Real Analysis

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Lecture - 52
Equicontinuous Family of Functions: Arzela - Ascoli Theorem

So, we had started the discussion about the equicontinuous families functions, let us recall the definition once again.

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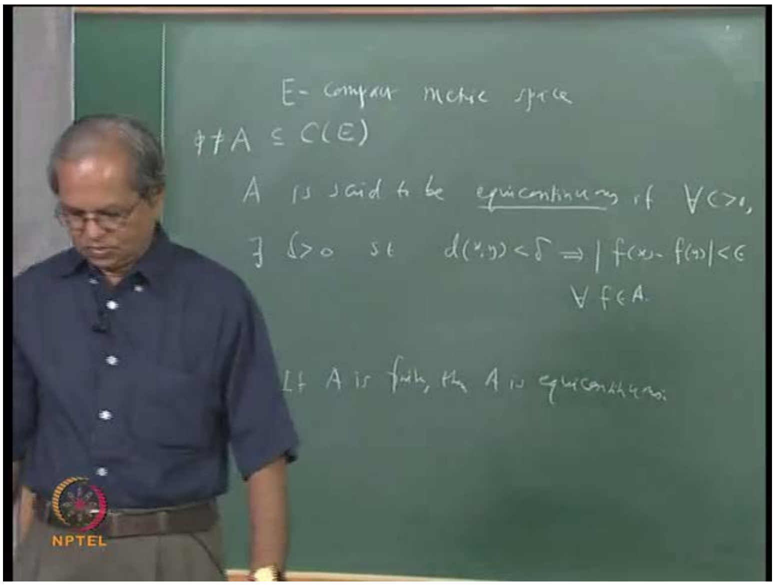

So, we have taken E as a compact metric space, and C of E as usual the space of all continuous harboral functions on E with the supreme metric, and A is sub subset of C of E, non empty subset. Then what we said A is said to be equicontinuous, if for every epsilon bigger than 0 there exists delta bigger than 0, such that wherever if we take any two points in the E distance between x and y less than delta. This implies mod f x minus f y less than epsilon and this should happen for every f in A.

So, suppose if it is just an arbitrary family, then given epsilon this delta may also depend on that function f, but if you can find the delta, which works for all functions, that is called equicontinuous family. Now, what are the obvious examples of an equicontinuous family of functions? One obvious example will be suppose, the family contains just one function, suppose A contains just one function then, it is obviously equicontinuous. What if it contains two functions?

You can find delta 1 for one function, delta 2 for f 2 and take the minimum of the 2 and then you can extend. So, one obvious thing to say is that every finite family is equicontinuous, so that is an obvious thing. We can say if A is finite then A is equicontinuous. We shall see non trivial examples of equicontinuous families little later, but before preceding that let us also see one example of family, which is not equicontinuous because if every family is equicontinuous then, there is no point in making such a definition. So, let us take the full C of E.

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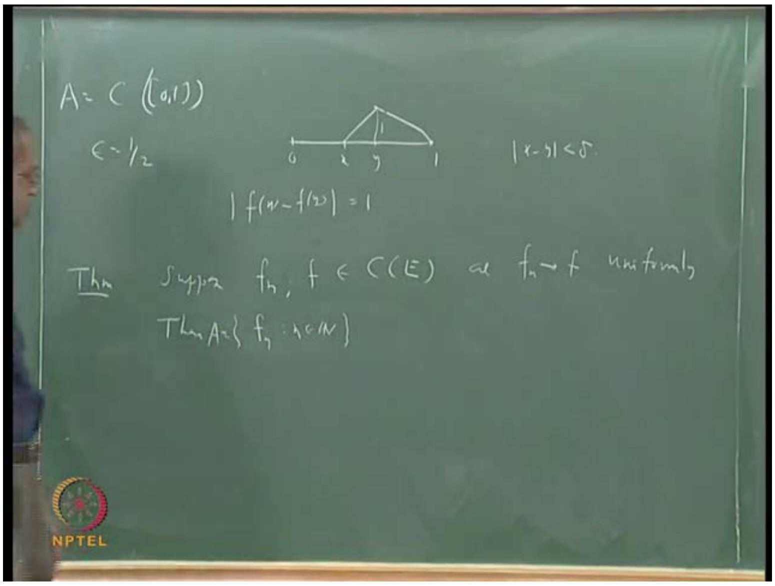

Let us take a particular case suppose I take C of 0 to 1 that is I am taking this whole as A. Now, what is the meaning of saying that a family is not equicontinuous, I mean for some epsilon this should fail, that is for some epsilon the whatever delta you give you can find some function. There are 2 points with distance between x and y less than delta, but mod f x minus f y is big.

We can actually show this in this case for any epsilon, but let us take equal to half then, for this we will we will show that whatever delta you take, you can always find a function and the points x and y such that mod x minus y is less than delta and but mod f x minus f y is bigger than this. Then this instead of giving the actual details I will just give geometric idea. Suppose, this is the interval 0 to 1, now suppose any delta is given then you choose any two points whose difference is less than delta.

So, suppose this is the point let us say this is the point x and this is the point y and such that, the distance that is mod x minus y is less than delta, whatever be the delta. Now, remember our family contains all continuous functions, so can I construct a continuous function, which is let us say 0 at x and 1 at y, that can be done always. For example, I can take this 1 here and 0 here so this part is continuous and remaining in y 2 1 and 0 to x what values it takes, I do not care.

So, I will I will make it 0 from 0 to x and I will take something like this from y to 1. This is a continuous function and whatever is that belongs to c 0 1 and for this what can you say about mod of f x minus f y f x is 0 and f y is 1. So, this is 1 and so that is bigger than half and this you can do for whatever delta is given. And you might think that we have since I have drawn this picture and also, this is something to do with space 0 to 1, but that is not the case.

You can do this is any metric space, given any metric space and let us say two distinct points. You can always consider the continuous function, which is 0 at one point and 1 at the other point. In fact there is more general things to what is called Urison’s lemma that is, if you are given two disjoint close sets then, you can always construct a continuous function which takes the value 0 at one close set and 1 at the other close sets, but any way that is not very important now. Is it clear that this is not an equicontinuous family?

Now, let us see further examples of equicontinuous families and also what this to do with uniform conversions is. The main thing is that if a sequence is informingly convergent then that set of functions forms a equicontinuous family. So, that is the first thing that we shall see. Suppose, f n and f they belong to C of E and f n converges to f uniformly. Then suppose you just take this set f n, n belonging to n. So, our set is A, A is f n, n belonging to n is an equicontinuous.

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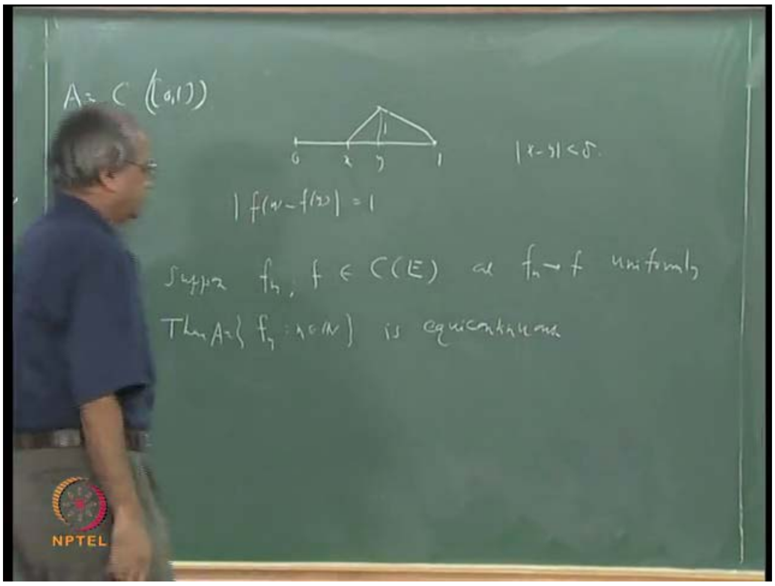

Let us let me begin the proof here remember because of this observation, if you take any finite number of functions that will always form an equicontinuous family.

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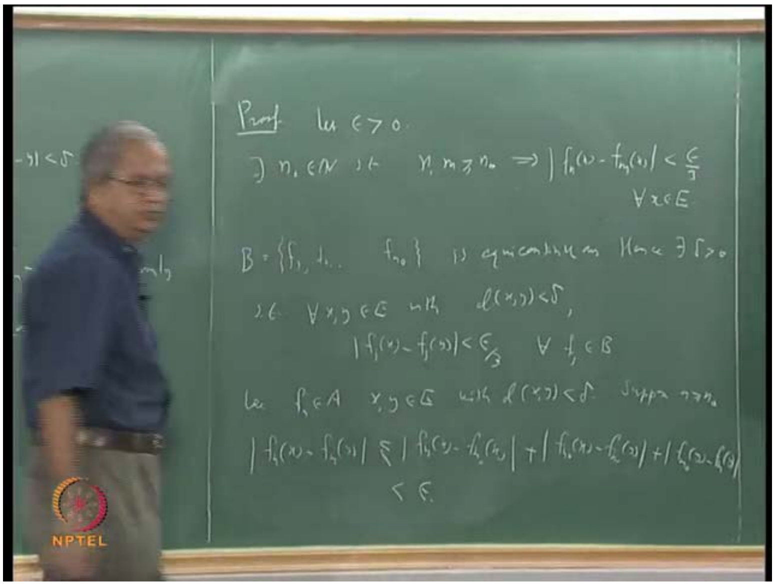

So, suppose there is some equicontinuous family and if you add some finite number of functions to that family, it will still remain equicontinuous. Even if you remove a finite number of functions from that family, it will also remain equicontinuous. Of course, removing will never cause a product, because if this works for every f in a it will work for any subset. So, once A is equicontinuous all subsets will be equicontinuous. So, what we have to do, we have to show that there exists a delta such that, all this things happen.

So, let epsilon bigger than 0, these are also one of the standard three epsilon proofs or epsilon by three proofs where basic idea is to add and subtract a few terms. Since f n converges to f uniformly f n also satisfies Cauchy criteria. So, let us say that there exists n 0 in n such that, n and m bigger than or equal to n 0. This implies that mod f n x minus f m x is less than I will take epsilon by 3 for every x in E. And since this is true for every n and m bigger than or equal to n 0, I can take this m equal to n 0 also, that is how we shall be using in the final.

Now, look at these functions f 1, f 2 etcetera, etcetera up to f n 0, suppose I call this as b. This is an equicontinuous family because it contains only a finite number of functions, so this is equicontinuous. So, b is equicontinuous therefore, hence there exists delta bigger than 0 such that, for all x y in E with d x y less than delta, we will have for all this functions. So, let me call f j of x minus f j of y less than epsilon let us take epsilon by 3 for and this should happen for every f j in b. So, that means for all f j in b or you can say for all j from 1 to 0, 1 to n 0, that is the same thing.

Now, what we want to show is that this is a delta, which will work for the whole family and for that we shall be using two things one is this. We already know that it works for this family, this finite number of functions and we will now show that this works for every f. Now, let take any f n in A and x y in E with distance between x and y less than delta and we consider mod of f n x minus f n y. See if n is less than or equal to n 0 there is no problem, this is already less than epsilon by 3.

So, we will consider only the case when n is bigger than or equal to n 0. So, let f n belong to A and let us say suppose n is bigger than or equal to n 0, that is the only case which needs proof. Then as I said we shall add and subtract, we will take f n x minus let us say f n 0. So, we shall write this as f n x minus f n 0 x, I will write this less than or equal to this plus mod f n 0 x minus f n 0 y and then finally, plus mod f n 0 y minus f n y. And now, you will see that the proof is more or less over. If you look at f n x minus f n 0 x that is less than epsilon by 3 because of this I will take m as n 0 here.

Similarly, for this last f n 0 y minus f n remember, the whole point is this; this inequality is true for every x in E. And what about the middle term f n 0 x minus f n 0 y that is because f n 0 is in this family and we have already chosen delta for which all this holds works for every f n. So, each of this is less than epsilon by 3 and so the whole thing is less than epsilon.

So, what we have seen is the relationship between uniform conversions and equicontinuity. Now, the obvious question is what about the converse? Suppose we take a sequence, which is equicontinuous then does it follow that it converges uniformly. In general it is not true, but suppose we take equicontinuous and bounded then we can only say that it has a sub sequence, which converges uniformly. And that is what is called or that is what is called one version of Arzela-Ascoli theorem.

So, let me repeat again suppose, the family is bounded and equicontinuous then it has a uniformly convergent sub sequence, this is given as one possible statement of Arzela-Ascoli theorem. Arzela-Ascoli theorem also has again since its being old, it has many statements. Ultimately one can show that all those are equivalent and also several proofs. So, the proof which I am going to discuss today is from this book, the author is Bela Ballobas, and the title is linear analysis.

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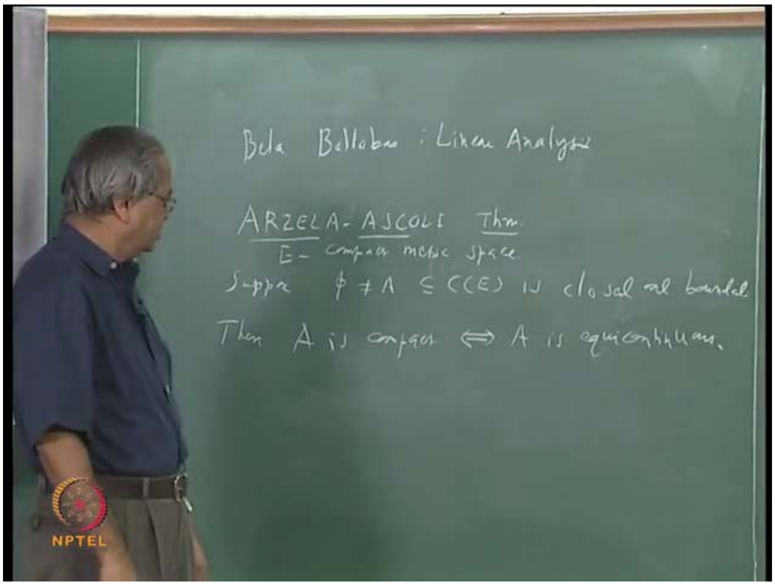

This is because the treatment given here is simpler than either Rudin or Siemens and that you will notice when you proceed with the proof. So, we go to this Arzela-Ascoli theorem, this is one theorem which is also quite useful, only thing is that all its uses or at least most of the well known uses are outside real analysis. It can be used to show about the existence of solutions of differential equations, integral equations etcetera, but anyway we should know this theorem.

So, let me again recall the statement suppose as usual let us take E as a compact metric space and take some non empty family, suppose contained in C of E is closed and bounded. Then what does the theorem says then A is compact if and only if A is equicontinuous. Let us now, look at the proof. Now, even before beginning with the proof, let me make a small remark here. We have already observed in yesterday's class that whenever it is a metric space saying that metric space is compact, one of the equivalence criteria is that it is complete and totally bounded.

Now, we already know that C of E is complete and we have assumed that A is closed, so that means A is complete .So, to show that A is compact all that we need to show is that A is totally bounded. So, in fact showing this is essentially same as showing that A is totally bounded if and only if A is equicontinuous. Let us take this part first this is relatively easy again both of these proofs have the same idea.

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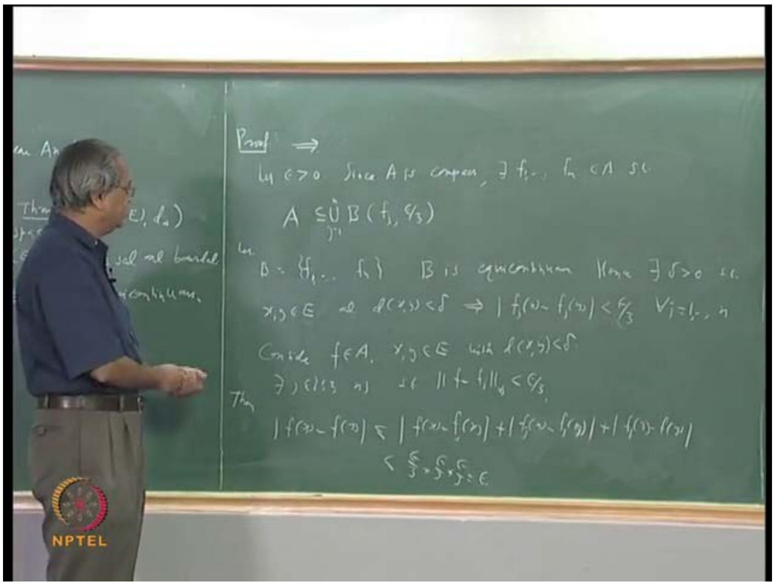

You have to have this epsilon by 3 at various places. Now, let us take epsilon start with epsilon bigger than 0. We want to show that A is equicontinuous that means we have to find the delta which satisfies the required, but we know that A is compact. And hence it is totally bounded and hence it is covered by a finite number of open balls with radius epsilon. What is the notation for the open ball, do you use this B a r or u?

So, we can say that since A is covered by a finite number open wall, what is the meaning of that? It means there exists a finite number of functions f 1, f 2, f n such that, if you take the balls with centre at those functions A is contained in that. So, since A is compact that is what I am using this totally bounded. Since A is compact there exists f 1 f 2 let us say f n in A such that, A is contained in open ball with centre at f j. And let us say radius epsilon by 3 and union j going from 1 to n.

Remember our metric is this given by supreme C d infinity, d infinity or norm suffix infinity metric given by this. So, by this ball what it is that, all functions such that mod of f j x minus f x less than epsilon by 3 for all x in E. Now, you look at this family f 1, f 2 in fact we are doing something similar to what we did there f 1, f 2, f n. Suppose, I call this family b then this is an equicontinuous set this is an because this is an finite, it contains only a finite number of functions.

So, b is equicontinuous, so hence there exists delta bigger than 0 such that x y in E and d x y less than delta. This implies for every function in this mod of f x minus f y is less than again let us take epsilon by 3 such that mod of f j x minus f j y is less than epsilon by 3 for all j equal to 1 to n that means for every function in this. Now, our idea is to show that this same delta works again using something similar because what is happening here is that, every function in A is close to 1 of this function. At these functions all are satisfy this, these are the two ideas that we shall use.

So, now let us consider f in A x y in E with d x y less than delta. We want to show that mod f x minus f y is less than epsilon that is our aim, but since f is in A f is in this union of this bars. So, there exists some j such that distance between f and f j is less than epsilon by 3. So, there exists j in this set 1 2 n or you can say that there exists some f j in b such that this norm of f minus f j suffix infinity, this is less than epsilon by 3. And then consider mod f x minus f y then, this is less than or equal to, again we are basically adding and subtracting f j x and f j y.

So, this is less than or equal to mod f x minus f j x plus mod f j x minus f j y and then plus mod f j y minus f y. And again notice that this first term and the last term that is less than epsilon by 3 because norm of f minus f j is less than epsilon by 3 because that is the supreme over mod f x minus f j x for all x in E. So, this and that is less than epsilon by 3 because of that what about the middle term? Middle term is mod f j x minus f j y that is less than epsilon by 3 because f j is one of the functions here and we have chosen delta corresponding to that.

So, each of this is less than epsilon by 3 for different reasons, but that is fine. So, taking a review of this whole thing, you will see what is what we have done. We started with an epsilon bigger than 0 and we produced a delta such that for every f in A and x y in E with d x y less than delta. We have showed that mod f x minus f y is less than epsilon that means that A is equicontinuous. Now, let us look at the other way that is what it means that, we assume that A is equicontinuous.

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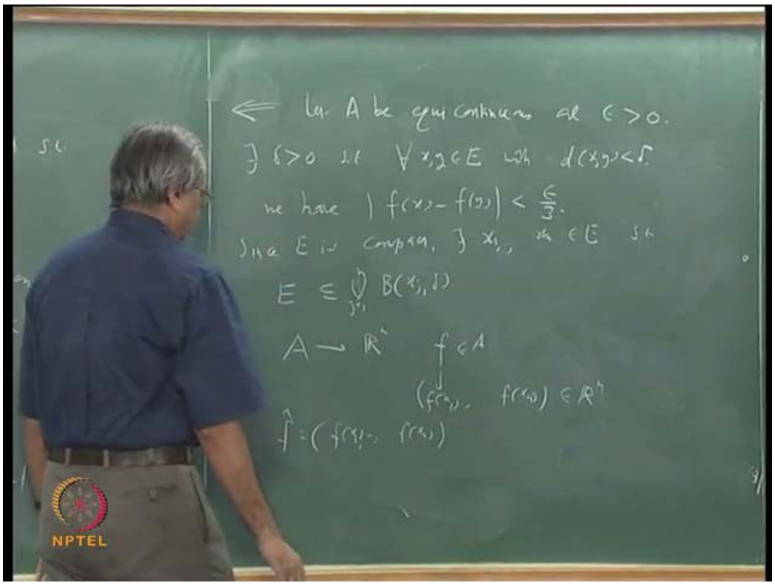

And we will show that A is compact again as I observed earlier showing that A is totally bounded is sufficient. So, let A be equicontinuous and epsilon be bigger than 0 and what is it that I wanted to do. I want to show that A is covered by a finite number of balls with radius epsilon or which is same as saying that there exists a very finite number of functions such that, A is contained in ball with centre at f j and radius epsilon, that is what we want to show.

Now to do that, we will use this equicontinuity. First of all let us say what happens, because of equicontinuity we can say that since A is equicontinuous there exists delta bigger than 0 such that, for all x y in E with distance between x and y less than delta. We have mod f x minus f y less than I will again take epsilon by 3 because we have to basically do the same thing add and subtract a few terms.

Now, once this delta is found there is one more thing that I want to observe and that is the following. Till now we have not used that explicitly, but this set E is metric space, E is compact. Since E is compact, E is also covered by a finite number of balls with whatever edges you want, I will take that radius as delta. So, E can be covered by a finite number of balls of radius delta. So, what is the meaning of that it means that there exists some finite number of points, suppose I call those points as x 1 x 2 x n, such that E will be contained in balls with centre at those points and radius delta.

So, we can say that since E is compact there exists x 1 x 2 let us say x n in E such that, E is contained in ball with centre at x j and radius delta and union j going from 1 to n. There are n points in E and the ball with centre at those end points and radius delta covers E completely. Now, what I want to do is that, I want to consider values of every function at those end points. Suppose, I take any f in A and I consider the values of f at x 1, x 2, x n each of this is a real number.

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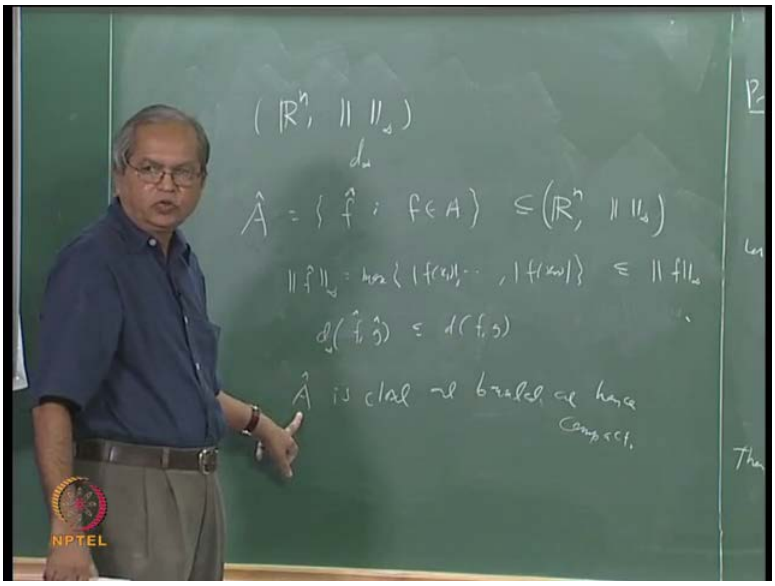

So, you will get a point in R n, you will get a point in R n. So, what I want to do is I define the map from A to R n and what is the map f goes to f in A, this goes to f x 1, f x 2, f x n, this is in R n. And in order to since we are referring to such n, I will give some notations. I will call this f hat, so what is f hat, f hat is this f x 1, f x 2, f x n. Now the idea is we shall look at the image of A in R n, but I also want to regard R n as a metric space.

So, I should give some metric on R n and there also I shall give this suffix infinity metric.

So, consider the metric space R n, R n with this norm suffix infinity or d suffix infinity. What is the meaning that if you take any f hat like this, its norm is supremum maximum of mod f x 1, mod f x 2 and mod f x n. So, suppose I take all such f hats and I will give some notation for this. Suppose we call that set as A hat, this is the set of all f hat for f in A, this set of all f hat for f in A. So, this is a subset of R n, and R n as I said we take R n with norm suffix infinity. So, first I want to say that this set A hat, see remember we started with the set A, which was closed and bounded. So, I want to say that this set A hat is also closed and bounded.

So, once we say that A hat is closed and bounded since it is in R n it will automatically imply that A hat is compact and hence totally bounded etcetera, but we will go to that a little later. First of all let us note one thing what is the norm of an element f hat here? Norm of f hat or norm f hat suffix infinity that is nothing but maximum of mod f x 1 mod f x 2 etcetera because there are n such numbers values at n points mod f x n. This is less than or equal to norm because this is supremum of mod f x for all x in E whereas, x 1, x 2, x n are some subset of E.

So, maximum of those n numbers is obviously less than or equal to supremum taken over r, which also means that if you take any two functions then, distance between say that b infinity f hat g hat is less than or equal to norm of distance between f and j. Now, does that show immediately that A hat is bounded, see what is the meaning of saying that A is bounded, A is bounded means there exists some number m such that, d of f g is less than or equal to m for every f and g in A. It will follow immediately from here that, if that is the case for d be distance between f hat and g hat is also less than or equal to that number m.

Will it also show that A hat is closed, how does one show that any set is closed? One can take any various definitions about the closeness, but one possible thing is that you can take you can consider, suppose some point is in the closure of A set then, there exists a sequence of elements in that set which converges to that point. So, suppose f hat is in the closure of A hat then there will exist a sequence f n hat, which converges to that and then you show that the limit f hat is also in the same set a hat.

So, that is elementary again it will follow from this. So, I shall leave that to you as an exercise, show that A hat is closed and bounded. Since A hat is also closed and bounded and it follows from this relationship between the two metrics. But A hat is a subset of R n that is a finite dimensional space and hence compact in R n every bounded set is totally bounded and hence compact. Now, it is compact and hence totally bounded, we can say that A hat is also covered by a finite number of open walls with radius again let us say epsilon by 3.

What is the meaning of that, it will get they will again exist a finite number of functions such that, if you take walls with the centre set say those functions, suppose those functions are f 1 hat, f 2 hat etcetera. Then A hat is contained in the walls with centre at those functions and radius, let us say again epsilon by 3.

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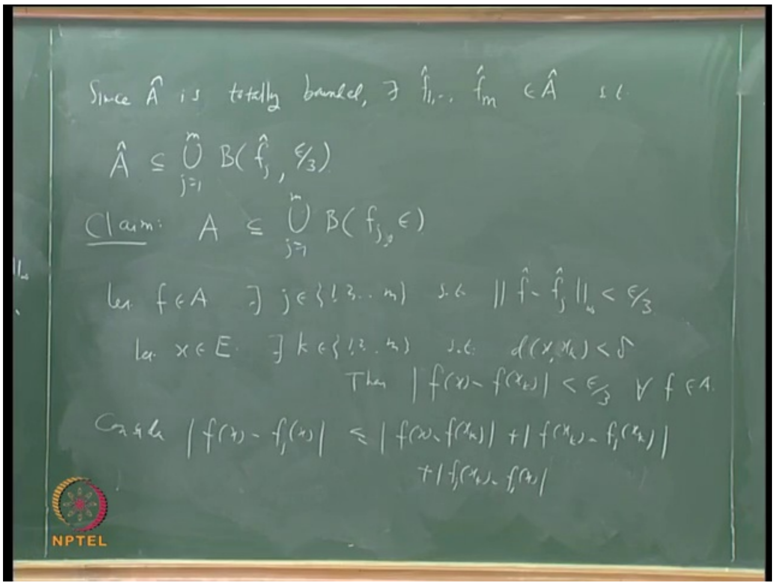

So, we can say that since A hat is totally bounded, there exists f 1, f 2, since they have taken n such things that number need not be same as this. So, let us say f 1 f 2, f m in A either you can say f 1 f 2 f in A or f 1 hat etcetera f m hat in A hat f 1 hat f m hat in A hat such that, A hat is contained in union j going from 1 to m union of open ball with set at f j hat and radius epsilon by 3.

That is not what we want, what we want is that A is totally bounded. So, what I want to say now is that, you just take this corresponding functions f 1, f 2, f m and take the ball with radius epsilon then, A is contained in that. So, let me first write that as a claim and we will give the proof later. Claim is this A is contained in union j going from 1 to m open balls with centre at f j and radius epsilon.

So, suppose we prove this claim, what it will be, that we started with epsilon and then we showed that for any epsilon A is contained in the finite number of open balls with radius epsilon. That will be in that A is totally bounded and hence compact. So, once this claim is proved, proof of theorem is over. Now, to show this means what, we have to take some let f be in A then, we have to show that this f there exists some f j such that, norm of f minus f j is less than epsilon.

Now, what is the obvious point to start with is, the corresponding effect f hat, f hat is in this. So, for f hat there exists some z such that, norm of f minus f j hat is less than you start with that j. So, we can say that let f b in A so there exists j in this 1 to m such that, norm of f hat minus f j hat suffix infinity, this is less than epsilon by 3, but this is not what we want actually? What we want is norm of f minus f j that should be less than epsilon.

What is the meaning of that, it means that if you take any x then mod f x minus f j x should be less than epsilon, that is what we want to prove. So, let x be in E since x is in E we may have to modify this argument here. We will come to that since E is contained in the open union of this open balls, x is close to one of these and maybe we have to take that particular. So, we can say that there exists for the time being, I will call it there exists k in this 1 to m etcetera such that, x belongs to ball with centre at x k and radius delta, that is same as saying that d x x k is less than delta.

Do you agree with this, given any point x in E, there will exist some x k one of this x 1, x 2, x n such that, distance between x and x k is less than delta because E is contained in the union of those n open walls. If distance between x and x k is less than delta, what we know is that for all of these whatever be the function in A mod of f x minus f x k that will be less than epsilon by 3. Remember what is important is that, in fact this is something I forgot to write here, that is the most important part.

Since A is an equicontinuous family mod f x minus f y is less than epsilon by 3 for every f in A. You will realize the importance of that when we go to this proof, mod f x minus f x k is less than epsilon by three for every f in A. In particular this is true also for f j that is the important thing here because f j is also because mod f x minus f x k less than epsilon by 3. And also mod f j x minus f j x k is also less than epsilon by three and that is the fact that we shall be using.

Now, what we want to see? Now, consider mod of f x minus f j x and we will add and subtract f x k and f j of x k. So, this is less than or equal to mod f x minus f x k plus mod f x k minus f j at x k and then plus mod f j at x k minus f j at x. What is to be observed now? Since the distance between x and x k is less than delta mod of f x minus f x k is less than epsilon by 3 for every f in A and in particular that applies to f j also, that is why this mod f x minus f x k and this mod f j x k minus f j x these two terms are less than epsilon by 3. What remains f of x k minus f j of x k, what about that f of x k minus f j of x k that is nothing but one of the value of f of x k minus f of x k minus f j of x k this is less than or equal to what we would have called norm of f hat minus f j hat.

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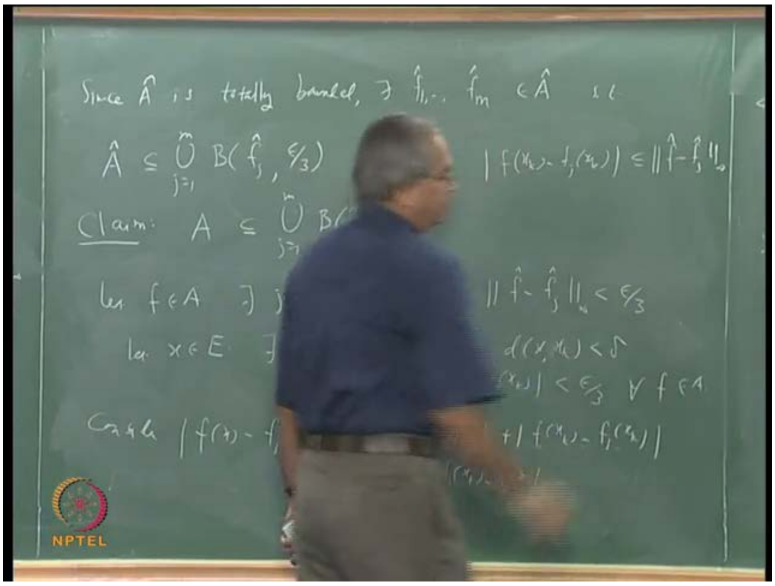

What is norm of f hat minus f j hat, that is the maximum of mod f x 1 minus f f j x 1 mod f x 2 minus, that is the maximum of those n numbers maximum of those n numbers and here you have got only one of those numbers. So, it is less than or equal to f hat minus f j hat suffix infinity and that is less than epsilon by 3 because of this, because A hat is contained in b. That is because of this mod f hat minus f j hat is less than epsilon by 3, that was our starting point.

This mod norm of f hat minus f j hat, which is the maximum of mod f x 1 minus f j x 1 mod f x 2 minus f j x 2 etcetera of those n numbers and that, is one of those n numbers.

So, each of this is less than epsilon by 3 plus epsilon by 3 or which is equal to epsilon. This proves the claim and hence the proof of this because this means that A is totally bounded and hence compact. That is what we wanted to show that is if A is equicontinuous then A is compact.

As I mentioned in the beginning, the statement as well as proof of this Arzela-Ascoli theorem, the way in which I have given here may not be same in every book. Because there are very various equivalence statements about the compactness, different books prefer to give their own versions of this theorem. Instead of saying that A is compact, what they will say is that the every sequence has a convergent sub sequence.

And that is what I said in the beginning that if it will follow from this that if you take A equicontinuous family and if it is bounded and then it has a uniformly convergent sub sequence. Now, before closing let us just take an example. See till now we have seen the examples of the families, which were equicontinuous and which were not equicontinuous. Of course, if a sequence is uniformly convergent sequence that is the best non trivial example of any equicontinuous family.

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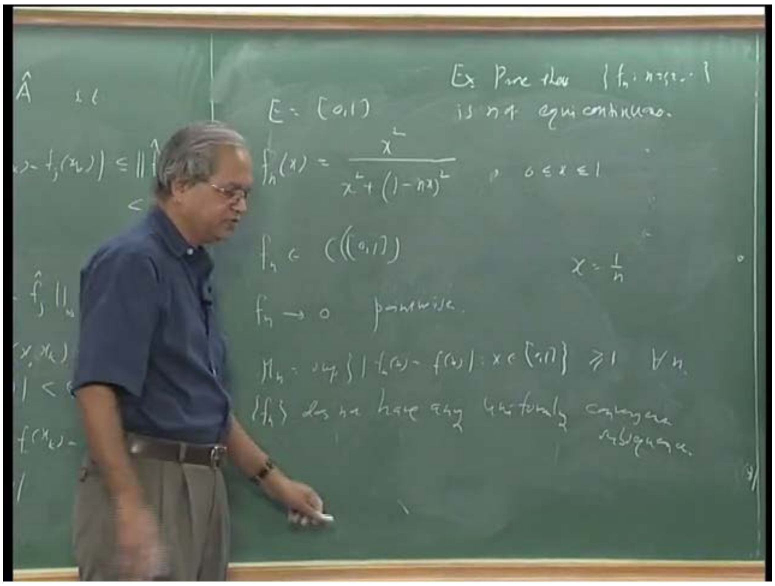

Let us now take something else suppose I define let us take E as the set 0 to 1 and suppose I take f n of x as x square divided by x square plus x minus n whole square 0 less than or equal to x less than or equal to not x minus n. Let me just change this slightly 1 minus n x whole square. Is each f n continuous, but that is no problem. Does the sequence f n converge to some function? Take a fixed value of x, does this converge to anything, what is that 0. So, f n converges to 0, we can at least say point wise, whether it converges uniformly or not. Does it converge uniformly, why does it converge uniformly? Let us check, as I said the best way to check this is compute estimates of m n, what is m n, m n is supremum of mod f n x minus f x for x is 0 to 1.

Out of which f x is 0, so we can forget about that. So, it is just supremum of mod f n x, but f n x is also each in 0 to 1, this is also a positive number, so we can forget about mod also. It is nothing but supremum of f n x for f x in 0 to 1. Now, suppose you take x equal to 1 by n, what happens to f n x, it becomes 1. So, I can always say that this is bigger than or equal to 1, do you agree m n is bigger. It may have some value, which is not very important as far as deciding uniform convergence.

So, m n x is bigger than or equal to 1 for all n. So, what does it say about the uniform convergence? So, it means that since this does not converge to 0, f n does not converge to 0 uniformly. Does it have any uniformly convergent sub sequence? Suppose instead of this f n suppose I have taken some f n k, is it clear then also this will be true, but if you take m n k that will also be bigger than or equal to 1. So, no sub sequence of f n also can converge uniformly.

So, you can say that this sequence f n does not have any uniformly convergent sub sequence. So, what does it say in terms of equicontinuity, then... Since it has no uniformly converged that follows, but in this case I would also give the definition and I will give that to you as an exercise. Check by not using this, by using Arzela-Ascoli theorem it follows that this is not an equicontinuous. As an exercise you try to prove without using that. Prove that this f n, n is equal to 1, 2 etcetera is not equicontinuous that by using only definition.

And the idea of the proof, I have given you already. To show that something is not equicontinuous again what we have to do, we have to show that for some epsilon there exists low delta and it is clear from here what epsilon you should take. Again take epsilon is equal to half and you know that f n x becomes one somewhere; it also becomes 0 somewhere at x equal to 0. And that is an enough material to show that it is not an equicontinuous family. And we will stop with that for today.