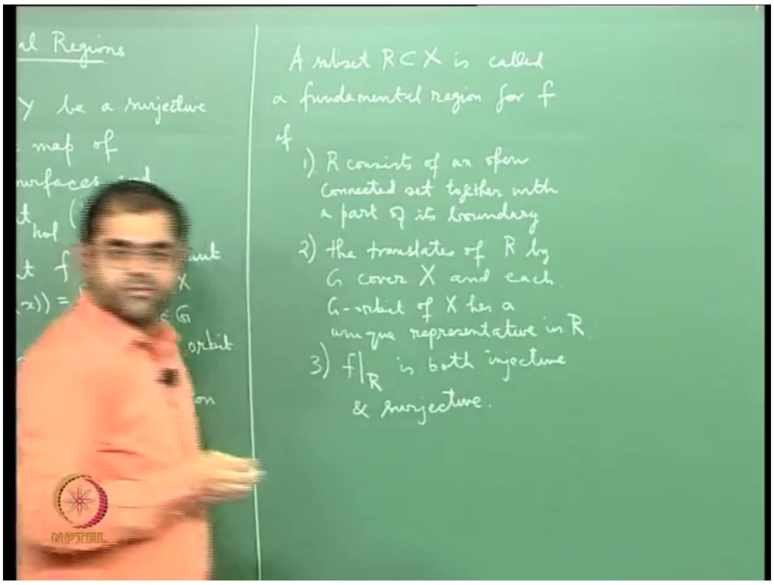

So, a subset R of X is called a fundamental region for f if number one well. So, let me write the following R consist of a R consist of an open connected set, open connected set together with a part of its boundary that is the first condition.

The second condition is the translates of a R by G cover X. So, this is the condition, that this is the condition that you know every G orbit hits R. If the translates of R by G cover X what you are saying is that if you take the orbit of every point in R and take then it should you should get all the points of X; that means, every orbit essentially has a representative in R, every G orbit in a X has a representative in R and I will put the extra condition that that representative is unique.

So, the translates of R by G cover X and each G orbit of X has a unique representative in R of course, when I say if I just say each G orbit of X has a unique representative in R that that already means that the translates of R by G cover X, but this is more I want only one representative right. And the third, the third thing, so this is the first thing. The third thing is f if I take f and restricted to R mind you that R is not a, it is not a open set or a close set because it is an open set the interior of R is of course, an open set. But there is a part of the boundary that is added. So, f restricted to R for example, you cannot talk about f restricted R being holomorphic because R is not an open subset, but the point I want to make is f restricted to R should be injective it should be injective as well as surjective. So, f restricted to R is both injective and surjective.

So, these are the conditions that define what is meant by the fundamental region for a map alright and. So, you see let me try to give examples of this.

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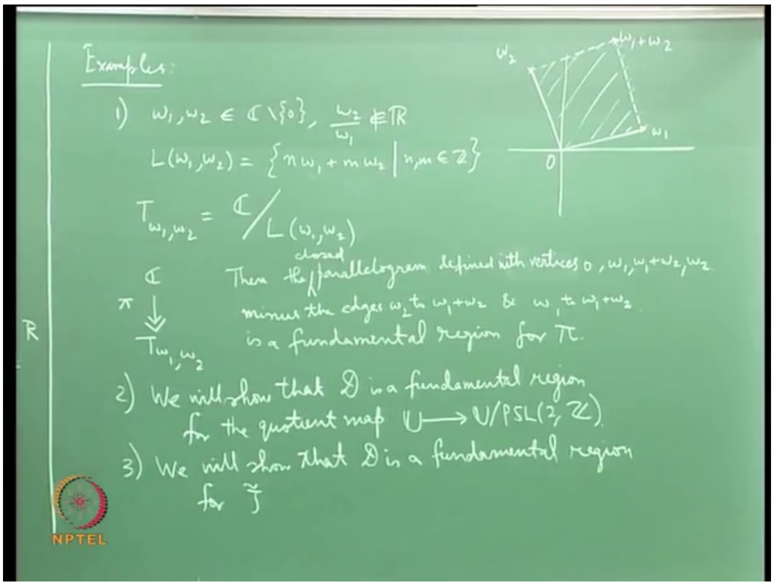

So, if you take. So, examples are well number one you take omega 1 omega 2 are two complex numbers, non zero complex numbers such that omega 2 by omega 1 is not a real number. Then you take the lattice you take the lattice L of omega 1 comma omega 2 which is lattice consisting of integral combinations of omega 1 and omega 2 lattice generated by omega 1 and omega 2. So, it is in omega 1 plus m omega 2 where n and m very (Refer Time: 23:50) integers then you know that I can define a torus using this lattice and namely I take I take T omega 1 comma omega 2 to be complex the complex plane modular this lattice. So, this gives me the complex torus that is defined by omega 1 and omega 2.

And if you now look at if you look at this map you see to T omega 1 comma omega 2 if you look at this let me call this map as pi the natural projection map the quotient map of course, you know that this is a. In fact, universal holomorphic universal covering with fundamental group of the torus being identified with the subgroup of mobius transformations that I given by translations by elements of the lattice

You of course, know that and the bind I want to make is that the if you take a portion of the fundamental parallelogram that is generated by omega 1 and omega 2 then that is a fundamental that is a fundamental region for this map this a holomorphic map it is the surjective holomorphic map. So, you see this is a surjective holomorphic map is to Riemann surfaces. The L omega 1 and omega 2 can be thought of as a subgroup of holomorphic automorphisms of see namely translations you think of every element of here as a translation by that element and then if you take of course, this map is a constant on orbits because it is actually the quotient map. So, it is constant. So, it is invariant and real omega 1 and omega 2 it is constant on orbits by this action. And for this map the fundamental region is part of the fundamental parallelogram you throw away to adjust into edges of the fundamental parallelogram and you keep the rest, then that will be a fundamental region, then the parallelogram defined by edges 1, omega 1, omega 1 plus omega 2, omega 2.

So, if I draw diagrams, well, here is when if this is omega 1 and that this is omega 2 then you get this parallelogram here this is omega 1 plus omega 2 and you take the parallelogram with edges 0 omega 1 omega 1 plus omega 2 and omega 2 and what you do is you take the whole interior. And take for example, take throw away this edge and this edge then that is the fundamental region.

In fact, let me say closed parallelogram, the closed parallelogram defined by these edges defined by the not edges I should say vertices defined with a, defined with vertices the closed parallelogram defined with vertices which means I take all the interior parallelogram and take also the edges the boundary and then throw out to two quoted minus edges. Namely for example, throw out the edge containing this edge and this edge minus with minus the edge omega 2 to omega 1 plus omega 2 and omega 1 to omega 1 plus omega 2.

So, you take this. So, you throw away. So, you throw away this. So, you throw away this and mind you when throw away this this point is also thrown out and you throw away this edge so this point is also thrown out. And then you take you take this, region then this is a fundamental region for the map pi, is a fundamental region the map pi because you see first of all you will see that a every every orbit has a unique representative here that is clear.

The second thing is that this map restricted to this is both, the map is restricted that is both injective and surjective and if I take translates of this by the lattice I will cover the whole complex plane. So, all the three conditions are satisfied, all the three conditions are satisfied. So, this is the fundamental that is a reason why it is called the fundamental parallelogram.

But of course, you want to really make it a fundamental region you have to throw out. So obviously, you know one is throwing out this because you know if I include this then for every point here there is another point there which is another representative. So, I throw out this edge. So, that I get a unique representative for points here and I throw out this edge. So, that I get a unique representative points here and of course, the lattice point will have all the 4, all the 4 points will be equivalent and relatives. So, I want only one of them. So, but I already thrown out these 3, I get only one fine.

So, this is one thing and you see if you look at; if you go back and if you go back to this. So, let us go back to this diagram let us go back to this diagram these also Riemann, this is the Riemann surfaces, this is the Riemann surface, in this a holomorphic map. And well you see the of course, what is acting on top is PSL 2 Z which is a subgroup of holomorphic automorphisms of U which is the all the holomorphic automorphism of U or PSL 2 R and PSL 2 Z which is a subgroup is a unimodular subgroup and the fact is for this map this is surjective holomorphic map for this map the fundamental region is actually this D that is a theorem that we are going to prove.

So, I am just trying to refresh everything in terms of that language. So, what I am trying to say is that sorry. So, the second and third statements I am going to write down actually claims we are going to prove them. So, a second statement is that this script D is a fundamental region for this quotient map. Two, we will show that script D is a fundamental region for the quotient map U to U mod PSL 2 Z we will prove this, this has to be prove and putting it as an example, but we have to prove it. So, that concern this map.

On the other hand look at this map, this is also a map between Riemann surfaces this is also holomorphic and surjective and again here I have this subgroup PSL 2 Z which is subgroup of holomorphic automorphisms of U and J tilde is PSL 2 Z invariant that claim is for J tilde the very same region D is again a fundamental region.

So, 3, we will show that D is a fundamental region for J tilde and once you prove these two it will fallow immediately that it fallow immediately that J is surjective, J is an isomorphism J is see because you see, you can think you can think about it for a minute you see if there a two points here. So, first of all J is already surjective I have to only show injectivity if I have two points here which are map by J to the same point then I can find two representatives there which are map by J tilde to the same point, but then those two representatives I can move them by PSL 2 Z into D because D is a fundamental region and its translates cover the whole upper half plane.

So, I will get two points here I will get 2 points here and J tilde taking the same value, but J tilde, but you see script D is also a fundamental region for J tilde. So, J tilde restricted to be D, D has to be injective so that will tell you those two points are actually equal and that will give me the infectivity of J. So, in that language, this is the language we are going to use to prove that actually J is a holomorphic isomorphism, fine.

So, let me having given you this language. Let me first go and show the first the second statement namely that U that script D is a fundamental region for PSL 2 Z for the unimodular group. So, in this definition I want to say something I just want to say that you know if you apply this definition to the case of a quotient that is for this quotient map when this quotient is a Riemann surface. If you get a fundamental region for this map then that is also called the fundamental region for G. So, the fundamental region for subgroup of holomorphic automorphisms of a Riemann surface is defined if it is exist it is defined to be D fundamental region for the quotient map with then you will have to assume that X mod G is also a Riemann surface in this is a holomorphic map.

So, in other words statement two is actually in statement two we are actually saying that for PSL 2 Z script D is a fundamental region right.

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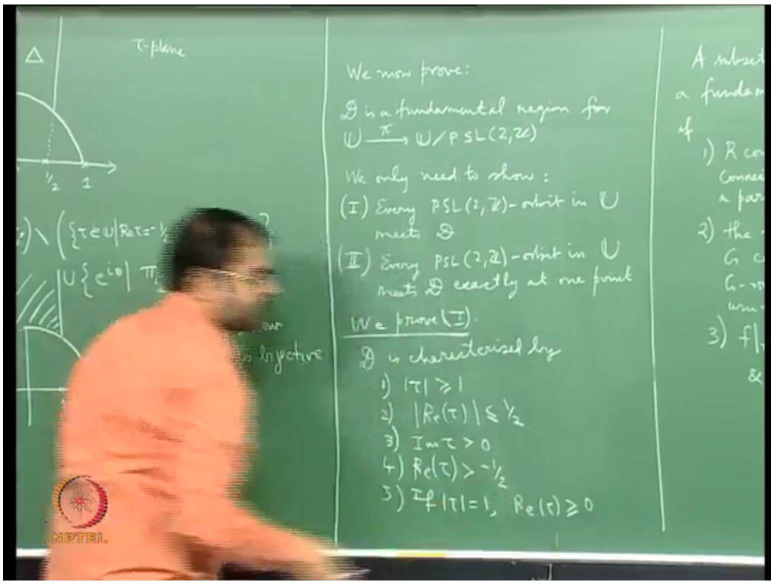

So, let me try to go to that we now prove script D is a fundamental region for U to U mod PSL 2 Z let me call this map as pi do not confuse this map with the pi in the first example. Let me call this map is pi we will proves.

So, the first thing that, so the first thing that I will need to show is that of course, the first condition is satisfied by D it is an open connected set together with the part of its boundary. So, D is an open connected set and what I have included apart from the interior is part of the boundary namely this arc of the unit circle fallowed by this ray.

So, it is, condition 2 is satisfied. Its condition 2 and 3 that I will have to verify. So, first thing let me verify that every every orbit of PSL 2 Z meets a script D. I will first prove that then I will prove that it makes in script D exactly at one point, we only need to show a one let me put roman letter one every PSL 2 Z orbit in U meets script D. Number 2, every PSL 2 Z orbit in U meets D exactly at one point.

So, of course, two is in, I mean the way I severed it to stronger than one I mean if I say every PSL 2 Z orbit in U meets the exactly at one point it also meets it meets it does meet D at one point, but essentially what I will to prove to what I will essentially prove is that if the orbit has two points in D then I will show the two points are equal. So, this is existence is uniqueness is existence is uniqueness.

So, how does one go about proving one? So, for this we will have to again go back and take a small detour into some properties as a lattice. So, you see, first of all let us write out what are the conditions that characterize the point of D. So, D is characterized by number 1 mod tau is greater than or equal to 1 it is, so this region is on exceed of the unit circle. Then the second thing is real part of tau lies between you know minus half and plus half, so it is here the of course, tau is in U. So, imaginary part of tau is greater than 0 right this is all in the upper half plane and then you see I want include I want include this arc and I want to throw, I want to throw out that arc and I want to throw out this line.

So, the condition is that slightly more restrictive condition in part two namely that the real part of tau is always greater than minus half. So, real part of tau is greater than minus half which means you know I could have written 2 and 3 together and set the real part of tau actually lies between is greater than minus half, but is less than or equal to half that throws out this line segment alright. And then I also want to throw out this arc. So, for that I will say that if mod tau is 1 which is the condition for tau would lie on the unit circle if mod tau is 1, then I will say that real part of tau is greater than or equal to 0. So, if mod tau is equal to 1 then real part of tau is greater than equal to 0.

So, these are the conditions that characterize when a point tau the complex plane is in D alright and what is that I have to prove I will have to prove that give me any point in the you start with the point tau now start with a point let us say tau 1 in the upper half plane I will have to show that the in the orbit of that point there is a point in script D.

So, I have to find a tau to which is translate of which is the image of tau 1 under PSL 2 Z element and sets the tau to satisfies all these conditions and I should this for any tau 1 in U.

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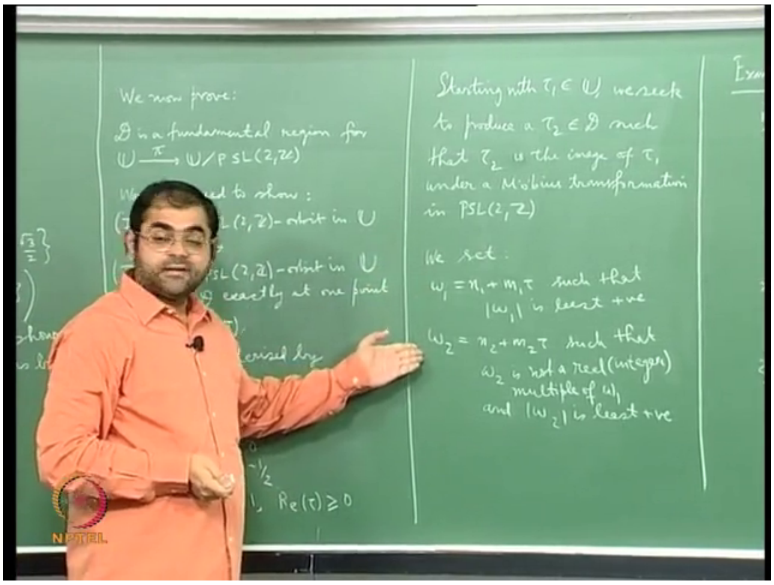

So, starting with tau 1 in U we seek to produce a tau 2 in D such that tau 2 is equal to tau to is the image of tau 1 under a mobius transformation in PSL 2 Z. Namely a unimodular transformation this what you have to do.

You start with the, Once you do this what you are saying is that. So, your just saying that the PSL 2 Z orbit of tau 1 which contains tau 2 intersects D in tau 2 this what you have to do. So, the point is that you have to you have to cook up this a suitable mobius transformation and so that the image of tau 1 under that unimodular mobius transformation is actually equal to tau 2 and that tau 2 lies in script D.

Now, so to beginning with to do that, so we begin as fallows. So, the first thing that we do is that we, you know this is, let me say something before I begin. So, you know this is on the same lines of the proof that I gave several lectures ago when I was trying to show that you know a discrete is at sub module of c is either 0 or it is generated by a single complex number non zero complex number or it is generated by integer multiples of two complex numbers with non real ratio. I proved this statement long back. And it is the proof is actually again an analysis of that proof and using that proof a little the ideas of that proof little cleverly.

So, we said such that mod omega 1 is least positive. So, you see, so what you do is that you look at that lattice generated by 1 and tau and you take the take a member of the lattice which is closes to the origin then there may be more than one, but you can take one of them. So, call omega 1 to be that alright. So, call this to be omega 1. Then what you do is now you look at all those elements in the lattices in the lattice which are not an real multiple of omega 1, namely not an integer multiple of omega 1 and among those choose omega 2 to be the closest one of to one, that is one that is closest to the origin. So, omega 2 is equal to n 2 plus m to tau such that omega 2 is not a real multiple real of course, it has to be integer multiple of omega 1 and mod omega 2 is least positive.

So, this is how I choose omega 1 and omega 2 and now the fact is that you should take the lattice generated by omega 1 and omega 2 then that lattice is exactly the same as L tau. So, what I have done is I have instead of taking the basis 1 comma tau for the lattice over Z I have found a new basis omega 1 comma omega 2 of the same lattice over Z.

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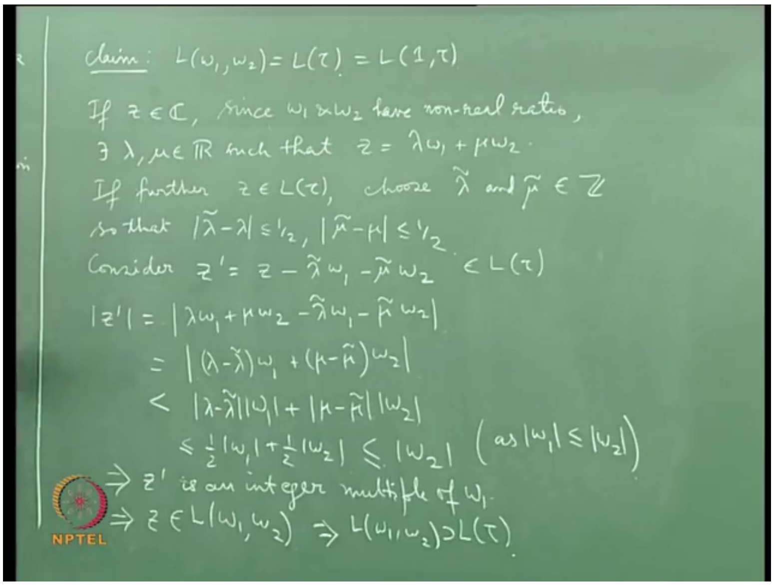

So, how does one establish that? So, the first claim is that the lattice generated by omega 1 and omega 2 is actually the lattice generated by tau of course, L tau is lattice generated by 1 and tau. So, the first claim is the lattice generated by, if you that is I am saying if you take all integer linear combinations of omega 1 and omega 2 you will get all integer linear combinations of 1 and tau of course, L tau is L 1 tau, L 1 tau.

So, how does one prove this? So, what one does is if Z is a complex number if Z is the complex number you see D, the first thing that you need to notice is that you know omega 1 and omega 2 they are not omega 2 is not a multiple of omega 1 by the varied definition because you are choosing omega 2 among the members in the lattice L tau which are not real multiples of omega 1.

So, you see omega 2 by omega 1 is a non real ratio. So, that is already that is already there in the definition here. So, these are two complex numbers one which is not a real multiple of the other therefore, these two complex numbers see it is they will form a basis for the complex vector space C over R over the real numbers, which means that any complex number can be written as a real linear combination of omega 1 and omega 2. So, since omega 1 and omega 2 have non real ratio that exist lambda mu in R such that is Z is lambda omega 1 plus mu times omega 2. You can write any complex number in terms of these two with real quotients because they form a basis of complex number of the field C over as a vector space over R.

Now, in particular if you take for Z and a member of L tau. So, if further Z belong to L of tau suppose Z is in L of tau alright choose lambda tilde and mu tilde integers so that the distance between lambda tilde and lambda is less than or equal to half distance between mu tilde and mu less than or equal to half. After all lambda and mu are real numbers and I am saying choose the closest integer to lambda and call it lambda tilde and choose the closest integer to mu and call it as mu tilde that could be two of them if lambda or mu happens to be midpoint of an integer an interval with n points integers fine.

So, choose this now consider Z prime to be well Z minus lambda tilde omega 1 minus mu tilde omega 2 look at this element, look at this element. Now, you see this element see zeta is in L tau and if you look at all these things notice that omega 1 is in L tau omega 2 is in L tau therefore, the lattice generated by omega 1 and omega 2 is certainly in L tau. The hard part that is the part that needs to proved is that L tau is contained inside this. So, this whole quantity namely Z prime is in L tau, this is in L tau and compute the modulus of Z prime if you compute the modulus of Z prime well and plug in for zeta lambda omega 1 plus mu omega 2.

So, I will get modulus of lambda omega 1 plus mu omega 2 minus lambda tilde omega 1 minus mu tilde omega 2, this is just modulus of lambda minus lambda tilde into omega 1 plus mu minus mu tilde into omega 2. And this is strictly less than modulus of lambda minus lambda tilde into mod omega 1 plus modulus of mu minus mu tilde into mod omega 2. This is actually less than or equal to by triangle inequality and you know in triangle inequality you will get an equality only when these two are one is a real multiple of the other which it is not, which it is not omega 1. It is not a real multiple of omega 2 they have, non real ratio therefore, it is a strict inequality in the triangle inequality. And well if you and you know now mod lambda minus lambda tilde minus lambda is less than or equal to half.

So, what it I will get is this is this is less than or equal to half mod omega 1 plus half mod omega 2, but you see the way I chosen mod omega 1 is a least and mod omega 2 is great greater than or equal to mod omega 1. So, this is less than or equal to mod omega 2 because mod omega 1 is less than or equal to mod omega 2 as mod omega 1 by every choice.

So, you see, now if you look at it I am I am having an element of L tau whose this element it is having length I mean it is having modulus lesser than mod omega 2 therefore, this element has to be here integer multiple of it has to be an integer multiple of omega 1, Z prime is an integer multiple of omega 1 that is what it means and see if. And now go back to this equation if Z prime is an integer multiple of omega 1 and you push Z to one side that will you that Z is in the lattice generated by omega 1 and omega 2.

So, I start with a Z in L tau and I have established that Z is in the lattice generated by L by omega 1 and omega 2 therefore, these equality (Refer Time: 54:45). So, this essentially shows that L omega 1 comma omega 2 contains L tau which was the hard point. This is already contained in L tau that it contains L tau based on here. So, they are equal. So, this claim is settle.

So, what does this tell you this tells you that omega 1 omega 2 is another basis for the over Z for the lattice generated by tau. Now, keep that in mind and well I will do one I will do something. So, what I can use it for reference later let me move all these conditions here it may write then here D is characterized by number one imaginate part of tau is positive mod tau is greater than or equal to 1. Number 2, I will write minus half less than or equal to strictly less than real part of tau less than or equal to half that combines this and this these two have combined as one and there is only one more condition mod tau equal to 1 implies real part of tau greater than or equal to 0.

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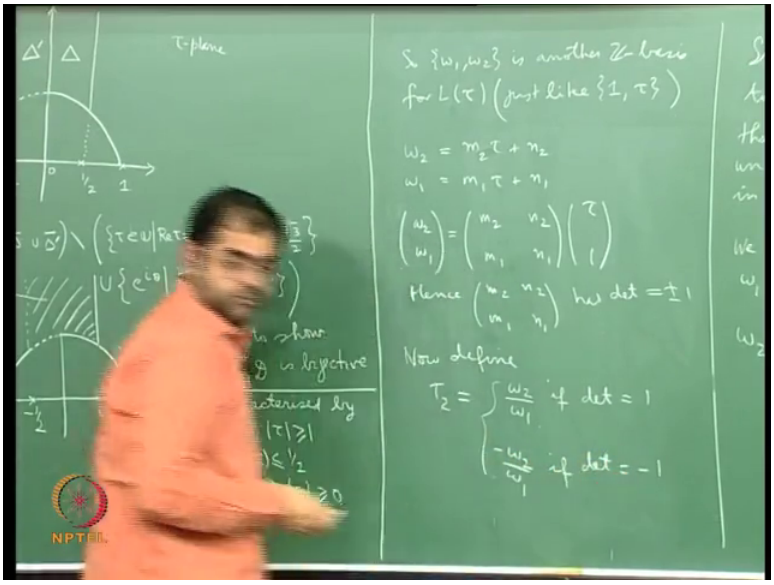

So, let me put that here. So, now, let us go back and look at what is written here. So, so you see, omega 1 comma omega 2 is another Z base for L tau just like, well 1 comma tau 1 comma tau is of course, a Z basis and what we have just shown is that omega 1 comma omega 2 is also Z basis.

Now, you just write this write this expressions for omega 1 and omega 2 see you will see omega 2 is m 2 tau plus n 2 and omega 1 is m 1 tau plus n 1 you can write it compactly as omega 2 omega 1 in metric form is equal to m 2 n 2 m n 1 tau 1, you can write it in this form. So, this would tell you that this matrix which will, this will give you a Z linear map from L tau to L tau this matrix will give you a Z linear map from L tau to L tau which is same as L tau is same as L omega 1 omega 2 and this matrix is mapping the basis one comma tau to the basis omega 2 comma omega 1.

So, the moral of the story is that this matrix is a Z isomorphism therefore, it is invertible in Z therefore, is determinant plus or minus 1. So, let me write that here hence m 2 n 2 m 1 n 1 has determinant plus or minus 1 because you see it is a matrix, see it is a map of L tau to L tau which is a Z linear map and it is taking a basis to therefore, is an isomorphism.

So, you see you can think about this we have use the similar argument earlier a specially we use such an argument to show how we should in fact we used to show that if you take tau 1 and tau 2, two different you know elements half the upper half plane then the complex tori defined by tau 1 and tau 2 are holomorhpically isomorphic if and only if tau can be moved to tau 2 by means of a PSL 2 Z element.

So, it is the same kind of reasoning that will have to essentially use here. So, this has determine plus or minus 1. And well, so what this tells you is that you know if, now, what we do is now define tau 2 to be. So, you do the following thing. So, what you do is you define it has omega 2 by omega 1 if the determinant above is 1 and you define it as minus omega 2 by omega 1 if determinant is minus 1 you make you define tau to in this way. Now, the claim is once you define it like this then my claim is. So, already the way we have defined tau 2, tau 2 is already the image of tau 1 under PSL 2 Z element.

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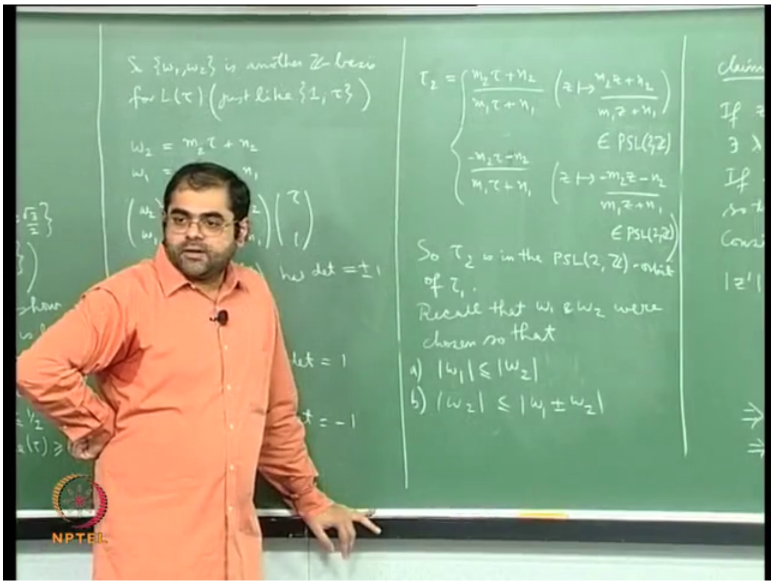

So, you see tau 2 is actually, tau 2 is in this case if the determinant is one it is well it is m 2 tau plus n 2 by m 1 tau plus n 1 with the mobius transformation is Z going to you know m 2 Z plus n 2 by m 1 Z plus n 1 thought of as an element of PSL 2 Z.

So, this is a first case when the determinant is 1. And or it may be you have to put a minus sign so that if you put a minus sign then the first row it is only the first row that gets. So, you have to observe the minus into the omega 2 or to omega 1. So, if you do that only one of the rows gets multiplied by minus therefore, the determinant gets multiplied by minus. So, if the determinant is minus one you will get plus 1. So, so it will be well minus m 2 tau minus n 2 by m 1 tau plus n 1 for example, which is where you know and Z going to minus into Z plus n 2 when this should be minus by m 1 Z plus n 1 this is a again in PSL 2 Z. So, these are the two cases.

So, what this tells you is that tau 2 is in the PSL 2 Z orbit of tau 1. So, tau 2 is in the PSL 2 Z orbit of tau 1. Now, the beautiful thing is that tau 2 is almost in D except that you will have to little bit work too very little work has to be done to get all these condition satisfy. So, I will explain that. So, we will have to show tau 2 is satisfy most of the conditions for being in script D. So, for that I will do a translation of a, I will have to again go back. So, you see it is a little involved I will have to again go back to the way in which omega 1 and omega 2 were chosen. So, recall that omega 1 and omega 2 were chosen.

So, that well mod omega 1 is less than or equal to mod omega 2, this is particular condition because we chose mod omega 1 to be the closest point in the lattice defined by tau to the origin and then we took away the although a lattice points on the line that joint omega 1 to the origin and from the remaining we try to we took omega 2 to be the closest point to the origin. So, this is a, this condition is already satisfied then, so this is a. And then there are two other conditions the condition is the other two conditions are mod omega 2 is less than or equal to modulus of omega 1 plus or minus omega 2. So, this is a second condition.

So, well I think this is fairly straight forward because you see if you take mod, if you take omega if you take omega 1 plus or minus omega 2 that is not, that is not a real multiple of omega 1. So, among those which are not a real multiple, which are not real, those among those members of the lattice L tau which are not real multiples of omega 1 omega 2 is of the smallest modulus therefore, this (Refer Time: 64:57). Now, this in terms of omega 1 and omega 2 translate everything in terms of tau 2 with this definition.

So, in terms of a tau 2 if you translate it what will get is a following. You translate everything in terms of tau 2.

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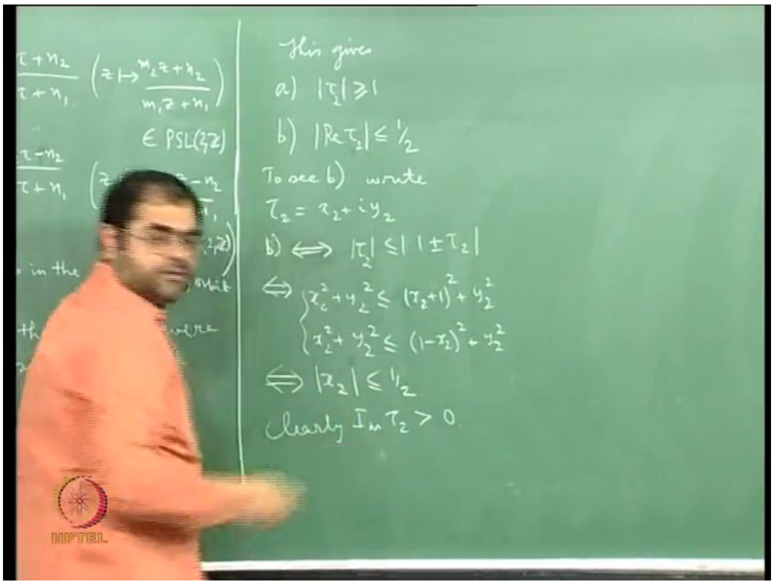

So, this gives I will call this as well a again. You will get mod tau is mod tau 2 is greater than or equal to 1 because tau 2 is either plus it is plus or minus omega 2 be omega 1 and mod omega mod tau 2 will be there therefore, mod omega 2 by omega 1 just greater than or equal to 1. So, you see already this, condition is satisfied. Then b, is well these two conditions are equivalent to real modulus of real part of tau less than or equal to half which is essentially this condition except that I need to do something to throw away the case when real part of tau is equal to minus half.

So, how does one I think? It is pretty easy to see essentially probably only it is just can you see in the following. So, this is tau 2. Well, how do you see this? To see b right tau 2 tau 2 as x 2 plus I y 2 right tau 2 is a x 2 plus y 2 and then you see b, b gives well if you divide throughout by mod if you divide throughout by mod omega 1 then you will get mod tau mod tau 2 is a less than or equal to one plus or minus mod tau 2.

Now, you plug in this if you plug in this you will get. In fact, b is equivalent b is of course, equivalent to this b is equivalent to this and that is equivalent to well x, well I square it x 2 squared plus y 2 squared is less than or equal to well I will get two things I will get let me take the first then I will get x 2 plus 1 the whole square plus y 2 squared and I will also get x 2 squared plus y squared is less than or equal to 1 minus x 2 the whole squared plus y 2 square this is what I will get. And of course, you know if I now expand everything out see I will get the x 2 square and y 2 square both sides will cancel the first one will give me two x 2 plus it will give me x 2 greater than or equal to minus half the second one will give me x 2 less than or equal to half. So, this will be equivalent to x 2 mod x 2 less than or equal to half and that is and of course, mod x 2 is mod modulus of real part of tau 2. So, it is fairly simple that these two translate to these two conditions.

So, I have got mod. So, I have got two of these conditions is only one problem namely I will have to ensure what happens when real part of tau is half and I also have to ensure that imaginary part of tau is positive.

So, well the first thing is you see the way I have written tau 2 tau 2 is a PSL 2 Z translate of an So, this I guess was I thought X one point is started with did we start with tau 1 or did we start with tau, tau 1. So, at some point I missed. So, this tau is supposed to be tau 1. So, I hope it you do not get confused. So, you know I started with that tau in the upper half plane I mean a tau in the upper half plane and I was trying to produce a tau 2 in script D which is in the PSL 2 Z orbit of tau 1, but somehow this tau 1 has changed to tau. So, please do not get confused.

So, this is the tau that I started within the upper half plane and I am trying to produce a tau 2 which is in the PSL 2 Z orbit of this tau and which is in script D. So, I am just trying to. So, you see since tau 2 is the image of tau under a PSL 2 Z element and fills tau is in the upper half plane the way I defined it tau 2 is already in the upper half plane. So, imaginary part of tau 2 is automatically greater than 0.

Clearly imaginary part of tau 2 is greater than 0 that is I mean that is a reason why we switch, we introduce the sign here then the determinant was minus 1. So, so the condition that imaginary part of tau is greater than 0 a is now satisfied for tau equal to tau 2 mod tau 2 is also greater than or equal to 1.

Now, I will have to look at only these two these two conditions. So, you see you do the following thing. So, let us look at this condition that. So, let us look at the last condition.

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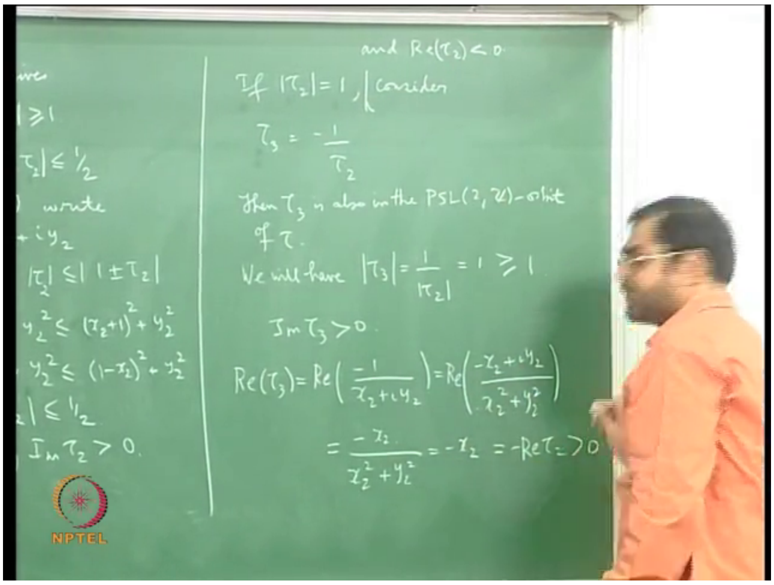

If mod tau 2 is 1, if mod tau 2 is 1 consider tau 3 to be minus 1 by tau 2. That is you are just applying another mobius transformation another PSL 2 Z element on the upper half plane.

So, we are just applying the transformation is Z going to minus 1 by Z which is a PSL 2 Z element. We are just moving it in the moving within the upper half plane and you consider this. Then tau 3 is also in the PSL 2 Z orbit of a tau because tau 2 is in the PSL 2 Z orbit of tau and tau 3 is the image of tau 2 under another PSL 2 Z element therefore, tau 3 also in the PSL 2 Z orbit of tau and of course, all these conditions that satisfied by tau 2 that so far we will also be satisfied by tau 3. So, we will have well mod tau three is equal to 1 by mod tau 2 and of course, I assume mod tau to is 1. So, this is 1 which is greater than or equal to 1. So, I have this condition mod tau 3 is greater than or equal to 1 of course, imaginary part of tau 3 is greater than 0 because it is in tau 3 is in the upper half plane because tau 2 is in upper half plane and this is also in the upper half plane then imaginary part of tau 3 is positive. And of course, you will have also and you see if you look at if you look at, if you calculate the real part of a tau 3 you will see that.

So, you see, real part of tau 3 yeah you have to just write it out this is going to be real part of tau 3 is minus 1 by tau 2 and tau 2 is X 2 plus I Y 2. So, if I calculate this I will get real part of well multiplying divide by X 2 minus I Y 2 I will get minus X 2 plus I Y 2 by X 2 squared plus Y 2 squared this what I will get this. And well this is going to be minus of a X 2 by X 2 squared plus Y 2 squared and that is minus X 2 because X 2 squared plus Y 2 squared is 1 mod tau 2 is 1 and this is going to be minus real part of tau 2.

So, you see well and this is going to be and this has to be greater than or equal to 0 it is it is it has to be greater than 0 if, you see I will have to see. So, let me explain something see for the condition 3, what I have to do is I will have to worry for the case when mod tau is 1 and real part of tau is negative. So, what I do is if mod tau is 1 and let me also and real part of tau 2 is negative then you do this then since real part of tau 2 is negative minus real part of tau 2 will be positive. So, I will modify tau 2 by a tau 3 by a PSL 2 Z element which will satisfy the condition that when mod tau 3 is 1 then the real part of tau 3 is positive.

So, the third condition is satisfy you know if tau is a I mean the tau 2 that you constructed or the tau 3 that you constructed the tau 2 that you constructed if it was on this line on this ray then all you have to do is just translate it by 1 and translation by 1 is also PSL 2 Z element. So, that is all you have to do. So, that, you can ensure therefore, that in the given any tau in the upper half plane there is a representative of its PSL 2 Z orbit in script D.

So, probably I should stop here and in the next lecture what I will try to do is I will try to show that there is only one such representative in script D and that is what will make script D into a fundamental region for PSL 2 Z. So, I will stop here.