An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1-dimentional Tori and Elliptic Curves

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Lecture – 41
A Region in the Upper Half-Plane Meeting Each Unimodular Orbit Exactly Once

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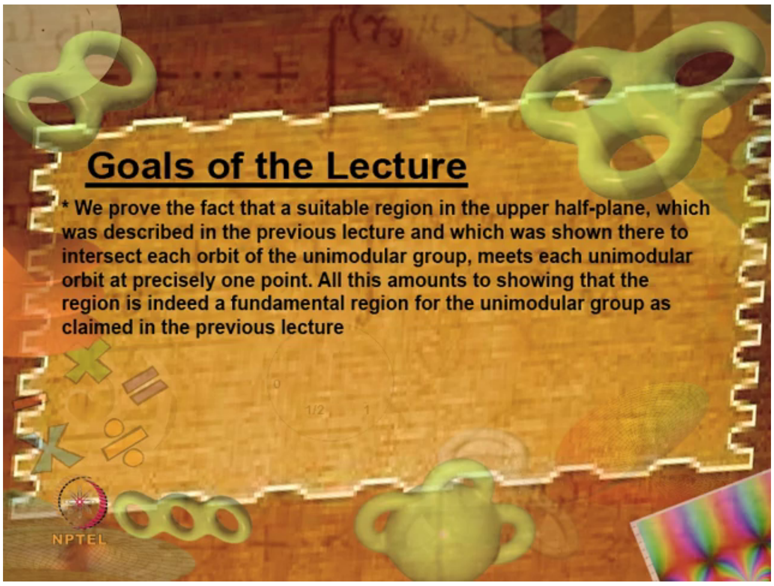

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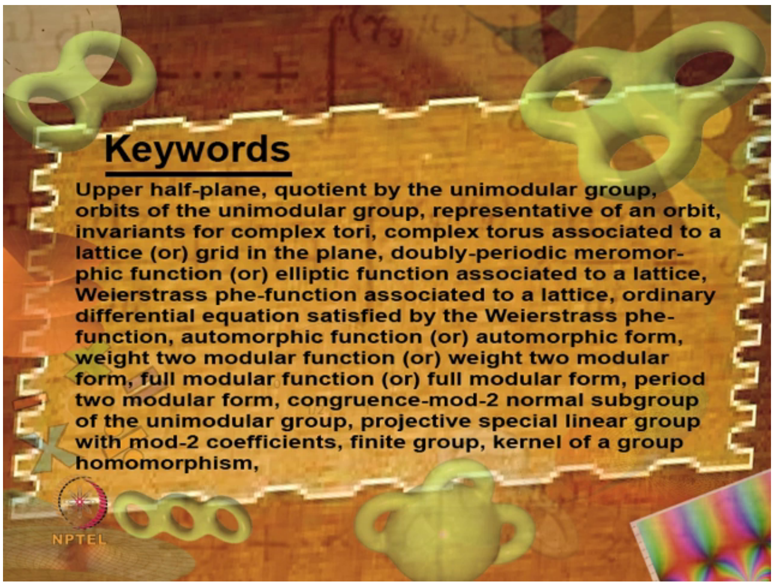

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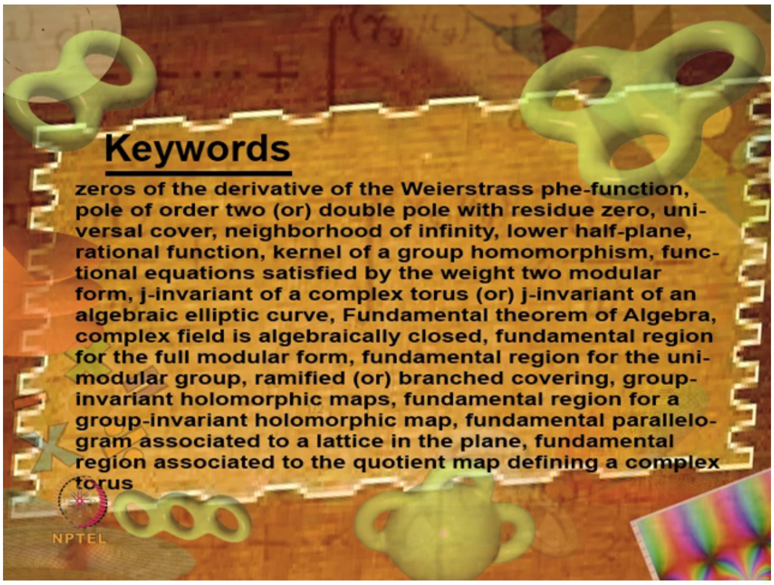

Ok. So, let me try to recall where we are at this point of our discussion. So, we have this region script d. So, let me draw that. So, you have well.

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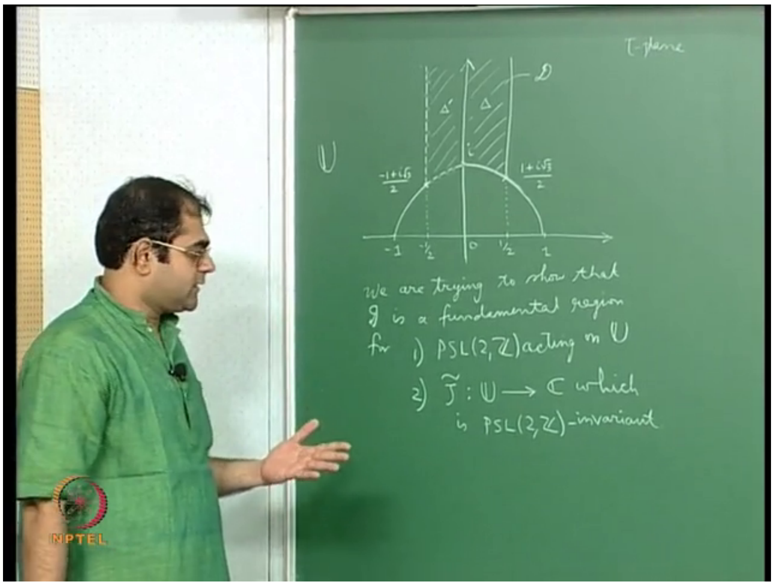

This is the tau plane and of course, this is the upper half plane and we take these points 0 this is the of course, this is the real axis this is the imaginary axis.

So, you have the point half and this is a point 1 and there is a point minus half there, is a point minus 1 and with 0 as the center, draws a draw a semicircle of radius 1 in the upper half plane. So, it will pass through i and. So, you have the semicircle and of course, we also take the vertical line passing through half, which is well you know going to be like this, and we have the vertical line passing through minus half we just going to be like this, and the region script D is actually composed of these 2 interiors. This is we called this delta, we call this interior is delta prime and we also add the boundary, but we take out the part of the boundary, we take out this vertical this ray and then we take out this arc of the semicircle, but we retain i ok.

So, this point. So, these 2 points are this point corresponds to 1 complex cube root of unity. So, this is. So, this point is 1 plus i root 3 by 2 and this is minus 1 plus i root 3 by 2 it is this point and script D is this shaded region and of course, we include we do not include this half line and this semicircular I mean this piece of the semicircular arc ok.

So, maybe I will just put I will just make it a dotted line. So, that. So, this is not included and this is not included, and then whatever we have whatever is shaded is he called it script D and the claim was we were trying to show that script D is fundamental region for the unimodular group PSL to Z acting on the upper half plane, and we also wanted to show that script D is also a fundamental region for the modular function J tilde ok.

So, let me write that down, we are trying to show that script D is a fundamental region; for number 1 PSL to Z the unimodular group acting on the upper half plane, and number 2 J tilde defined from the upper half plane to c which is PSL to Z to Z in variant. This is what we are trying to show and we are trying to show this because this will prove that the function that J tilde defines by passage to the quotient from U mod PSL to Z 2 c will turn out to be a bijectory holomorphic map and.

So, it will be a holomorphic isomorphism and so, our of course, our aim has been to try to show that the Riemann surface structure on U mod PSL to Z, is by holomorphic to the complex plane which has; the interpretation that the set of isomorphism holomorphic isomorphism classes of complex tori is identified with c as a Riemann surface, and the identification is done by a single invariant which is the J function, and the J function comes from passage to the quotient mod PSL to Z from J tilde ok.

So, this is what we are trying to prove. So, what we proved last time, we have already proved we have already shown.

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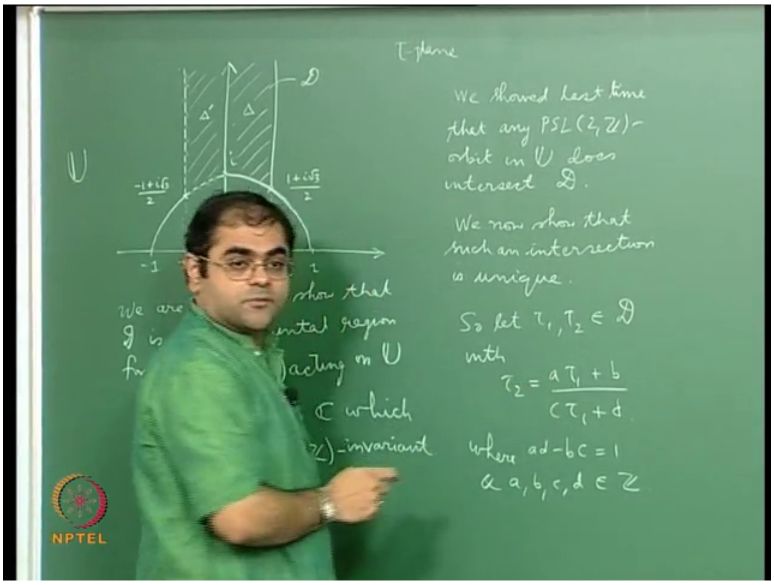

We showed last time last time that; what we showed was you take any PSL to Z orbit in U, it does intersect script D this is what we proved last time. We showed last time that any PSL to Z orbit in U, does intersect D what does it mean? It means give me any element in the upper half plane, and you take trance you take image translates of that that element by the elements of PSL to Z, that is apply on that element all possible unimodular mobius transformations, you will get various other elements in the upper half plane and all these elements put together will be the orbit of that element, and what we proved is that that orbit intersects D.

So, there is a point in D whose PSL to Z orbit is the same as any given PSL to Z orbit. So, this is what we proved last time. Now what we need to prove what we will try to prove next is that we will show that that intersection is unique namely that for every PSL to Z orbit there exists 1 and only 1 point of script D, where it intersects script D. So, this will tell you that the script D consists of a full set of representatives of the orbits of PSL to Z on PSL to Z on U. In other words it will show you that the restriction of the quotient map from U to U mod PSL to Z if you are restricted to script D it will be bijected ok.

So, let us try to prove uniqueness right. So, we now show that such an intersection is unique. So, how does one do that? So, we start with. So, what we will do is we will start with 2 elements in script D, and assume that one is the image of another by an element of the unimodular group. So, they will both be points in a single orbit, PSL to Z orbit which lie in d script D and then we will show that they are one and the same ok.

So, let tau 1 tau 2 be in script D and with tau 2 is equal to a tau 1 plus b by c tau 1 plus d, where ad minus bc is 1 and the abcd are integers. So, what are assumed is that this mobius transformation which is represented by the matrix abcd with determinant 1, and is thought of as an element in PSL 2 Z, when that acts upon tau 1 you get tau 2 this what I assumed. So, what I assumed is tau 2 is the image of tau 1 under that element of PSL to z.

Namely the mobius transformation defined by ez going to a ez plus b by c ez plus d and of course, the condition is ad minus bc is 1, and I have and abcd are of course, integer entries. So, that makes it that makes this an element of PSL to Z a unimodular transformation. What I will have to show is that, so; that means, the orbit of tau 1 is the same as the orbit of tau 2, the ally in the same orbit of PSL to Z and all I have to show is that I have to show tau 1 is equal to tau 2 .

So, we look at various cases and this and prove this. So, what will do is well yeah. So, yeah the first thing that that one needs to start with is. So, from this expression we will get an expression for imaginary part of tau 2.

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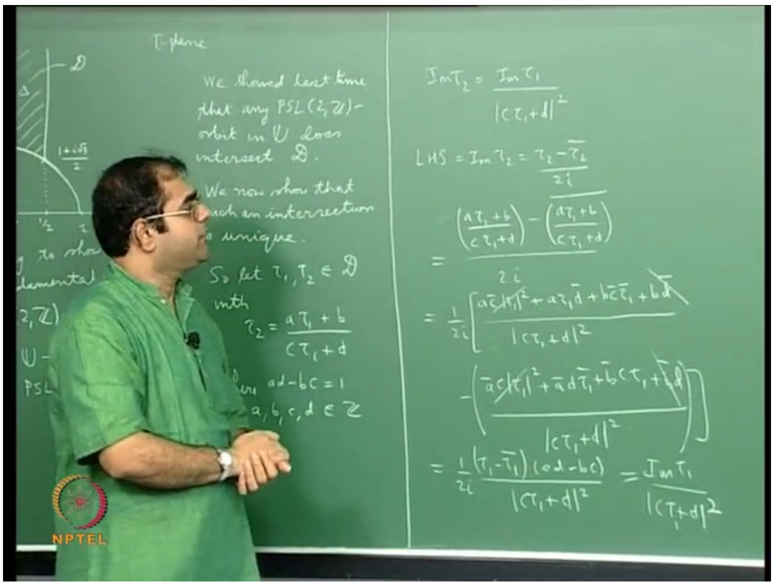

So, you see imaginary part of tau 2 will be well if you calculate it, it will be imaginary part of tau 1 by modulus of c tau 1 plus d the whole square you will get this expression. This is just by direct calculation, see because the left hand side is measuring part of tau 2 is tau 2 plus tau 2 minus tau 2 bar by 2 i.

So, this is going to be well plug in this expression for tau 2. So, you will get a tau 1 plus b by c tau 1 plus d minus a tau 1 plus b by c tau 1 plus d the whole bar by 2 i, this what you will get and that turns out to be well 1 by 2 i what I will get here is of course, I will get this complex number into its conjugates. So, in the denominator I will get mod C tau 1 plus d the whole squared this complex number into its conjugate and the numerator I am going to get a. So, if I calculate I will get a c bar mod tau 1 the whole squared. So, it is a tau 1 into c bar tau 1 bar ok.

So, it will be ac bar tau 1 tau 1 bar is mod tau 1 the whole square, then I will get a tau 1 into d bar, a tau 1 times d bar and then I will get b into c bar tau 1 bar, and I will get b into d bar, these what I will get when I multiply this with this and then I will get minus well minus same denominator, c tau 1 plus d the whole squared I have to multiply now this with this, I will get ca bar or a bar c into mod tau 1 the whole square, and I will get d a bar d into tau 1 bar and I will get b bar c tau 1 and I will get a b bar d this what I will get and well I think I can cancel out a few terms can I. So, I will have to make use of the fact that of course, I will have to make use of the fact that abcd are integers. So, they are real numbers. So, the conjugates of abcd are abcd themselves. So, you to make use of that and if you make use of that you can see that this term and this term will cancel right there is a minus sign outside here and similarly this bd and this is bd, this is also bd. So, that will also go and well I will be left with ad into tau 1 minus tau 1 bar, plus ad minus bc into tau 1 minus tau 1 bar.

So, I will simply get 1 by 2 i tau 1 minus tau 1 bar, into ad minus bc divided by mod C tau 1 plus d the whole squared, but of course, ad minus bc is 1 and tau 1 minus tau 1 bar by 2 i is imaginary part of tau 1. So, it is just imaginary part of tau 1 by mod C tau 1 plus d the whole square. So, that is how I get this right.

So, why is this? So, this is this helps us to reduce to certain case. So, you know of course, tau 1 and tau 2 are in the upper half plane. So, the imaginary parts are positive. So, I without loss of generality I will assume that say the imaginary part of tau 1 is say less than or equal to imaginary part of tau 2. I will assume that imaginary part of tau 2 is greater than or equal to imaginary part of tau 1 ok.

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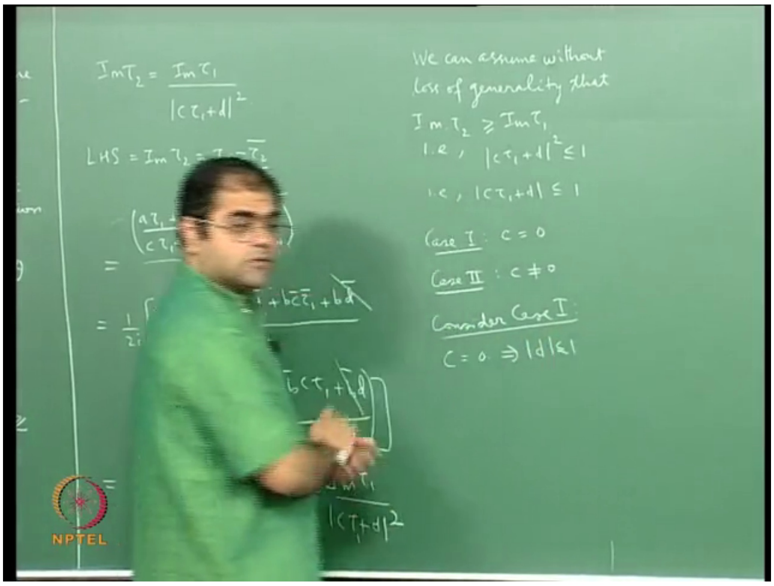

So, we can assume without loss of generality, that imaginary part of you know tau 2 is greater than imaginary part of tau 1, we can assume this. So, that I am. So, that amounts to assuming that mod C tau 1 plus d the whole squared is less than or equal to 1, from this expression it amounts to assuming that mod C tau 1 plus d the whole squared is less than or equal to 1 ok.

So, of course, what will happen if you imaginary part of tau 2 is, lesser than imaginary part of tau 1. So, you will just interchange the roles of tau 1 and tau 2. Interchanging the roles of tau 1 and tau 2 means you will have to write tau 1 in terms out of tau 2, and when you write tau 1 in terms of tau 2 you will just write the inverse of this mobius transformation, and the inverse will also be a PSL to Z elements and instead of abcd you will get some other coefficients a prime b prime c prime d prime, and that will also be a PSL to Z element and that will be the inverse of this ok.

So, there is no harm in assuming that imaginary part of tau 2 is greater than imaginary part of tau 1 there is there is no loss of generality. So, we assume this, now we will have to look at we have to look at this carefully. So, this is same as saying mod C tau 1 plus d is less than or equal to 1 and well. So, we look at various cases, we look at cases when a specifically at cases when c is 0 and c is not 0 ok.

So, let me follow whatever written down here. So, yeah so for first let us. So, we will assume case 1 c is 0 and of course, case 2 will be c not equal to 0. So, let us. So, consider case 1 c is equal to 0. So, if C is 0 what will happen is if C is 0 you will get mod d is less than or equal to 1 you will get mod d is less than or equal to 1.

On the other hand you know ad minus bc is 1. So, c is 0 will tell me ad is 1.

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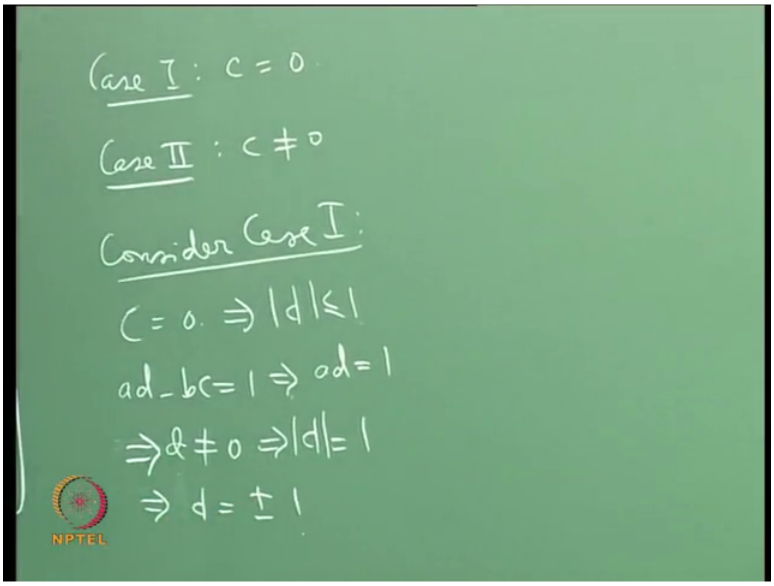

So, ad minus bc equal to 1 will tell me that c is because c is 0, it will tell me ad is 1, mind you ad is 1 therefore, d is not 0. So, d is a nonzero integer with modulus less than or equal to 1. So, it has to be modulus 1 it can be only plus or minus 1. So, this implies. So, this implies d is not equal to 0, and this implies that d is just mod d is just 1 this implies mod. So, it implies d is plus or minus 1.

So, this is what will get, and then what happens then well will have. So, if you look at this expression first of all I will get imaginary part of tau 2 is equal to imaginary part of tau 1. So, this will imply imaginary part of tau 2 be imaginary part of tau 1 you will get this and well you will also get what else.

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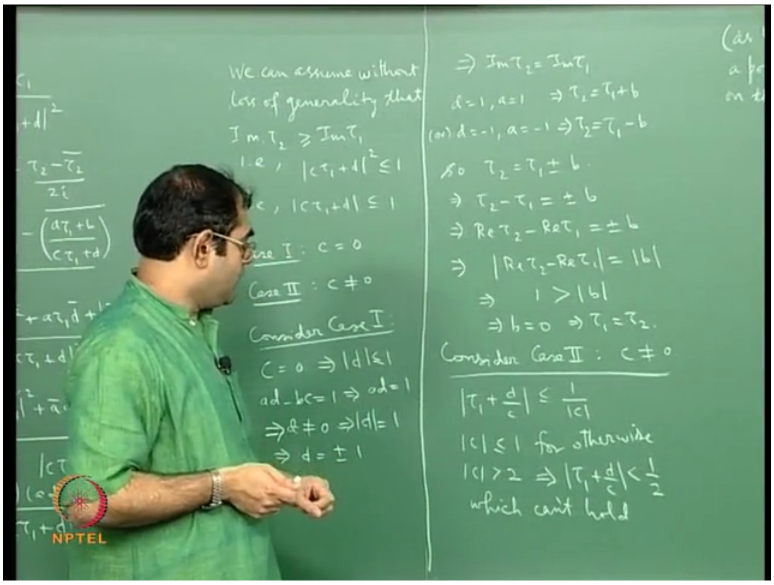

So, and I will also get here tau 2 yeah of course, I should also write here when corresponding to d equal to plus 1, I will get a equal to plus 1, when d is minus 1 I will get a is minus 1.

So, I will get d is equal to 1 a is equal to 1 or d equal to minus 1 a equal to minus 1 these are 2 cases I get; and well then what happens of tau 2. So, in this case tau 2 will be well if I put it here a is 1. So, I will get tau 1 plus b divided by c is 0 d is 1. So, I will get tau 1 plus b. So, I will get tau 2 equal to tau 1 plus b and in this case I will get tau 2 well if I plug in I will get tau 1 minus b because I will have to put a equal to minus 1 and I will have to put d also as minus 1. So, I will get tau 1 minus p.

So, what I will get is basically I will get tau 2 equal to tau 1 plus or minus b, this what I will get corresponding to whether d is plus 1 or minus 1, but this will tell me that tau 2 minus tau 1 is plus or minus b and, but then there is real. So, this will also tell me that real part of tau 2 minus real part of tau 1 is equal to plus or minus b, this is what you will get; and you see, but you see there is a restriction on the real part of tau and tau is in d there is a restriction.

So, you see in fact, that should tell you that b is actually 0, that is because you see the real part of any tau in the script D has to be greater than minus half, and less than or equal to half and you take 2 such tau 1 and tau 2 there, the distance between them the distance between the real parts has to be strictly less than 1. So, what this will tell you is that this will tell you that if I take modulus of this. If I take module I on both sides you see this side is strictly less than 1. So, the left side is strictly less than 1, and that will be strictly greater than the right side and b is an integer.

So, b is an integer with modulus strictly less than 1. So, it has to be 0. So, this will imply b is equal to 0 and if you put it back here you will get tau 1 equal to tau 2. So, that disposes of the case when c is equal to 0, that is case 1 all right. So, we will have to go to the case 2 which is c is not equal to 0. So, if c is not zero, then I look at this inequality and I try to then I can divide by c. What I will get is, I will get mod tau 1 plus d by c is less than or equal to 1 by mod C since c is not zero, I can I can divide by c by mod c, this what I will get.

And now again the claim is that mod C has to be less than or equal to 1 for otherwise mod C is greater than 2, will tell you that modulus of tau 1 plus d by c is strictly less than half. If mod C is less than or equal to 1 if it is not less than or equal to 1 that means, mod C is greater than 2, 1 by mod C is less than 1 by 2 and therefore, mod tau 1 plus d by c is strictly less than 1 less than half what does it say? If you look if you go back to this diagram it says that the distance of tau 1 from d from minus d by c. Minus d by c is on the real line, because d and c are integers nonzero integers ok.

So, I mean c is nonzero its a. So, distance of tau from the real line from a point on the real line it is less than half that can never happen, because you see the shortest distance of a point from script D to the real line is this and that is root 3 by 2, see this length is root 3 by 2. This is the shortest distance of a point of script D to the real line and therefore, you cannot have this inequality can never hold.

So, which is which cannot hold as.

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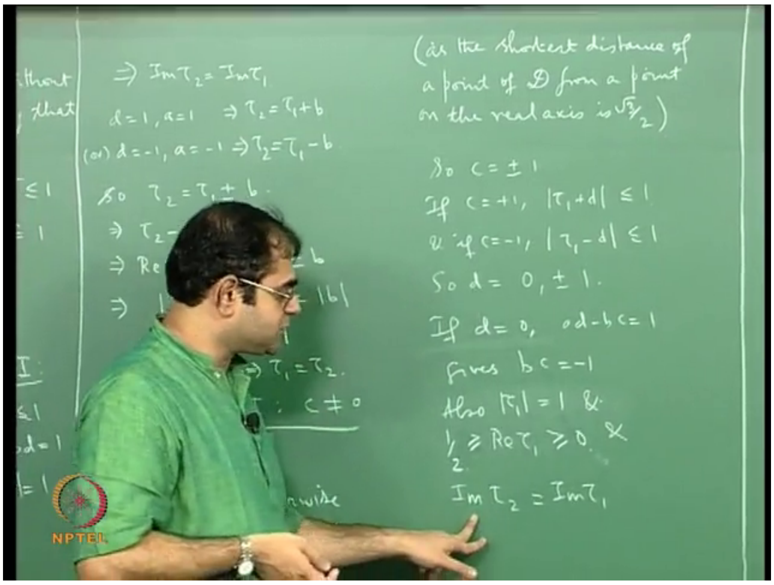

So, let me write it here as the shortest distance of a point of script D from your point on the real axis is half is root 3 by 2. The shortest distance is root 3 by 2 that is attained only at this point, not even at this point because this point has been removed. So, if they. So, this is the point which will have a shortest distance shortest distance would be root 3 by 2. So, this can never happen.

So, the moral story is mod C is less than or equal to 1. So, and c is c is of course, not 0. So, it means that mod C is 1. So, c is plus or minus 1. So, c is equal to plus or minus 1 this is what you get and. So, what will happen is you will get if c is 1, then you will get mod tau 1 plus d is less than or equal to. So, c is 1 is less than or equal to 1 will get this, and if c is minus 1, then mod tau 1 c is minus 1 and mod tau 1 minus d is less than or equal to 1 this is what we will get.

So, let us try to interpret this. What this says is the distance of tau 1 to the integer d is less than or equal to 1, and this says the distance of tau 1 to the integer d I mean. So, this is a distance of tau 1 to the integer minus d, that is less than or equal to 1 and this is this is of tau 1 from the integer plus d is less than or equal to 1. So, if you look at that if you look at that diagram, then see there are only there are only few possibilities for d. So, tau 1 is somewhere here, the first inequality says the distance of tau 1 from d is less than or equal to 1 ok.

So, see of course, d can be 0 or minus 1 or plus 1 all right there are there are 3 possibilities and of course, if and you know if d is minus 2, then the distance if d is minus 2 d is less than or equal to minus 2 or d is greater than plus 2, then the distance of d from tau is certainly greater than 1 therefore, the only possibilities are d equal to 0 plus 1 or minus 1 ok.

So, what I want to tell you is that. So, d is only 0 or plus or minus 1, there are only 3 possibilities and. So, will have to look at each of these cases of course, you know if. So, if we look at d equal to 0, then I will I can again use I can again use the fact that ad minus bc is 1. So, I will get ad minus bc is equal to 1 gives bc equal to 1, and well of course, d is 0 I will get bc is equal to 1 and sorry bc is equal to minus 1 and yeah of course, I should say before I said all this see if d is 0 see the condition is mod tau 1 is less than or equal to 1 ok.

And see mod tau 1 is less than or equal to 1 so; that means, tau 1 is in d and the distance of tau 1 from the origin is less than is less than or equal to 1. So, it means it is actually equal to 1, and it means that it has to lie only on this semicircular. So, that is a deduction that we have to do. So, you see. So, I should say here. So, even before I write this, I should say also mod tau 1 is has to be 1 and real part of tau 1 is greater than or equal to half and of course, real part of tau 1 is also supposed to be less than or equal to; no I what I want to say is real part of tau is greater than or equal to 0 and less than or equal to half yeah.

So, I should say it is greater than or equal to 0 and less than or equal to half. So, you see tau 1 is lying here on this on this semicircular arc it that is what it says, and I have to make use of this I have to make use of this you see c is. So, you know here is c is plus or minus 1 d is 0. So, this is just mod tau 1 the whole squared and that is 1. So, I will get imaginary part of tau 2 is equal to imaginary part of tau 1. So in fact, I will also get. So, that is the main point.

So, imaginary part of tau 2 is equal to imaginary part of tau 1 is what I will get. So, in this case we will essentially show that tau 1 and tau 2 are equal they are equal to i. So, we lets compute what is what tau 2 is.

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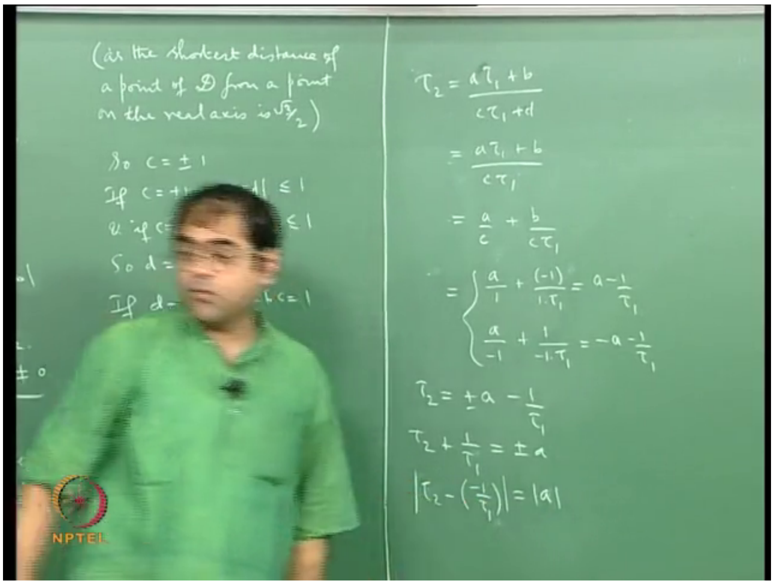

Tau 2 is by our assumption it is a tau 1 plus b by c tau 1 plus d and our d is d is 0. So, I will get a tau 1 plus b by c tau 1, which is a by c plus d by c tau 1 and this will turn out to be if I put the choices for c are plus or minus 1. So, I put c equal to plus 1 if I put c equal to plus 1 then b is minus 1. So, I will get a by 1 plus b is minus 1 by 1 into tau 1 and this is going to give me a minus 1 by tau 1 if I put c equal to minus 1, then I will get b equal to 1. So, I will get a by minus 1 plus 1 by minus 1 times tau 1 this is going to be minus a minus 1 by tau 1.

So, in all put together you get tau 2 equal to plus or minus a minus 1 by tau 1. So, tau 2 plus 1 by tau 1 is equal to plus or minus a now. So, you see the. So, interpret it as well if you take modulus of tau 2 minus 1 by tau 1, I will get mod a and this mod a is an integer ok.

So, this is a claim is a is actually 0, first of all minus. So, tau 1 is here, tau 1 is somewhere on this ark. So, minus 1 by tau 1 is therefore, on this piece of the ark and tau 2 is somewhere else and the maybe I should have taken before taking the modulus, I should have taken real part and then I should have taken modulus. So in fact, what I will do is instead of taking. So, what I will do is at this point, let me first take real part.

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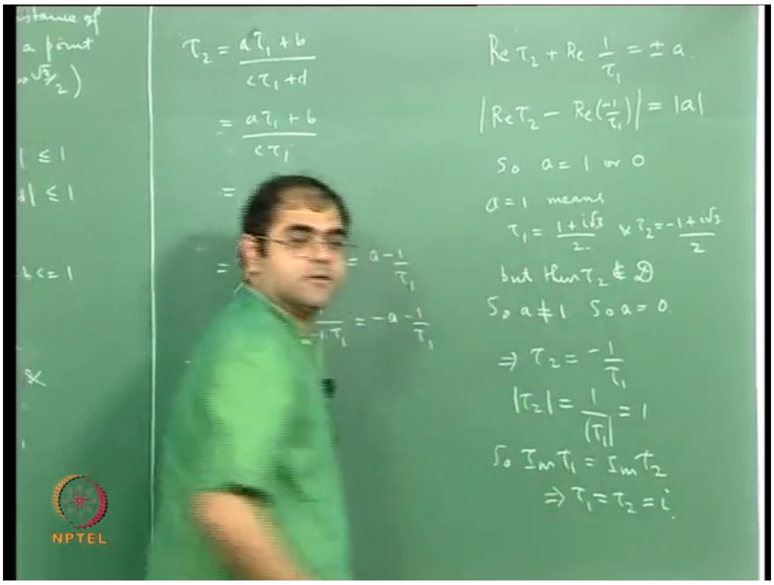

So, I will have real part of tau 2 plus real part of 1 by tau 1 is plus or minus a and now I take modulus I will get real part of tau 2 minus real part of minus 1 by tau 1 is equal to mod a.

And now you see. So, tau 1 is here minus 1 by tau 1 will be its reflection here, and tau 2 is somewhere in this region in script D and if you take the real parts and you want the distance between them to be equal to an integer all right and the therefore, the only possibilities are either the distance can be 1 or it can be 0. So, if. So, you see a is 1 or 0 and what I want to say is, I want rule out the case when a is 1 when if a is 1 then I will have they have see if a is 1 then they have to lie on these 2 vertical lines and 1 is here.

So, that means, and of course, I also have imaginary part of tau 1 of course, I also have imaginary part of tau 1 is equal to imaginary part of tau 2. I mean the only possibility is that 1 of them is 1 plus i root 3 by 2 and the other is minus 1 plus I root 3 by 2. A equal to 1 means tau 1 has to be 1 plus i root 3 by 2 and tau 2 is minus 1 plus i root 3 by 2, but then tau 2 is not in d because this has been removed from d.

So, a is not equal to 1. So, a is 0 and if a is 0 of course, this will tell you the tau 2 is minus 1 by tau 1 and you will also get mod tau 2 is equal to 1 by mod tau 1 which is one. So, you will get mod tau 2 is also 1 and combined with the fact that tau 2 and tau 1 have the same imaginary part will. So, imaginary part of tau 1 is equal to imaginary part of tau 2 will tell you that tau 1 is equal to tau 2 is actually equal to i ok.

So, that settles a case when d is 0. Now we will have to look at the case when d is plus or minus 1. So, the point is that when if the case in the sub case when c is not equal to 0, d equal to 0 corresponds to the tau 1 equal to tau 2 equal to i and d not equal to 0 which means d is equal plus or minus 1 is will correspond to tau 1 equal to tau 2 equal to 1 plus I root 3 by 2 that is what we have, that is what we will show essentially ok.

So, let me look at that case also.

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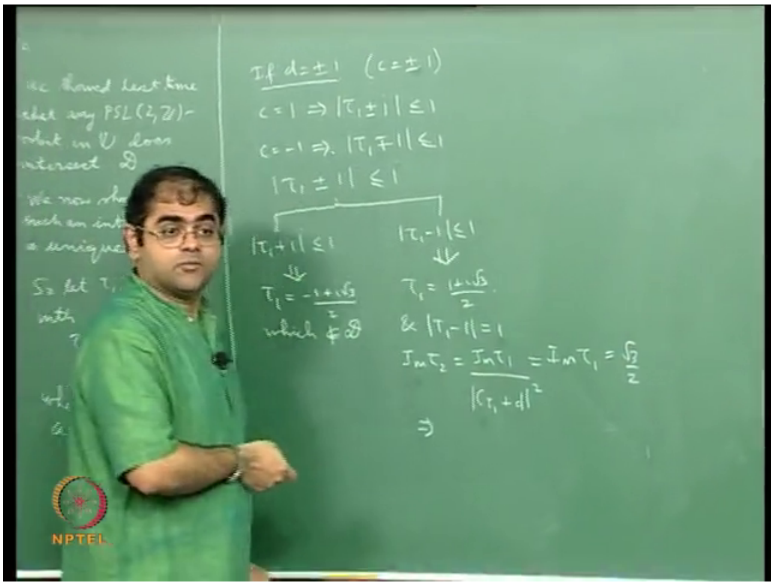

So, I will have to look at if d is equal to plus or minus 1 then I will have. So, I also have c is equal to plus or minus 1 and of course, I have both of this. So, if c equal to. So, if you go here, c equal to 1 corresponds to mod tau 1 plus d is less than or equal to 1. So, I will get c equal to 1 corresponds to mod tau 1 plus yeah. So, I will get mod tau 1 plus or minus. So, d is plus or minus 1 is less than or equal to 1 and c equal to minus 1 also will give I guess will give me the; c equal to minus 1 and also give me mod tau 1 plus or minus 1 is equal to is less than or equal to 1 ok.

So, in any case the condition I have to look at is mod tau 1 plus or minus 1 is less than or equal to 1. So, you think look at two possibilities mod tau 1 plus 1 is less than or equal to 1, and mod tau 1 minus 1 less than or equal to 1. So, what does this tell you? This tells you distance of tau 1 from minus 1 is less than or equal to 1 is at most 1.

So, here is my. So, tau 1 is there somewhere in d it is distance from minus 1, that distance has to be less than or equal to 1 well. Of course, you know if I draw a circle centered at minus 1 radius 1, it will pass exactly through this so; that means, that the only solution is tau 1 is this, but that is not possible ok.

So, this implies that tau 1 is minus 1 plus i root 3 by 2 which is which does not belong to script D. So, this case is not correct and if I take this case this, this is of tau 1 from 1 is less than or equal to 1. So, you know this means that tau 1 has to be 1 plus I root 3 by 2. So, this will give me tau 1 is equal to 1 plus I root 3 by 2 ok.

And then I will have to say that tau 2, I will also have to say that tau 2 is equal to tau one. So, for that what I will have to do is maybe look at whatever assumed in this case, if I look at the imaginary parts yeah. In fact, see if tau 1 is this then this quantity is actually equal to 1.

So, this implies that mod tau 1 minus 1 is actually equal to 1, and then now if you calculate the imaginary part of tau 2, is actually imaginary part of tau 1 divided by mod C tau 1 plus d, the whole squared and in this case I am going to get. So, mod. So, this is 1. So, this is exactly imaginary part of tau 1. So, what you will get is tau 2 and tau 1 they have the same imaginary part and the imaginary part is root 3 by 2. There are only 2 I mean the only solution to that is that tau 2 is also equal to 1 plus I root 3 by 2, because this is the only point in script D whose y coordinate is root 3 by 2. So, that will tell you that so in fact,. So, imaginary part of tau 2 is imaginary part of tau 1 which is equal to root 3 by 2, but there is only 1 tau in script D with imaginary part root 3 by 2.

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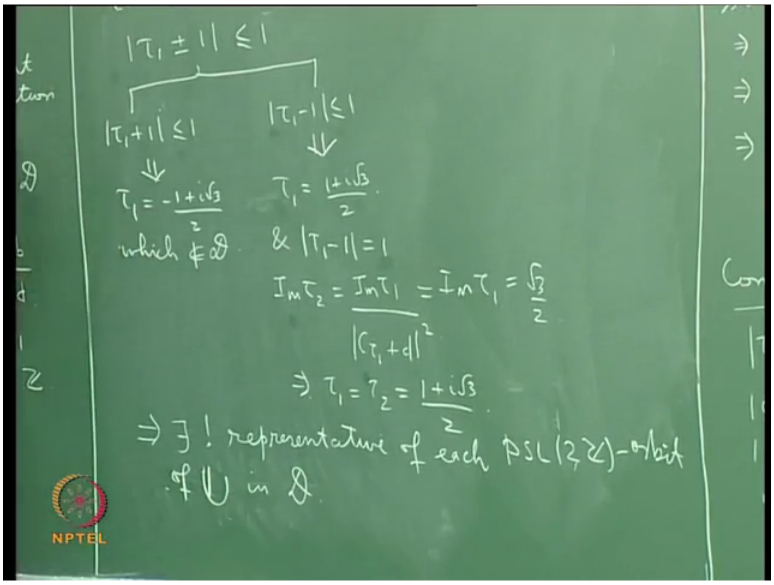

So, this will give you tau 1 equal to tau 2 is equal to 1 plus i root 3 by 2 .

So, as a result we have proved that there is only one orbit, I mean there is only 1 representative for each orbit in script D. So, what we have proved is that script D is a fundamental region for PSL to Z, the our definition of fundamental region was it should be an open set along with a part of the boundary which contains, exactly 1 representative of each orbit and such that the translates of this by the elements of the group, which is PSL to Z should cover the upper half plane that we have already checked all right.

So, this completes the proof that. So, this implies that there exists unique representative of each PSL to Z orbit of U of the upper half plane in script D. Now in other words script D is a fundamental region for PSL to Z all right. Now I want now the next step I will have to show is that for J tilde also script D continues to be a fundamental region all right.

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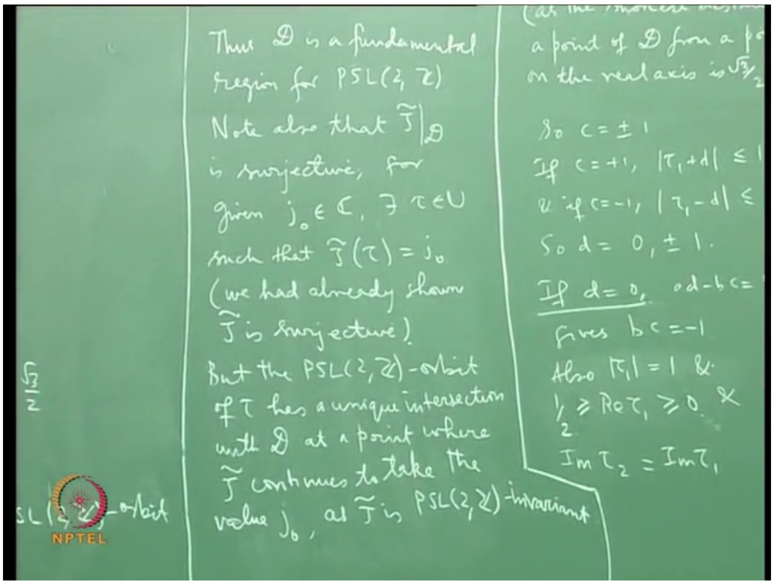

So, what one can tell immediately is that. So, you have to prove J tilde restricted to the script D is both injective as well as surjective; again they surjectivity part is immediate the hard part is injectivity part.

So, let me write this down yeah thus script D is a fundamental region, for PSL to Z note also that J tilde receipted to script D is surjective. See we have to prove that script D is a fundamental reason for J tilde. So, you have to show J tilde receipt to script D is both injective and surjective, I am saying surjectivity is immediate because you see why is that true that is because J tilde we know is already surjective. So, there is some and you if you remember that was proved using fundamental theorem of algebra and. So, J tilde is already surjective on the upper half plane ok.

So, there is a point given any value J tilde will take that value, at some point in the upper half plane, but then I can take the orbit of that point that will have unique representative in script D, and the value of J tilde there will still be the same because J tilde is PSL to Z invariant therefore, I can always given any value complex value, I can always find a point in script D where J tilde takes that value therefore, J tilde receipt to d. So, surjective ok.

So, let me write that down for given J naught in C, there exists tau in U, such that J tilde of tau is J naught, we had already proved shown J tilde is surjective, but the PSL to Z orbit of tau has a unique intersection with D at a point, where J tilde continues to take the value J naught as J tilde is PSL to Z invariant. So, the moral story is that we are able to show that J tilde receipted d is surjective immediately, the harder part is to show that J tilde receipted to script D is injective. So, we will do that in the forthcoming lecture.