An Introduction to Riemann Surfaces and Algebraic Curves: Complex 1 -dimensional Tori and Elliptic Curves

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Lecture - 09
A First Classification of Riemann Surfaces

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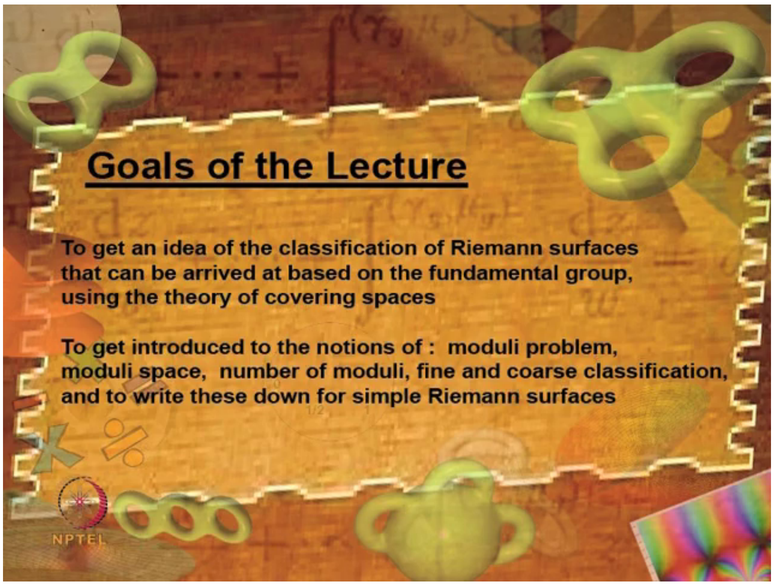

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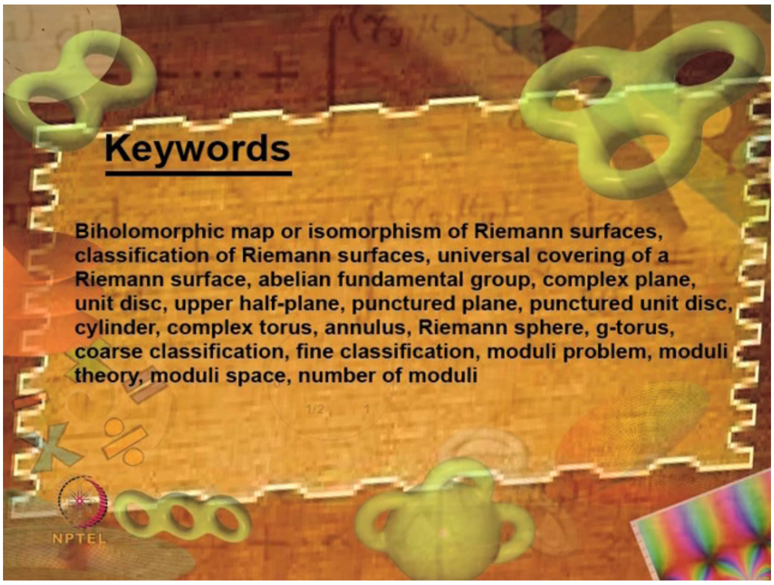

So, what I am going doing now is, to tell you the fundamental theorems that you can obtain the first theorems that you can obtain about classification of Riemann surfaces using covering space theory.

So, I will state the theorems, so that you get a feel of what kind of results you can expect and then later on one will start we can start proving these theorems. So, let me begin like this. So, here is a theorem.

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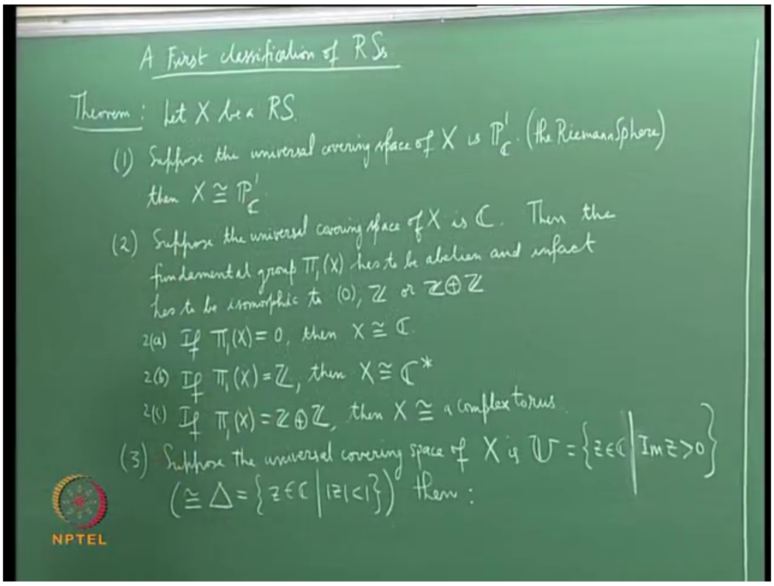

So, I will just write it as a first classification of a Riemann surfaces.

So, as I was telling you the last time that any Riemann surface admits a universal covering, which also becomes a Riemann surface and the universal covering simply connected. So, the universal covering has to either be the; it has to be one of the 3 simply connected Riemann surfaces namely the Riemann sphere or the complex plane or the upper half plane which is the same as unit disc.

So, that; that means, that every Riemann surface is going to be one of these 3 modulo certain sub group of automorphism, which are going to be Mobius transformations and that sub group is going to be isomorphic to the fundamental group of the Riemann surface that you are trying to study of the remain surface whose covering space you are you are looking at ok.

So, I will discuss all the 3 cases. So, let me write the first case let X be a Riemann surface, suppose the universal covering space of X is P1 namely the Riemann sphere ok. So, suppose you take Riemann surface and suppose universal covering space is P 1, P 1 is a of course, this is the Riemann sphere then X is isomorphic to the Riemann sphere itself. So, in other words the only Riemann surface which has universal covering space the Riemann sphere has to simply the Riemann sphere. So, you do not get anything in this case right.

So, then therefore, the next case that we have to look at is when the universal covering space is the complex plane. So, of course, again let me repeat, the universal covering space of X can either be the Riemann sphere or it can be the complex plane or it can be the unit disc which is by holomorphic to the upper half plane.

So, let me consider the next case suppose the universal covering space of X is the complex plane. So, this is the next case then what happens is that the fundamental group the first fundamental group of X has to necessarily be Abelian. So, that is a condition that will follow from covering space theory, and it has to be Abelian and it can be only certain possibilities; only one of 3 possibilities which I am going to write down. So, the 3 possibilities are of course, the fundamental group is trivial which is the 0 group or the fundamental group can be Z or it can be Z direction 3 there is there is no more freedom.

So, let me a write this case, then the fundamental group pi 1 of X has to be Abelian and in fact, has to be a isomorphic to the trivial group Z or Z direction Z. And what do you get in these cases if the fundamental group of X is 0 of course, sometimes we write just 0 or we write 0 with a bracket it is it means the same thing, then X has to be just isomorphic to see.

So, whenever I write this isomorphism I mean by holomorphic isomorphism, it is an a isomorphism as Riemann surfaces not just a topological isomorphism, it means isomorphism as Riemann surfaces. Mind you we are trying to classify Riemann surfaces of isomorphism. So, all the isomorphisms I am talking about here or all isomorphism as Riemann surfaces.

So, the first fundamental group is 0 then X has to be just the complex plane up to an isomorphism holomorphic isomorphism. If the first fundamental group of X is Z then X has to be essentially a cylinder, it is it has to be a cylinder with the Riemann surface structure obtained by fixing a nonzero complex number and take going the plane modulo a integer translates of that nonzero complex number, which give which gives you as I told you in earlier in an earlier lecture Riemann surface structure on the cylinder.

So, then X has to be just C star and of course, you know the first fundamental group of C star is Z and the other possibility is that if the first fundamental group of X is Z direction of Z this is the first fundamental group of X Z direction Z then X has to be isomorphic to a complex torus then X is isomorphic to a complex torus ok.

So, by a complex torus of course, I mean you take 2 nonzero complex numbers, which as vectors are linearly independent over r so that they are the code terminus the you take the parallelogram, but they form in the complex plane. And then you are essentially looking at the complex plane modulo translations integer by translations by integer multiples of these 2 complex numbers. And as you know that that is going to give a Riemann surface structure, on the torus and that is what I call as a complex torus. So, they X is isomorphic to a complex torus.

But of course, please also remember at this point that where I say X is isomorphic to a complex torus, it could be different complex tori. So, there is a point of view that if you look at various non isomorphic complex tori structures that you can put on a real torus I told you there can be many. In fact, I told you that there are as many as complex numbers please remember. So, at this point what I want to stress is that you know when I say X is a isomorph into a complex torus, that complex torus could be one among many depending on X. So, these are the cases when the universal covering space it is the complex plane.

Now, the only thing that is left out is the case when the universal covering space is the upper half plane, which is the same is unit disc. So, let me write down this case which is slightly more involved, suppose the universal covering space of X is u the upper half plane, which is the set of all complex numbers such that you know real part of the imaginary part of that complex number is greater than 0. This is upper half plane, which is by the way isomorphic to the unit disc you know that you can always find a Mobius transformation that maps the open unit disc to a to an open half plane. So, this is set of all Z belonging to c such that mods at less than 1 this is the unit disc.

So, this is only case that is left, when the universal covering space of X is the upper half plane. Then we look at 2 cases the case when the fundamental group is Abelian and the case when the fundamental group is non Abelian.

So, what happens? Again when the fundamental group is Abelian you get only 2 possibilities and these 2 possibilities are as follows. So, let me continue here maybe I will draw a line here somewhere.

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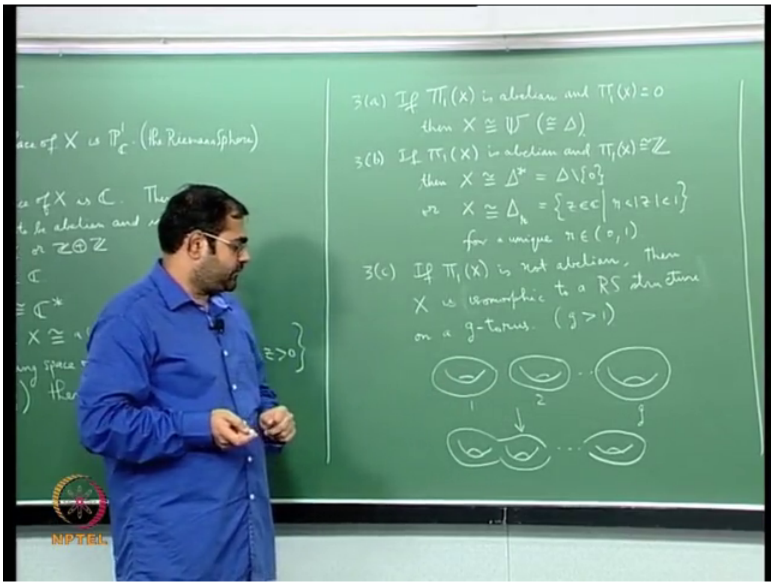

3 a if pi 1 of X is Abelian then of course, and pi 1 of X is 0, then what you are going to get is again the just the u upper half plane or the unit disc always, the case pi 1 is equal to 0 is always going to give you back the simply connected case.

So, it is not hard to guess in all these cases whenever pi 1 is 0, you have to get the universal cover. So, that is what is happening. So, then X is isomorphic to just u it is isomorphic to delta, 3 b if pi 1 of X it is Abelian and pi 1 of X is isomorphic to Z, then X is isomorphic to either delta star, which is delta minus the punctured unit disc or X is isomorphic to delta sub r this is the annulus with outer radius one inner radius r less than 1. So, this is the set of all is Z in c such that r is less than mod Z is less than 1 for a unique r real number r belonging to the open interval 0 1 ok.

So, and of course, the point I am can to say is that, you can ask whether there are any other possibilities there are no other possibilities. So, if pi 1 is Abelian these are the only

2 possibilities and in some sense; what this means is that you take a Riemann surface with pi 1 isomorphic to Z direct some Z that forces, that the universal covering space has to be C and your Riemann surface has to be a torus complex torus ok.

So, what is the power of such a statement? The power of such a statement is one can define what is meant by an arbitrary Riemann surface structure. Just using the general definition of a Riemann surface, what is a guarantee that any other Riemann surface structure on the real torus has to be of this form? See a general Riemann surface structure on the torus could be could have been anything, but the beautiful thing is the underlying topological space the fundamental group depends only on the underlying topological space. So, even if I had put some other Riemann surface structure on the complex torus, it is fundamental group being still Z direct some Z that will force that the covering space is c, because that is the only case where you get Z direct some Z ok.

And then this theorem will tell you that it is the Riemann surface structure that you put has to be some complex torus structure, gotten by fixing 2 nonzero complex numbers you know which are linearly independent over reals.

So, this is the amazing power of this theorem that is what you must realize. So, somehow the property of the underlying topological space of the Riemann surface being a torus, that forces the condition that it has there is more other way of getting a complex Riemann surface structure, other than the way you got it for a complex torus. So, that is the power of the theorem that is the power of the statement that is what you have to realize.

Then of course, the situation when and of course, you know let me also add that if you change r here, then the delta r also changes in (Refer Time: 17:19) isomorphism of some class. In fact, if you take 2 different rs then the delta rs are not by holomorphic they are not by holomorphic ok.

So, you change the radius then it will be in a different holomorphic isomorphism class that is the reason I say for a unique r. So, then the other thing that I have I am left out with 3 c if pi 1 of X is not Abelian then and if you then this is the left out case and of course, I am assuming X is compact and what this case will cover will be all the complex structures that you can put on g dimensional tori ok.

So, let me write that if pi 1 f X is not Abelian, then X is isomorphic to a complex a Riemann surface structure on. So, I will put this in bracket let me remove the balance makes this isomorphic to Riemann structure on a g torus. So, g torus is you know it is something that is that look that is gotten by taking you know g copies of a torus, you take g copies of the real torus 1 2 and so on up to g. And what you do is you join them, you stick them together namely you cut out an open disc here you cut out an open disc there and then the boundaries of this the boundary circles of these 2 open discs you just stick them and you what you get is a g torus is something that you look that will look like this. So, this is the g torus ok.

So, if pi 1 is not Abelian and of course, I am assuming X is compact, then you get a Riemann surface structure on a g torus all right. So, this is a first classification and what it tells you is that somehow you can see that you will it is a kind of clue that it is better if we start by looking at you know Riemann surface structures on a topological space whose fundamental group is Abelian ok.

So, because the Abelian case puts lot of restrictions and essentially and of course, you know I should say g greater than 1, because g equal to 1 is the case of a torus. And if it is if g is equal to 1 then the fundamental group is Z cross Z direction Z and that is certainly Abelian. So, when I say it is not Abelian g has to be greater than 1 ok.

So, this is what you get and the way one proves this theorem is basically if by assuming you do some covering you us assume the fundamental uniformization theorem for simply connected surfaces, which is actually you know which is essentially the statement 1 2 if I you want I can call this as 2 a, 2 b, 2 c. So, it is 1 2 a and 3 a essentially.

So, if you assume the fundamental uniformization theorem, which says that the only simply connected Riemann surfaces are either the Riemann sphere or the complex plane or the unit disc or which is same as the upper half plane, if you assume that and then you use properties of universal coverings and covering spaces and you also use a proper study Mobius transformations then you can reduce the rest of the statements

So, I will write that down.

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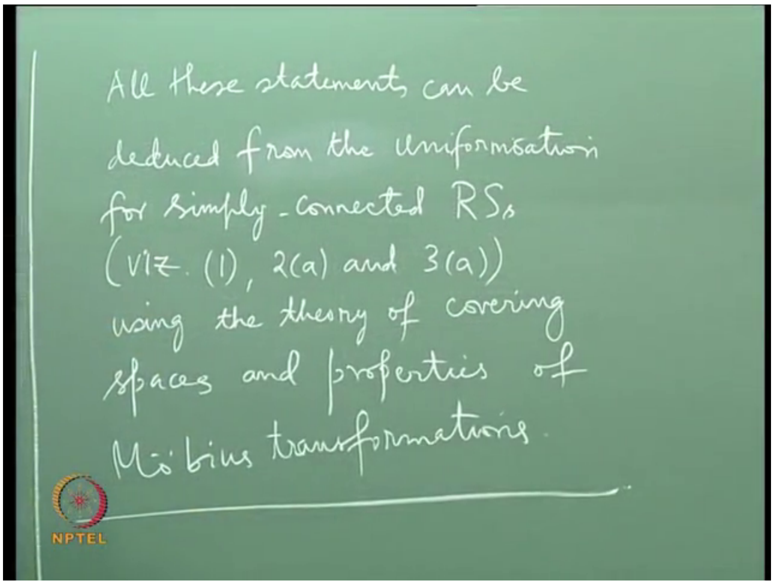

So, all these statements can be reduced using, reduced from the uniformization for simply connected Riemann surfaces, which is essentially 1 2 a and 3 a; namely 1 2 a and 3 a using the theory of covering spaces spaces and properties of Mobius transformations ok.

So, this is the key to proving all the statements in the theorem. So, you divide the proof of this whole theorem into 2 pieces, the first statements are essentially the fundamental theorem uniformization of simply connected Riemann surfaces you assume that, and the rest of the statements can be gotten by covering space theory and by studying Mobius transformations. So, this is what we will do in the forthcoming lectures, and then of course, what will remain is to prove the fundamental uniformization theorem, that is uniformization for the simply connected Riemann surfaces, that will involve some more work some analysis will be involved in that and we will do that that as well ok.

So, this is one aspect of. So, this is to give you an idea of what kind of theorems you can expect. Because you see whenever we have structures in mathematics of a certain type, you are always interested in finding out what happens to the isomorphism classes, I mean how will you distinguish one structure from the other. So, this is basic classification problem that always comes in end, this is how the how a first solution looks like

But then there are; but you can see that this is a in some sense this is a course classification because in some sense it is a classification that basically involves only the fundamental group. Roughly you see everything is controlled by the fundamental group of the underlying real surface.

But then we need to have a finer classification for example, you take a Riemann surface with fundamental group Z direction Z, then it could be any complex torus and there are complex tori which are not isomorphic to each other. So, among these complex tori, you still have to do a classification and trying to understand that is what leads you to study various complex structures on a real torus and such a problem is an example of a moduli problem. So, you have an underlying object and you put new structures on that, and then you want to see how many different structures you get.

So, you take a real torus and you make it into a Riemann surface, there are so many ways of making it into a Riemann surface, and then the moduli problem will tell you that if you take the set of isomorphism classes of Riemann surface structures on as real torus, that is various structures of this type, then that is I told you that is bijective to the complex numbers and in fact, I told you that set itself acquires the structure of Riemann surface ok.

So, it is amazing and a similar thing happens also with the Riemann surfaces of this type, these are compact Riemann surfaces after all you can see this is clearly compact as a manifold compact as the topological space. So, I have not yet discussed what a manifold is. So, which I will do, but a Riemann surfaces is a one dimensional manifold in any case complex manifold, I will explain that later, but do not be worried about it. The point I want to make is it this is compact and trying to count how many independent complex parameters you need to classify all the different Riemann surface structures on this is called is part of the moduli problem for various Riemann surface structures, that you can put on a certain g torus a real g torus ok.

When I say real; that means, no Riemann surface structure has been fixed and you are trying to put various Riemann surface structures making it into a complex g torus. And then you want to know what is if you take the set of isomorphism classes, is this going to give you something.

So, again it is beautiful that it is it does give you a space and then this is a decent space we can talk about the dimension of the space, and you know dimension is you in a very naive it is a number of independent parameters. And then if you calculate the dimension there is a formula for the dimension which is known to Riemann, and that formula was justified in the 1960s and that leaves you into the problem of final classification. So, what I am trying to tell you is, the reason why I call this as the first classification is because there is still finer classification that needs to be done and the final classification essentially is done when you study Moduli theory ok.

So, at least for the case of a complex torus, I will try to outline how you can do this final classification because it involves reasonably easy methods, but of course, for genus g greater than 1 the methods are a little more advanced, and I will let us see if we can try to outline what happens in this case also.

So, what I am going to do next is I am going to tell you about I am going to draw another diagram I am going to rub out rub out all this. I am going to another diagram, trying to tell you to give you a pictorial view of this classification. So, let me rub this off. So, this is just to you know help you aid your memory k to.

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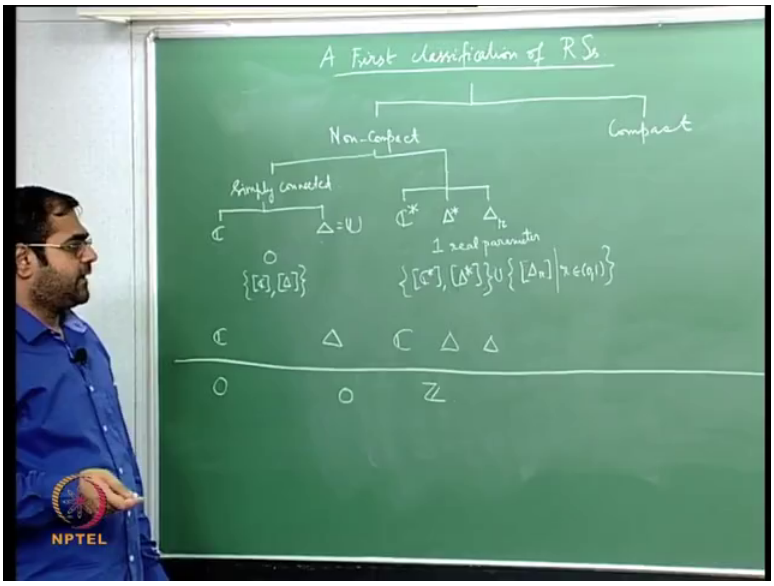

So, you take you take Riemann surfaces you can take them down into non compact Riemann surfaces and then you can break them down into the other possibility is your compact Riemann surfaces ok. And I am going to look at several things, I am going to write here number of moduli and then I am going to write moduli space then I am going to write universal covering, and then I am going to write first fundamental group ok.

So, of course, in both in the both in the non compact and the compact case simplest is the case of simply connected surfaces which is a fundamental sample. So, if I take the simply connected one, then you if it is non compact you know you are going to get either you know the complex plane or you will get the unit disc, which is the same as upper half plane and in the. So, maybe I have to rewrite that. So, that it looks better.

So, let me write one line and then. So, maybe I should write it here. So, let me write the non simply connected once, let me write a few of them, which I am interested in namely C star delta star, delta r and you see for all these cases if I look at. So, I put here the number of moduli.

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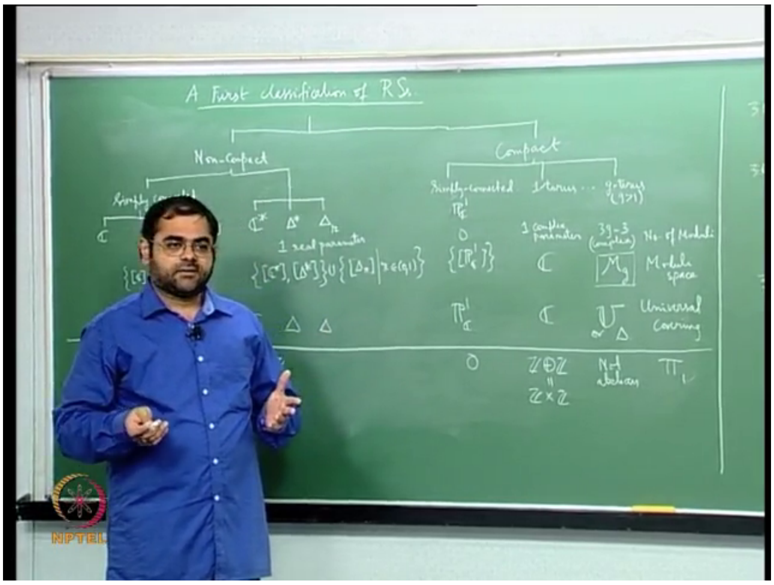

And then I put the moduli space and then I put universal covering and then I put pi 1.

So, if you look at it roughly the moduli space is supposed to be the set of isomorphism classes of Riemann surface structures of that particular type. So, if I take if I look at the complex plane, it means look at the set of isomorphism classes of Riemann surface structures on the complex plane and you know that on the complex plane actually I can put 2 Riemann surface I can force the structure of the unit disc on the complex plane, and that will not be isomorphic to the complex plane ok.

So, if you are if you look at the, if you fix the underlying surface 2 be r 2 then you will essentially get 2 Riemann surface structures. So, what happens is that you get a degenerated situation. So, in this case the let me draw a line here. So, in this case what happens is the first fundamental group is of course, for both the first fundamental group is 0 that is because both the simply connected and both are simply connected. So, the universal coverings are themselves. So, the universal covering for C is just C, the universal covering for delta has to be delta mind you these 2 are homeomorphic ok.

But as holomorphic universal coverings they are not the same, because the holomorphic structures are all different. So, see whenever I am talking about universal coverings mind you these are all universal coverings in the holomorphic sense, because I told you if you take a Riemann surface and you take the underlying topological space take the universal covering. Then because the underlying topological space below as a Riemann surface structure and because the universal covering is a local homeomorphism you can transport the Riemann surface structure locally to the space above. And in that way you can make the space above the covering space we can make it into another Riemann surface. So, whenever I say universal covering of a Riemann surface, it means that the top space is also Riemann surface in a unique way such that the covering map is a holomorphic map. So, please remember that

So, in this case the universal covering is complex plane here it is the unit disc and the well the number of moduli. So, the number of moduli is defined in the following way; whenever you take the set of isomorphism classes and if that set is discrete for a discrete set we never defined dimension, we always think of it as being of dimension 0 ok. So, usually the number of moduli is defined to be the dimension of a moduli space, and a moduli space is some nice structure on the set of isomorphism classes of Riemann surfaces Riemann surface structures on a given topological space ok.

So, in this case what happens is that the underlying structure is you know just the plane. So, if you take the set of isomorphism classes you will get only 2 points, you will get one equivalence class that corresponds to this another equivalence class that corresponds to that. So, the moduli space will contain only 2 points, and then the number of moduli is you know I we agreed that if there are only if the modulis space is discrete you better give the number of moduli you better give it a 0 because it should be the dimension the dimension of a discrete set should always be 0.

So, excuse me. So, in this case what happens is I will be a basically get moduli 0 and the moduli space is just consisting of it is a set consisting of just 2 you know to 2 elements. So, it is just going to be these 2 elements and when I put the bracket I mean holomorphic isomorphism class. So, here what I am doing is I am looking at Riemann surface structures on r 2, and there are only 2 I can put there are only 2 Riemann surface structures on r 2 ok.

So, mind you when you when I am talking about moduli space basically I am fixing the underlying topological space, and then I am looking at isomorphism classes of Riemann surface structures that you can put on the topological space. So, in this case it is r 2 mind you these 2 are both homeomorphic to r 2, if you go back to example 1 and 2 you will notice that and there are, but these 2 are not holomorphic which you know already ok.

So, this is the story for this is a very simple case, then in the in the case of C star delta star delta r and of course, there are further examples I am just giving some examples. So, in each of these cases the fundamental group is all always Z, the fundamental group is the same for all of them and all of them are homeomorphic they are all homeomorphic ok.

So, in this case if you look at the; but if you look at the universal coverings the universal coverings are all going to be different. So, for C star the universal covering is C, for delta star the universal covering is delta for delta r also the universal covering is delta and mind you delta is the same as u. So, u will be the universal covering or delta will be the universal covering for these 2 whereas, c will be the u universal covering for C star. And in this case if you look at the number of moduli, I will write it as 1 real parameter for the following reason because the set of isomorphism classes of Riemann surface structures, that you can put on one of these spaces which are all anyway on one of these spaces which are all homeomorphic.

If you think of them as the same topological space then there are 3 possibilities, you get one isomorphic to C star, you get one isomorphic to delta star and then you also get these 2 r 2 discrete points and then you get a continuous family, and that continuous family is the isomorphism class of the delta rs where r belongs to 0 comma 1. So, this is what happens and. So, you know if you look at the dimension of this, the only thing that will contribute dimension is this real parameter varying over the interval and therefore, you get one as a real parameter.

And if you want to think of it as a complex parameter if you treat it as half, but let us not do that. Then I come to the case of compact Riemann surface. So, again, let me make space for writing that down, let me write this in a little to the right the compact case . So, in the case in the compact case what happens is you have first of all again the simply connected.

So, the simply connected case and then of course, the next thing is one torus, then you can go on dot dot dot and then you have a g torus, the way I have defined it here and what happens to this situation. When it is simply connected of course, the fundamental group is 0, and the universal covering space is itself. So, simply connected case is of course, compact simply connected case is Riemann sphere is just P 1 it is a Riemann sphere and the universal covering is P 1 itself, and you know there is no you do not have any freedom. So, when the number of moduli is 0 and the moduli space is single point namely the isomorphism class of P 1 that is all you get ok.

The next possibility is that of a one torus this is a complex torus, and what happens is that the fundamental group is Z cross Z or Z direction Z is same as Z cross Z under addition; and in this case what happens is the universal covering space is the complex plane and you are getting the complex structure on that torus by going modulo you know this group of translations by integer multiples of 2 nonzero complex numbers whose ratio is not real and in this case what happens is that the moduli space is C that was theorem that I told you.

The set of isomorphism classes of complex structures on one dimension on a one torus which is a s 1 cross s 1 a real torus, is isomorphic to C and that is it actually acquires a Riemann surface structure, and as a Riemann surface it is isomorphic to c and. So, the number of moduli will be 1 complex parameter.

So, let me write it as one complex parameter and. So, what is left out is the case of g torus, I have told you that if you go to a g torus g greater than 1 I told you the fundamental group is not Abelian and in this case the universal covering space is the upper half plane ok.

So, it comes in the third case, you know in the statement of the theorem which covered the case when the universal covering space is upper half plane. So, the universal covering space is upper half plane is the same as a unit disc mind you. So, I wherever I write delta I can also write u, u or delta and of course, here again it is delta or u.

And the beautiful thing is that that is a moduli space mg which was known to Riemann and the only thing is that this moduli space had the structure of what is called as a complex analytic space and this is something I will try to explain. But in any sense it was it is a decent space and it depends on 3 g minus 3 complex parameters ok.

So, this is this is Riemann's formula. So, Riemann in his amazing work says that the various isomorphism classes that you can of Riemann surface structures that you can put on a g torus has to depend if you construct it as a space, then that space viewed as a space over the complex numbers has to be of dimension 3 g minus 3. So, this is Riemann's formula and of course, to justify it there was following Riemann the work of Tyche Muller and then the work of Lipman Bers and so on who justified this formula, and now a proper proof of this formula is very accessible and I will see if I can try to give a proper proof of this formula ok.

So, this is the picture of classification of Riemann surfaces, and the you must realize that the to begin with to have said so much that the first classification is already very interesting, but actually today the field of research that is very active today is trying to study this object a moduli space. How to properly define it and then study it is properties you know with respect to topology, with respect to analysis and in so many ways.

So, this is what is of current research interest. So, I will stop here.

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1. Note the following phenomenon with respect to statement no.2 of the Theorem in the lecture. If X is a Riemann surface such that the underlying topological space of X has fundamental group isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (under addition), then X is not only forced to be topologically isomorphic (i.e., homeomorphic) to a real torus, but also forced to be holomorphically isomorphic (i.e., biholomorphic, or isomorphic as Riemann surfaces) to a complex torus. This shows how a simple topological condition (in this case the fundamental group being isomorphic to $\mathbb{Z} \times \mathbb{Z}$) can have amazing implications for more complicated structures (in this case the structure of a Riemann surface) on a topological space.

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The above also tells us that in order to classify more complicated structures on a space, we may begin by first looking at the effect of imposing conditions on underlying structures, for example assigning various possible values for invariants of the underlying structure. In this lecture we got a first classification of Riemann surfaces by looking at the various possible values of the fundamental group (a topological invariant) of the underlying topological space.

2. Use Riemann's theorem on removable singularities, the Riemann mapping theorem and the Liouville theorem to show directly (as far as possible) that the various Riemann surfaces classified in the lecture are not isomorphic to each other. (Take two of the Riemann surfaces, assume there is given a holomorphic isomorphism, and use the above theorems to deduce a contradiction i.e., that such a holomorphic isomorphism could not have existed in the first place.)