An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 05
A Riemann Surface Structure on a Torus

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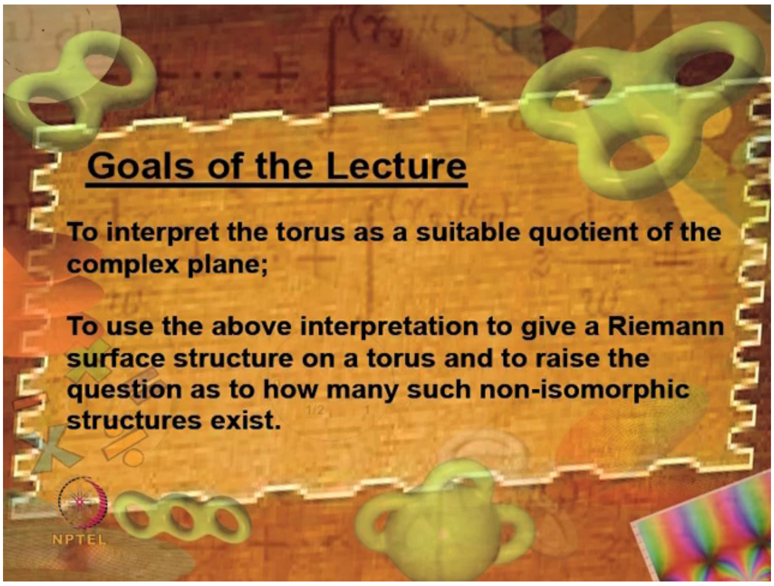

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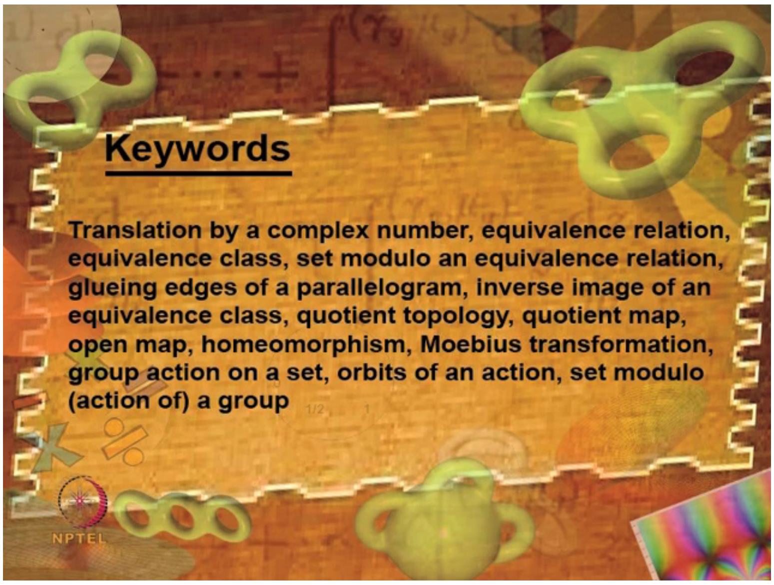

So, welcome to this fifth lecture on Riemann Surfaces and Algebraic Curves. So, let me briefly recall what we did in the last lecture: it was trying to give a Riemann surface structure on the cylinder.

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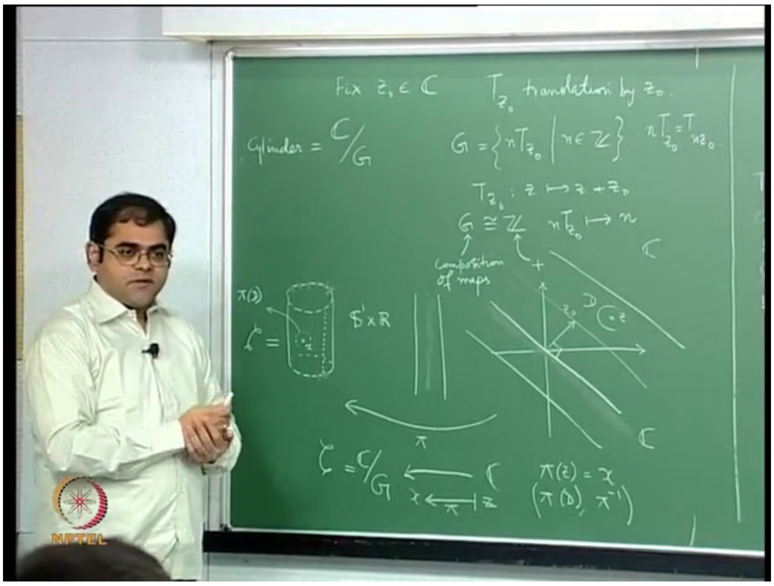

So, the way we did it was essentially we took the complex plane, and we went modulo the group of automorphisms of the complex plane, which isomorphic to the integers under addition, and for the generator of the group we took a translation by a fixed vector. So, what we did was we took we fix a complex a number z naught, and then we take the group then we denote by T sub z naught translation by z naught.

So, if you take translations by z naught and then you take 2 such translations, then you get translation by 2 z naught. So, in this way what you get is the set of all possible integer combinations of integer multiples of this translation, namely n times T z naught where n is an integer, this gives you a certain group and this each element of this group is a translation. In fact, you know that translation n times T sub z naught is a same as T sub n z naught because T z naught is actually the map z going to z plus z naught.

So, this group G is isomorphic to the integers and of course, under. So, when I say group here the question is under what operation, it is under the operation of composition of mappings. So, here we have composition of mappings and here it is under the usual addition. So, there is an isomorphism and this isomorphism is gotten by sending simply sending n times T z sub z naught to the integer n. So, this is just a group of translations by integer multiples of fixed complex number, and you go modulo of this group and you get the cylinder ok.

So, this is a cylinder at least topologically and then I told you that once you do this you can transform the cylinder into Riemann surface by giving a system of charts. So, diagrammatically the situation was that. So, if I recall that that is. So, here is my cylinder it is an infinite cylinder. So, I will put this is dotted lines to say that it extends in both directions. So, it us a it is basically s one which is a circle cross r, the r being thought of s z axis and the s 1 corresponding to the unit circle on the x y plane all right and what we did was, we cut across this cylinder and we obtained and an infinite strip like this we obtain infinite strip. And then basically since we wanted to have charts from small neighborhoods around a given point on the cylinder to the complex plane, we had to somehow think of this strip we had to somehow find the connection between the strip and the complex plane. So, essentially what we did was we just spread we just repeated this strip on the complex plane, then we realize that you repeating this trip do you see the complex plane.

So, basically you have the complex plane C and then if they are vectors if they are vector corresponding to the complex numbers z naught is this then we think of this strip as this strip. So, let me draw the cylinder carefully. So, its. So, this is this is ninety degrees if you want and. So, this is the strip here, and then you have various copies of this strip that that give rise to that essentially the union actually gives you the complex plane and the way to come back is essentially to identify 2 points on the complex plane, if they are translates by and it is a multiple of z naught and that is the same as actually going modulo the group G all right and that is how we got this complex structure on the cylinder making it into a Riemann surface.

Now the next point that we can ask is a well in fact, the complex structure. So, what did we do I mean if you choose a point lets use a point say x on the cylinder and what is the coordinate chart. So, remember a coordinate chart was required at the point x. So, that you can identify it with an open subset of the complex plane, and then you can transport the notions of holomorphicity from the complex plane to this point to the neighborhood of this point. So, what we did was given a point here we chose another point here in the complex plane let me call it z, and this map is precisely the if I may say the quotient map which I call as pi and this is the.

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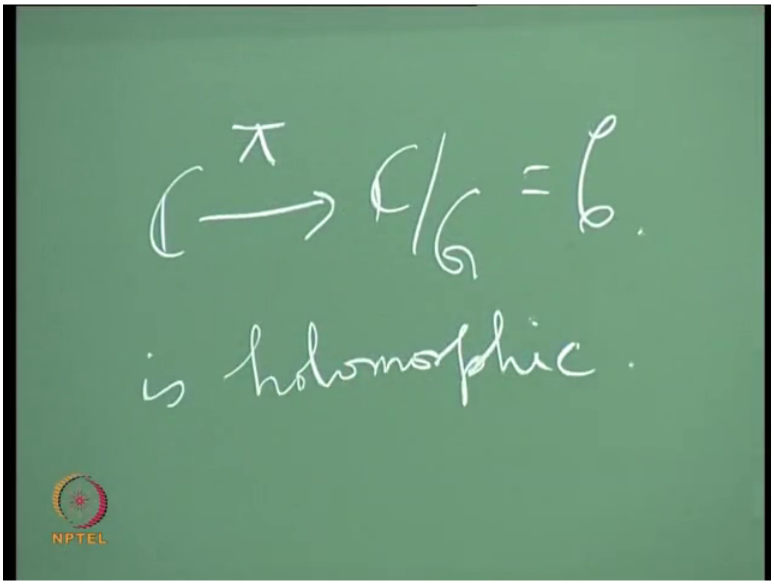

So, this is my cylinder here, I will denote it by script C and this map is just the map from C to C mod G which is the which is cylinder and which sends every point to it is equivalence class under G. Equals class under G means it is essentially the orbit of the point under G, which is a and what is a equivalence class it is simply the collection of all translates of that point ok.

So, you send a point to the collection of all of it is translates, which forms an equivalence class here and all these equivalence classes together as a set give you precisely the cylinder C script C. So, this is my map and given a point x on the cylinder, I choose a premade z and then here is my map pi. So, pi of z is x. So, given a point here I chose a point there and. So, let me write that down pi of z is equal to x, and then what we do is that you choose a sufficiently small disc surrounding this point, let me call that is D and of course, you choose the disc to be sufficiently small. So, that it is far smaller than this then the width of the strip which is smart z naught of course, I am not assuming z naught to be 0, I am really taking a nonzero vector here which a which is implied, but anyway let me say it.

So, that is a small disc here small enough disc, and if you take pi of that disc that is you take the image of this disc under pi, then you will get an open neighborhood of x. And the reason this is true is because pi is actually an open map pi is an open map it takes open set to open set. So, if I take a disc like this in fact, you take any open set to an open set, but I am taking a small enough disc because I want this map to be injective, because if the radius the disc is large then there would be many points which go to the same point here ok.

So, pi of D will be this open set surrounding x which is which looks like a disc and why does it look like a disc? Because if I take pi from D to pi of D that will actually be a homeomorphism because I can I it is an open map and it is injective and the inverse is also continuous. So, what I can do is that I can take my inverse, I can take the map from the I can take the inverse map pi inverse, which goes from pi of D to d and that will give me a homeomorphism of pi of D with an open subset of the complex plane namely D all right and this is going to this pair is going to give me a coordinate chart.

And then I explained that you can get charts like this at various points and so we have a covering of the cylinder by charts now of course, that is not completely enough to give you a Riemann surface structure, what we required is that D starts are compatible whenever they overlap and that is also a condition that we checked last time, we found out that the transition functions are precisely translations by a certain integer multiple of z naught and these are of course, by holomorphic maps. These are certainly injective holomorphic max the maps there they certainly holomorphic isomorphism.

So, since the transition functions holomorphic, you have compatibility of the charts. So, this is an atlas and once this atlas is prescribed the cylinder becomes a Riemann surface. And you can also see that the natural map from C to C mod G which is the cylinder this pi is a holomorphic map. So, these also something that you can see that is obvious because for a map to be holomorphic basically holomorphicity is a local property. So, I will have to check that this map is only locally holomorphic, but locally from a point on C to a point on the cylinder, I actually get the maps corresponding to the inverse of these pi inverse namely pi restricted to D and these are certainly holomorphic maps ok.

So, the moral of the story is that this is a holomorphic map and now you can again ask same question as we asked in the earlier cases, if you go back and look at the earlier lectures when we were trying to give Riemann surface structures on the complex plane, then I told you that there are only 2 possibilities; namely the complex plane and the unit disc. And the in the case of the Riemann sphere, if you ask the question is the question that how many Riemann surface structures you can give on the Riemann sphere. Then again the answer then the answer is that you can you only one up to isomorphic. There is only one on the on the real 2 sphere there is only one Riemann surface texture that you can give which is given by the Riemann sphere to be accurate ok.

So, no matter what atlases you use whatever Riemann surface structure that you impose on the real 2 sphere, it is going to be only isomorphic to the Riemann sphere the Riemann surface structure on the Riemann sphere. So, that is the case of the sphere and you can ask the same question for the cylinder. You can ask how many Riemann surface structures I can give can I that one that it is possible to give on the cylinder so that this map is holomorphic. And the answer to that is it there is only one up to isomorphism. So, of course, these are all facts that require further techniques for their proof, but I am just telling them because you get a flavor of what happens ok.

So, let me state this theorem, the set of holomorphic isomorphism classes Riemann surface structures on cylinder C is a singleton and it is represented by the natural Riemann surface structure on punctured plane C star, which is C minus large ok.

So, let me again explain the statement of the theorem. So, even before I do that let me again say what is that we are trying to think of; we have a surface real surface that we can imagine in three dimensions to begin with, and then we find various ways of turning it into a Riemann surface. And then the question is how many different Riemann surface structures is it that we can get. And the answer to that is in the case of cylinder the answer to that is there is only one. And what is that one? That one is there is a very special representative for that and that is a C star which is C minus large. So, roughly the if you want heuristically understand why this could be true, the reason is that I have already told you that this G is isomorphic to G. So, it is essentially C mod G the cylinder is exactly C mod G alright and if you know if you remember a little bit of complex frescoes in complex analysis, you know C mod G is actually C star.

Do you know how it is in the in the following exact sequence. So, you have 0 let me write this Z, C, C star 1. So, I will explain this notation. So, you see here is a complex plane and here is an map z going to e power let me say i z take the z going to e power i z then of course, you know that this map is surjective because every non-zero complex number has a logarithm. So, this map is surjective and in mathematical notation that is what it means to say that this whole sequence is exact at this point ok.

So, sequence of groups and group homomorphisms, you said to be exact at a certain point if the kernel of the outgoing map is equal to the image of the incoming map. So, if I want exactness at this point, then I the condition is that the kernel of the outgoing map should be the image of the incoming map. The kernel is outgoing map is everything mind you I have put one here is one here represents the multiplicative the trivial multiplicative group under multiplication, which has only the identity element for multiplication the trivial group, but with operation is multiplication and mind you this map the C star is also a group under multiplication.

And this map the constant map C start to this singleton group is; obviously, going to have kernel C star because everything is going to one. So, the kernel of this map is the whole of C star and the condition for exactness is that the kernel of this is the image of this, and that is the same as saying that the image of this is going to it has to be everything, which is true. That is what exactness means here and of course, this is convene the C here and the Z here are being considered as groups under addition, and of course, when you exponentiate the addition gets transformed into multiplication.

So, on this side the first three groups are all groups under addition and what about exactness at this point? The exactness at this point means that the kernel of this map has to be the image of this map. So, is C star identity element is 1, the only way e power i z is z should be a multiple of 2 and pi all right, but then let me normalize this by dividing 2 pi. So, instead of sending it z to e power i z, I let me send it to z to e power i z 2 pi. So, there is a small adjustment I have to make sure that I get integers here. So, let me put it as 2 pi i z ok.

So, what is the kernel of this map? The kernel, this map is by definition all the elements here which go to the identity element here, the identity element here is one. So, I want all these z as a e power 2 pi z is 1, and this will happen only if z is an integer. So, this is one if and only if z is an integer and; that means, that Z has to come from here. So, I am just saying the kernel of this is equal to image of this and that is the exactness at this point. So, let me write that down. So, we say that this is a short exact sequence of groups and. So, it is exactly at this point.

So, the kernel of this is the image of this and it is also exactly at this point, because the kernel of this is 0 which is the image of the 0 group here. So, this is a short exact sequence right and what this tells you is that C mod. So, there is a map the image C star kernel Z. So, what it tells you that C mod z is actually C star if you go by this map, if you go by this is the sequence C mod Z is just C star as groups, and that is exactly this is happening at the level of groups, but this exactly was also what happens in the level of Riemann surfaces. So, that is the point the point is that you take C modulo this group of translations which corresponds to a copy of Z. So, essentially you are looking at C mod Z and C mod Z C star and therefore, the cylinder the actually C star as a Riemann surface and by that I mean the cylinder is holomorphically isomorphic to C star right.

So, let me write a few more words. So, exactness here is let me call this is exp this map has exponential map his exponential is surjective and exactness here, here is due to the fact that e power 2 pi iz is 1 if and only if z is an integer and well exactness at this point is just infectivity of this map ok.

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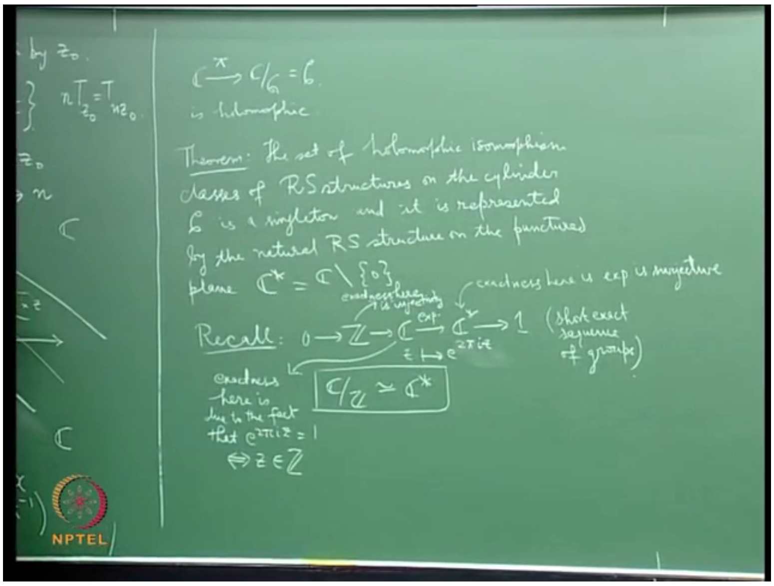

So, I just I am just understand to elute to the fact that C mod z is C star and of course, mind you this map this map is also a holomorphic map C to C star is an exponential map and it certainly holomorphic map. So, we are in this situation you have a map from C to the cylinder Riemann surface structure on the cylinder which is holomorphic map, but immediately the cylinder does not look like C star if you want to begin with. The cylinder that does not really look like C star, but you can see is homeomorphic to C star because cylinder is just s 1 cross r and that r you know the real line r is homeomorphic. In fact, diffeomorphic to any interval and you choose that interval to be 0 comma 1.

So, if you think of it is s one cross 0 comma 1; I mean it is C star can be thought of as D given by polar coordinates, namely one coordinate theta which comes from s 1 the other coordinate given by the distance to the origin which is essentially a real number. So, you can actually seeing think of C star as s 1 cross the interval 0 comma infinity, but that is a same as s 1 cross R because the interval 0 comma infinity is diffeomorphic to r you can always use the time function you know to and it is inverse, to map any open interval on to any other open interval or to the whole real line. So, they are all they are all the same ok.

Fine; so let us go on to look at this. So, let us go on to look at the case of a torus. So, this is what I do. So, here is a. So, I sort to the torus.

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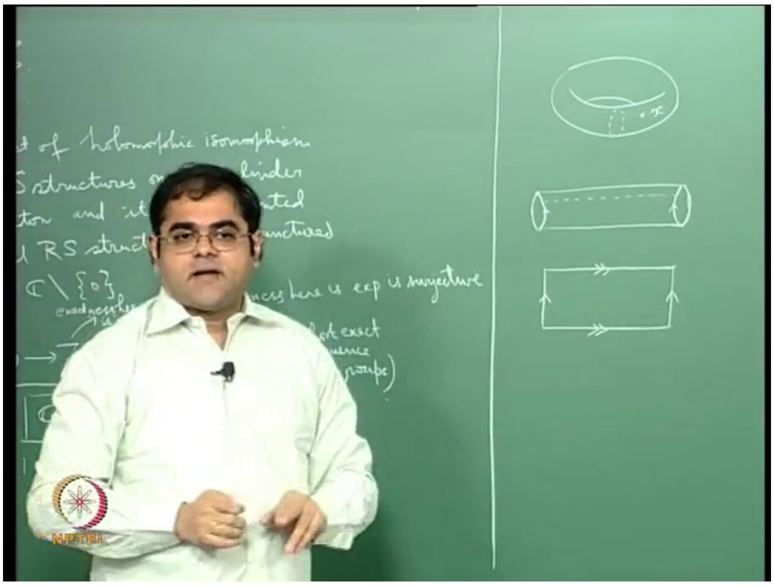

Sort the torus and somehow given a point on that on that torus let me call that as x, I want to get hold of a coordinate neighborhood around this which looks like an open subset of the complex plane that is how I get a chart.

So, how do I connect this to the complex plane? So, that the idea is very simple is very similar it is very simple and it is very similar to what we did here. So, what you do is you take this torus and you first cut it along vertically like this you make a cut, and what you will get up get is essentially the cylinder you will you will get a you will get the cylinder here, but of course, mind you that this is not the infinite cylinder. So, I will have to these one end of it which is with the boundary circle is here, and there is another end of it which is kind of without the boundary and I am supposes if I identify this circle essentially with this I paste it, then this is your torus. So, let me also include this boundary circle and say that you know these 2 circles need to be identified ok.

Then I get the torus, but this is still not the plane. So, what you do is you make just like the cases cylinder you make one more cut and you open it up what you get is rectangle. So, this circle has become this edge of the rectangle this circle has become this edge of the rectangle and this line here has split into 2 edges that corresponds to these 2 edges. So, all I am saying is that you can go back by simply identifying these 2 edges will give me the cylinder, and then in that process this is transformed into the circle and then identify the circles you get back torus ok.

Now, but the point is I want the plane. So, the natural way is to do exactly as we did there, just like we propagated the strip here to that is we repeated the strip to get the whole plane, it is obvious that you know you just have to repeat this rectangle to get the whole plane. And then if you go back to this philosophy of trying to get the cylinder as a quotient of the complex plane, in this case also you can get the torus as a quotient of the complex plane. The only thing is now that your group is not controlled by one vector it is controlled by 2 vectors, namely you get a copy of z direct some z or z cross z ok.

So, let me write that out.

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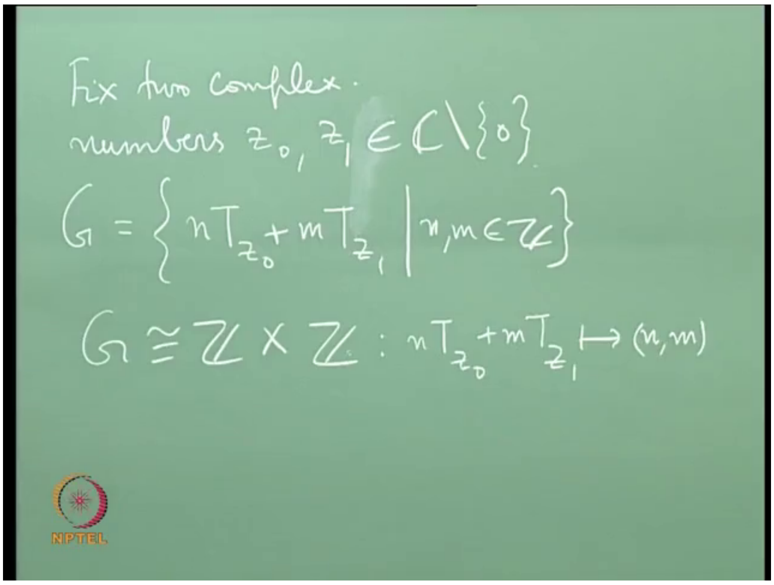

So, what we do is we take you fix 2 complex numbers, we fix 2 complex numbers let me call them as z naught z 1. Of course, I should mention there that z naught is non zero. So, you know let me put C minus 0 be very strict. So, that I do not end up taking z naught to be 0 which will then give mean nothing, because I will have translation by 0 which is just the identity map and I do not want that. So, in the same way I fixed 2 complex numbers here which is non-zero just to be sure that there are no such hiccups and what we do is again look at translations by these 2 complex numbers. So, you look at the group G which is given by translations by integer multiples of these 2 complex numbers. So, it is a group G is given by n T times T sub z naught plus m T sub z 1 where n and m are integers ok.

So, this essentially means translate by n times z naught then translate by m times z 1, and the order in which I do it is immaterial because they commute and I get a group and the group G again in this case isomorphic to Z cross Z and again what is the group operation on the left side and the left side it is composition of mappings. So, composition of translations is again a translation and on the right side it is addition it is the product group under addition and you get an isomorphism by simply sending any such element n times T z naught plus m times T z 1 to the ordered pair n comma n which is an ordered pair of integers ok.

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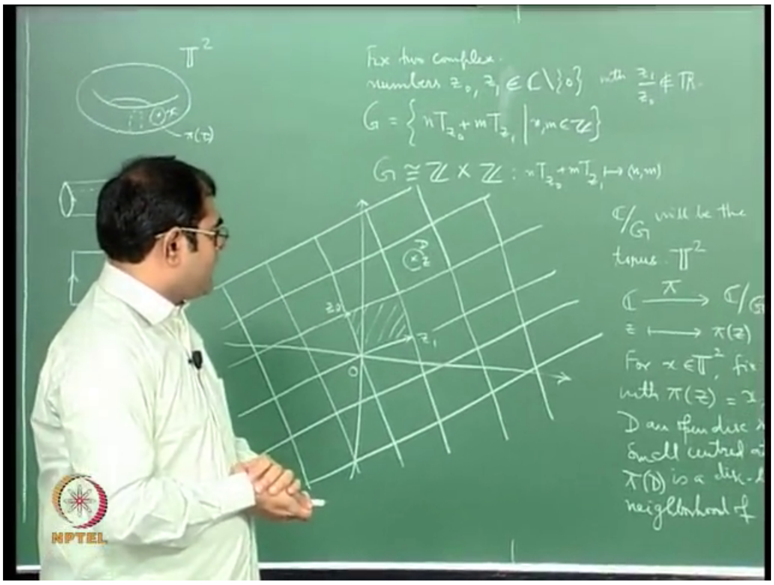

So, now what we do is that you can now you have the whole complex plane. So, let me draw it again and. So, we have. So, here is my is z naught and here is my is z 1 and you know I get this whole plane divided into you know grid formed by rectangles, whose edges are given by you know z naught and z 1 and their translates. So, of course, you know I really want a rectangle and to be more strict I must make sure that z 1 and these 2 vectors are not the same direction that is also important because if 2 if both the vectors are in the same direction then I am not going to get any I am not going to get rectangle.

So, I had to put that condition there as well. So, let me put that with let me write z 1 by z naught not a real number, basically I want 2 linearly independent vectors 2 complex numbers which are linearly independent over r, these 2 vectors should not be multiples of one another I do not want that in that case I would not get a rectangle basically I want to get a rectangle and. In fact, the point is it may not even be a rectangle in general it could be just a parallelogram this angle need not be 90 degrees. So, you could have even a parallelogram, but anyway it does not matter. So, you see the point is that the whole complex plane is divided into a grid like this of various translates of this parallelogram and this parallelogram is called the fundamental parallelogram and the whole complex plane is just translates of this so in fact, I will have. So, I should also draw this line and it will go on like this ok.

So, here is my here is the whole plane divided into rectangles, and again if I now take the whole complex plane and go modulo this group.

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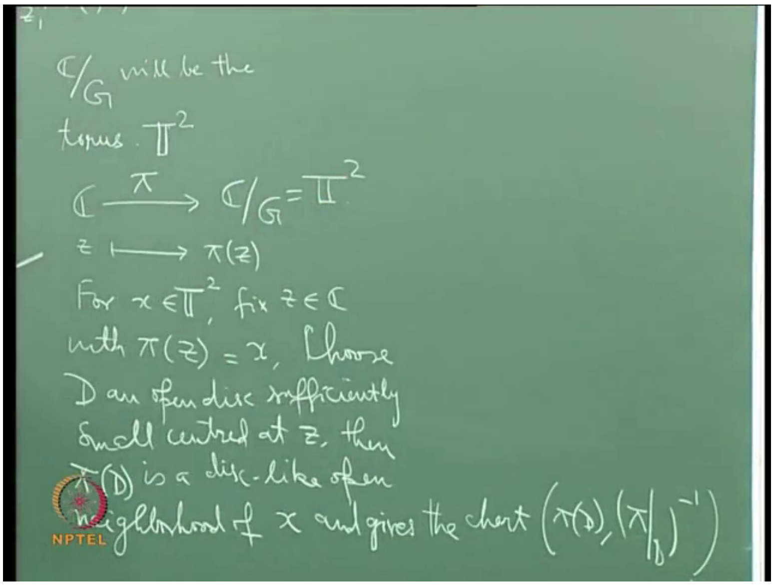

C mod G what I will get is a torus because I am just identifying points, which are translates of each other by an integer multiple of z naught and a integer multiple of z 1. So, C mod G will give me will be the torus, let me call this torus as. So, this is let me call it is T. So, this will be the torus T and I am thinking of T as a real surface. So, it is 2 dimensional real surface. So, let me put T 2, but it is not a bit confused with T cross T or something like that ok.

So, this is my torus. So, when I write T 2, I mean real 2 torus they considered as a subset of the real three space and when I say C mod G will be the torus T 2, what is going to happen is what I mean by that is that there is a natural map from C to C mod G which again I call as pi all right and which will send any complex number z to the equivalence class of that complex number right and then the set of equivalence classes is going to give me essentially at least topologically the torus. So, again I can again you will see that it is very easy to give a coordinate chart at a point, namely we do exactly as we did there.

So, what we do is you give me a point x on the cylinder choose your point z, if you want any point z in the complex plane such that said goes to x under this map pi, and choose sufficiently small disc surrounding this point. Now make sure that this disc is smaller than the rectangle in which it lies of course, if the if this point is going to lie in one of the edges the rectangle it does not matter, but make sure that the size of this the radius of the disc is very small sufficiently small, then you can check that again pi will be an open map and therefore, pi of this disc will give you an open disc like neighborhood here pi D and then you have a coordinate chart.

So, you get for x in the torus fix z in C with pi f z is equal to x choose D an open disc sufficiently small centered, I said then pi f D is disc like open neighborhood of x and gives the chart. So, let me continue here and gives the chart let me see let me write here itself, the chart pi of D comma pi residue to D inverse. So, by from D to of pi of D is going to be any homeomorphism, that is because you have chosen D sufficiently small; and since it is a homeomorphism it is inverse is also homeomorphism and this is this homeomorphism is what is going to give me and identification of pi of D with D and that is a chart.

And again you can check that this collection of charts are going to give you an atlas that is because you can check that wherever they intersect the transition functions will now be just translations by a multiple of n T z multiple of z naught and some integer multiple of z 1 and this of course, these are certainly going to be by holomorphic maps I mean holomorphic isomorphisms. Therefore, the compatibility of these charts is going to give you an atlas and with this atlas you are going to end up making C mod G that is the torus into a Riemann surface ok.

So, this is T 2 and of course, it is a Riemann surface structure that depends on your choice of z not an z 1. So, the situation is the situation is similar to that of a cylinder in trying to get hold of the Riemann's surface structure, but the question is again there we can ask the same question how many such Riemann's surface structures, that you is it that you can put on the on the torus, which are all you know non isomorphic that is distinct Riemann surface structures. How many distinct Riemann surface structures are there on the torus that you can actually put? So, there is an answer to that. So, let me write that down. So, let me write a couple of statements here.

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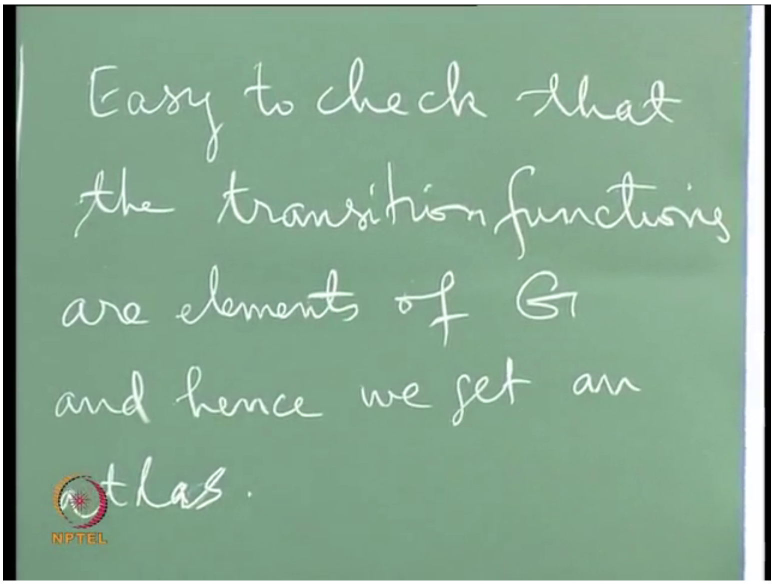

So, I just put on a side here and right easy to check that the transition functions are elements of G and hence we get an atlas ok.

So, I just I just record that the both. So, now, let me go back and give you this theorem, how many Riemann surface structures distinct Riemann surface structures, which you can put on a torus. So, again this theorem also involves the proof of this theorem again inverse some more further techniques more advanced techniques that we will develop, but it is important that you should know what you get; what you would get eventually let me write that down.

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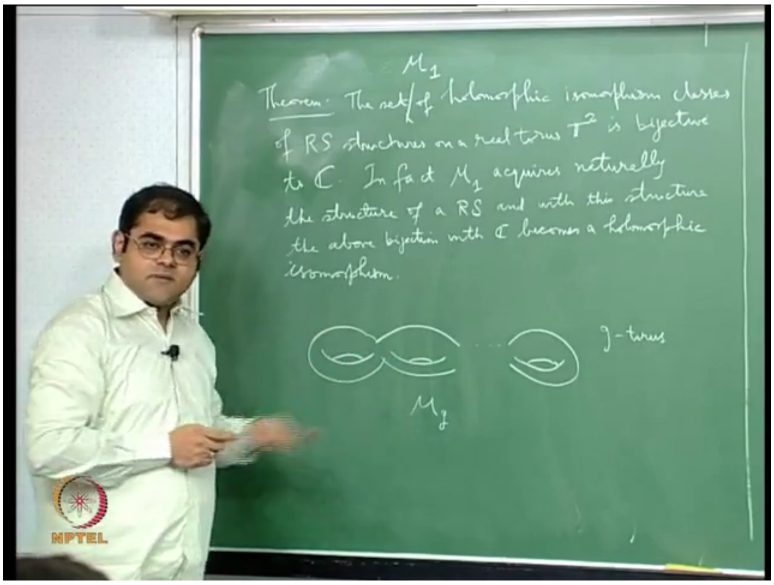

So, here is the theorem, the set of holomorphic isomorphism classes; classes of Riemann surface structures on a real torus T 2 actually based it to the complex plane, you will see the complex plane. Which means that you know you can give as many distinct Riemann surface structures on the torus as there are complex numbers and this more to this story. In fact, what happens is in the set of iso holomorphic isomorphism classes of Riemann surface structures are a real torus that is it itself becomes a Riemann surface ok.

The amazing thing is that the set of isomorphism classes of Riemann surface structures that itself becomes a Riemann surface, and if you consider that Riemann surface structure then this bijective map from that Riemann surface to see is an isomorphism of Riemann surfaces. So, much more happens. So, let me write that. In fact, let me call this set as something, let me call them call it as M1. In fact, M1 acquires naturally the structure of Riemann surface of a Riemann surface, and with this structure the above bijection with C becomes a holomorphic isomorphism. So, that is the amazing thing. So, let me end by just giving you of few words about the notion of modulo.

So, this was this go the term modulo goes back to Riemann and he was looking at Riemann surface structures on a torus. So, this is the one torus, but you know I can also have G torus namely something that looks like this a torus with so many holes, say with G holes. So, you know I can I can look at things like this. So, this is a G torus. So, it is just G of these toruses just stuck to each other and the way you stick it is by removing a disc like neighborhood from both and just sticking it sticking the boundaries together.

So, this is a G torus and Riemann was trying to look at the various Riemann surface structures that you can the complex structure that can put on this. And the set of isomorphism classes of these complex structures gave rise to a certain set let us call it as Mg, which is m one when you put G equal to 1 and Riemann found that you see that this space itself had a nice structure. He found that this space had just had a structure by which you can speak of holomorphic functions on the space only thing was that this space was no longer a Riemann surface namely it was not one dimensional it was higher dimensional. So, you need a higher dimensional analog of Riemann surface and that is called a complex manifold ok.

So, Riemann found that the this set of holomorphic isomorphism classes of Riemann surface structures on this namely this one naturally became a complex manifold and that was very amazing and so, it has been and in fact, he also found that the dimension he found a fall for the dimension of the space and he called it a modulo space ok.

So, the reach area of modulus theory actually investigates such questions namely you take an underlying real surface or a or a more generally even a real object which could be higher dimensional manifold, and try to put various complex structures on it and then ask this question that whether the set of isomorphism classes under that complex structure of the various complex structure that you get, whether that is it has a natural structure and it is amazing that it has always been soon I mean it is it is god given and it is amazing, that trying to look at the parameter the set of isomorphism classes. That set automatically has some geometric structure, and that is the kind of motivation to study modulo theory.

So, I also keep making several remarks during the course of these lectures, to give you also some idea about modulo theory of Riemann surfaces.

So, we will stop here.

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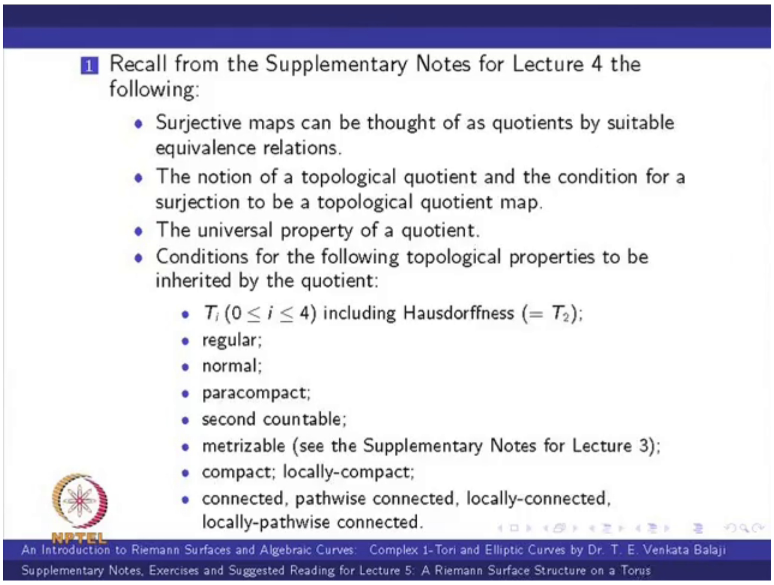

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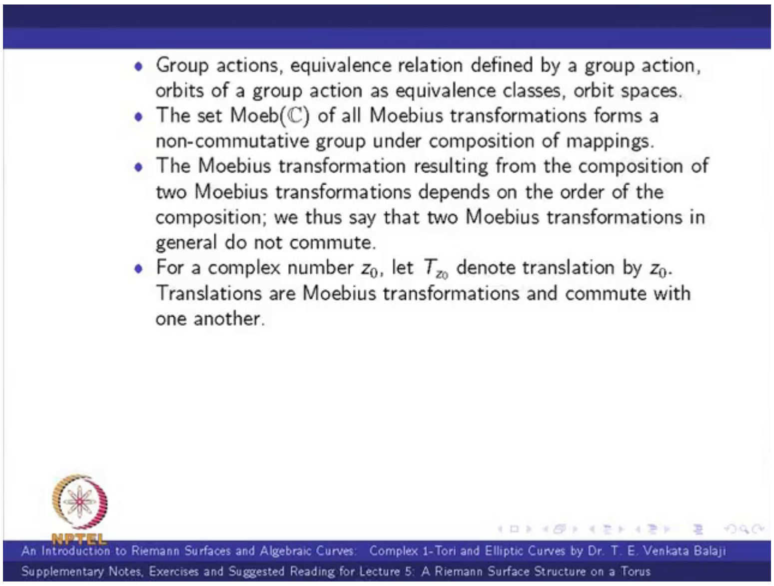

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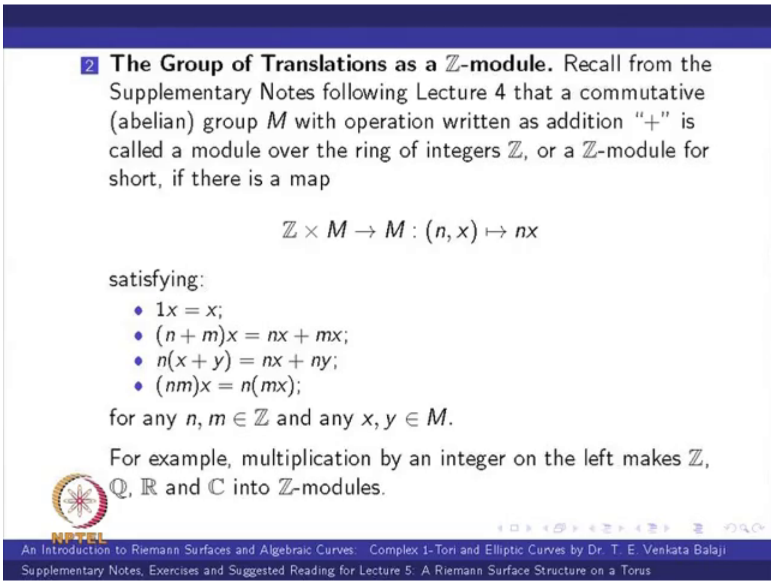

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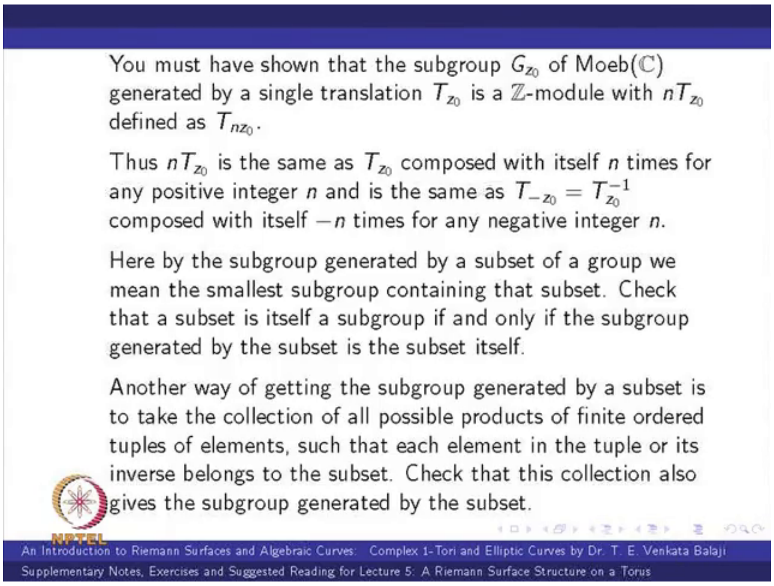

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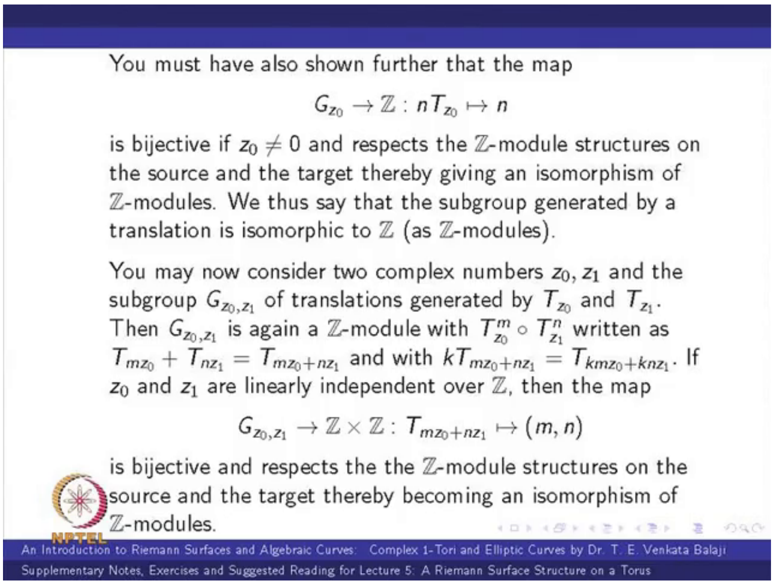

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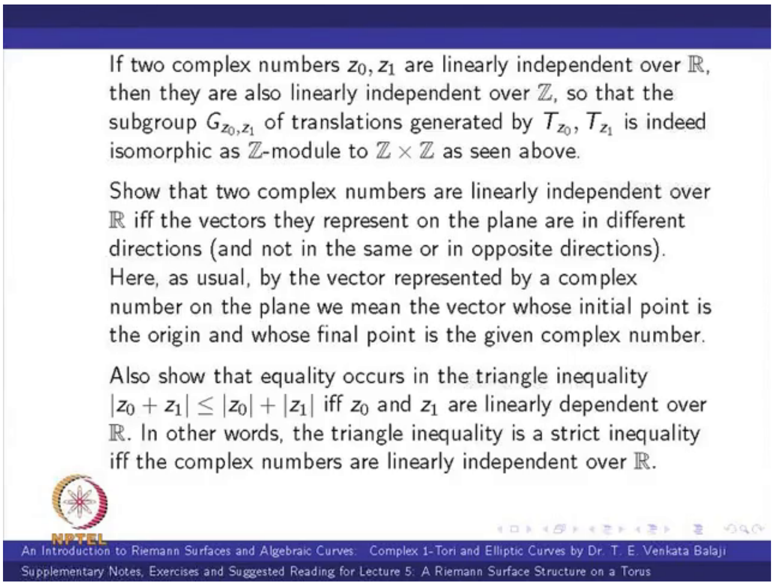

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In general, let G be the subset of Moeb($\mathbb{C}$) consisting of all translations. Then G is a subgroup which is commutative (abelian). It is further a $\mathbb{Z}$-module. The natural map $\mathbb{C} \to G : z \mapsto T_z$ is an isomorphism of $\mathbb{Z}$-modules.

We thus say that the identification of a complex number z with the translation T_z it defines results in the identification of $\mathbb{C}$ with the group of translations as $\mathbb{Z}$-modules.

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3. **Torus as a Quotient of the Complex Plane by the Subgroup generated by two Translations.** Show that the group of Moebius transformations acts on the complex plane in the natural way: if g is a Moebius transformation and z is a complex number, then $g \cdot z$ is just $g(z)$.

Show that the quotient of the complex plane, by the subgroup of Moebius transformations generated by two translations T_{z_0}, T_{z_1} where $\{z_0, z_1\}$ is linearly independent over $\mathbb{R}$, is a Riemann surface which as topological space is homeomorphic to a torus.

The main technical point of this exercise is to verify that the quotient is Hausdorff, locally-compact and second countable; the Hausdorffness being concluded by checking that the equivalence relation defined by the group action is a closed subset of the product $\mathbb{C} \times \mathbb{C}$ and that the quotient map is an open map.

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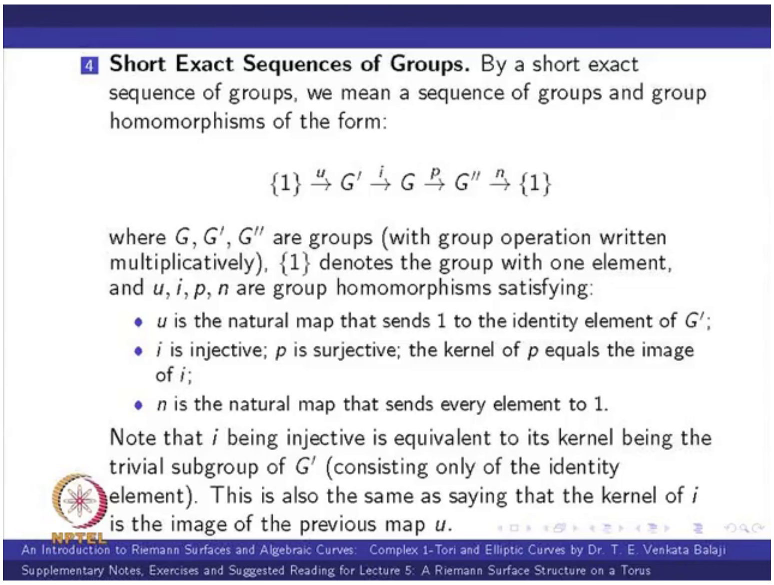

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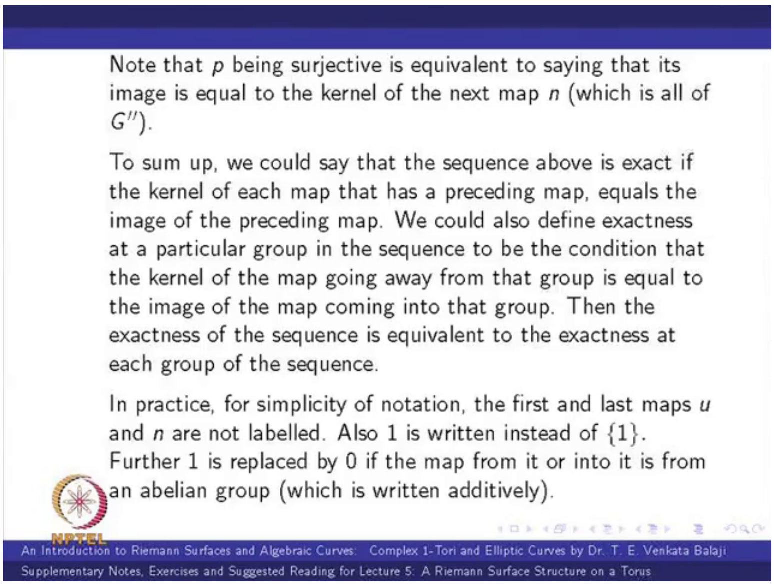

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