An Invitation to Mathematics
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Unit
Combinatorics
Lecture – 13
Permutations and Cycle Type

Welcome back, so what we will talk about today is more about Permutations. So, so far what we have been interested in doing is well counting permutations or counting combinations and so on. Now, what we like to do is to study permutations in a, you know slightly different way. What we like do to is to try and understand permutations themselves, rather than just count the total of permutations.

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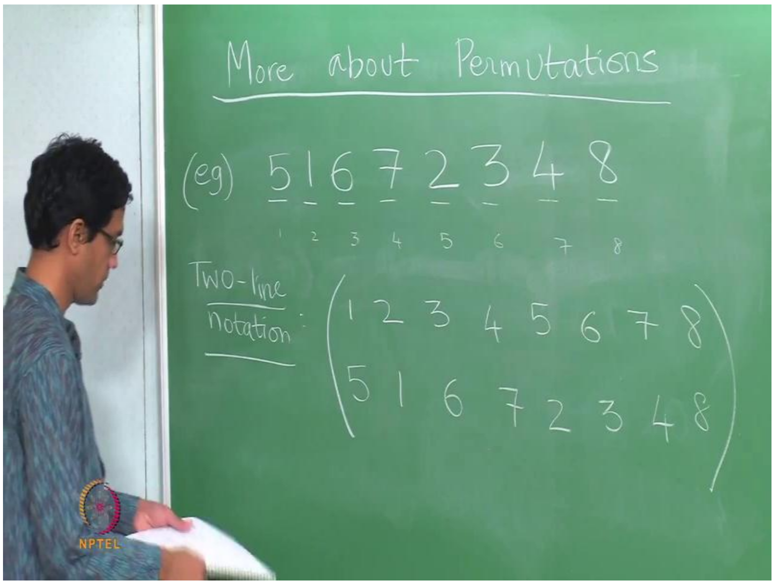

So, recall what is an example of a permutation? So, well a typical permutation is just a rearrangement of the numbers. So, let me write down an example 5 1 6 7 2 3 4 8, so here is an example of a permutation of the numbers 1 through 8. Now, so sometimes we can think of these as being the various positions, so there are 8 positions which we are filled with these numbers in some order. So, often there are various notations which make it somewhat easier to think about permutations.

So, one of them is what is called the two line notation, so two line notation, so this permutation here would be represented as follows. So, we think of this first thing here as really being position 1 and the second blank here has been the position 2, the third is position 3 and so on. So, we think of each of them as being a certain position 5, 6, 7, 8 and then we write down what... So, the first line here is the positions, so in position 1 position 2, 3, 4. So, we first the top row here, the top line is just a position numbers and the bottom line is the entry in that positions.

So, for example in the first position here we have the number 5, the second position has number 1 6 7 2 3 4 and 8. So, this notation here is sometimes called the two line notation for the permutation. So, we think we wrote out first is often called the one line notation, where we do not explicitly write out the positions themselves, but only keep track of the numbers in the various positions. So, this is another way of representing the same permutation.

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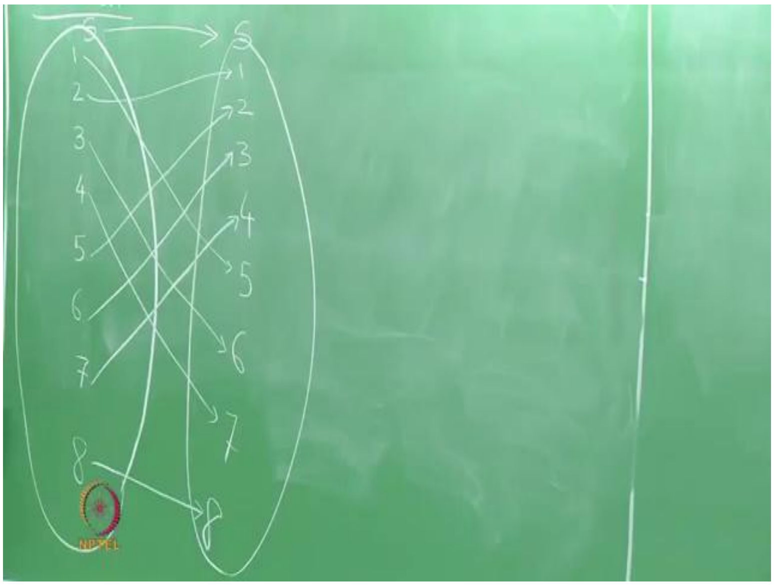

Now, observe, so there are also pictorial way you can think of a permutation as really being a function. So, what do you mean by that, so imagine I have the set consisting of the numbers 1, 2, 3, 4, 5, 6, 7 and 8. So, that is my set, so it mean write as a set and a permutation, so let me call this set as S. A permutation can be thought of as giving you a function from the set S to itself, so I have the set S to the set S.

So, what is this given permutation think of it as the first set S as sort of being like the position numbers and the second set S sort of representing the increase themselves. So, what we have is something it change 1 to 5, 2 to 1. So, let us write this out this sense 1 into the number 5, 2 maps to the number 1. So, this function sends 3 to 6, 4 to 7, 5 to 2, so 2 3 4 and 8. So, it goes to 2, it goes to 3, 7 goes to 4 and 8 goes to itself 8.

So, this picture here that we have drawn is well another way of thinking of the same permutation, it is a function from the set 1 through 8 to itself and further it has the following nice property that it is a one to one and onto function.

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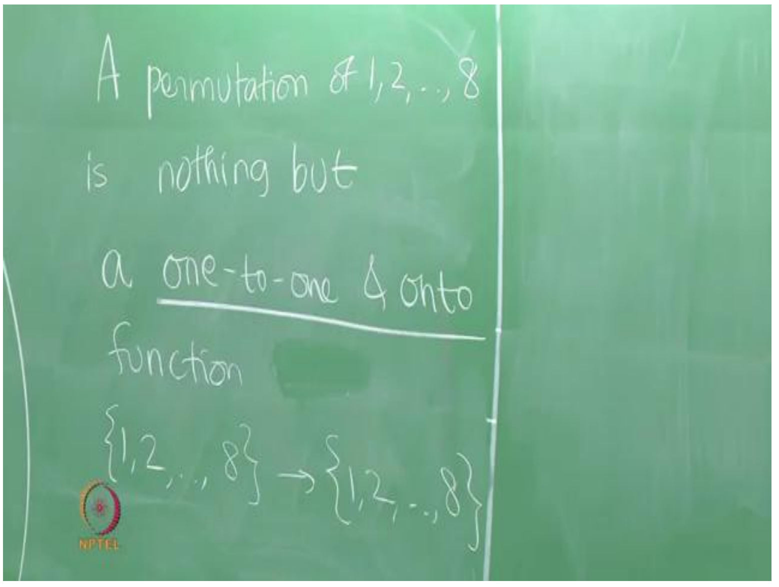

So, well what is the permutation? So, observe a permutation of in this example the numbers from 1 to 8 is nothing but a function from the set consisting of the numbers 1 through 8 to itself. So, let me from it is the function from 1, 2, 3, 4, 5, 6, 7, 8 to itself, but it is more it is in fact one to one and onto function that is a key observation here. It is these two put together are what sometimes called a bijection or a bijective function. One to one of course, means that two different elements of S here do not map to the same element.

So, everything maps to different elements and onto means that every single element in S here has something which maps to it. So, it is sometimes called the one to one corresponded, everything here maps to different things there and everything there does get map from something. So, it is a one to one and onto function and so sometimes such diagrams are nice to have.

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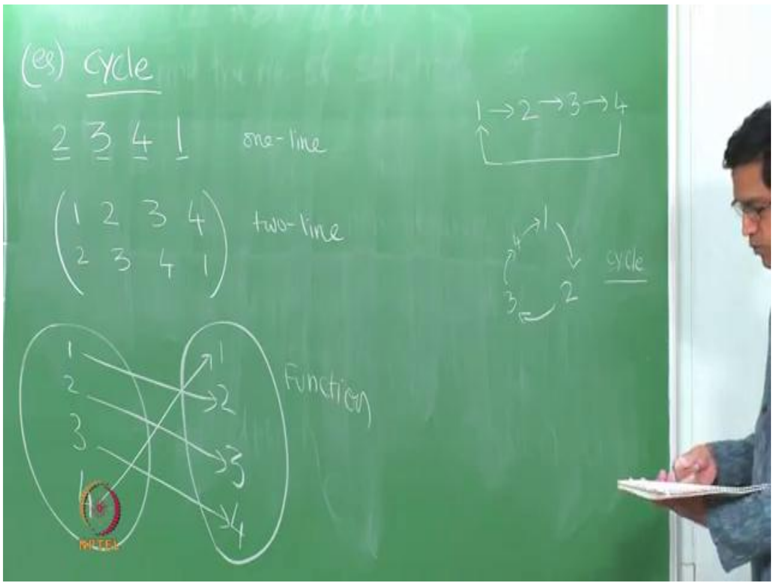

Now, one on the simplest examples of a permutation general, it is what is called cycle. So, what is an example of a cycle, so let us assume we have the numbers 1 2 3 4, so here is an example of a cycle 2 3 4 1. So, what is this? So, this is the one line notation, so now, I written this out in one line notation, the same thing in two line notation would be the following. So, I have 4 positions, so I have 1 2 3 4 those are the 4 position numbers and position 1 maps to 2, in position 2 I have 3, in 3 I have 4 and 4 I have 1, so same permutation in two line notation.

And so again you could also think of it in terms of the function, so what is it from the function point of view. So, I have a diagram as I did before, I have 1 2 3 4 the set 1 2 3 4. So, here 1 goes to 2, 2 goes to 3, 3 goes to 4 and 4 goes to 1. So, those are the three different notations or ways of thinking about it, but in this case this yet another thing one can do. So, we have 1, so imagine you know what happens, so think of it from the function point of view or the two line notation point of view.

Let us start with 1 and look at what 1 maps to, so 1 maps to 2 and now I am a 2 and I look at what 2 maps to, 2 maps to 3 according to this, 3 maps to 4, it is then maps back to 1. So, here is yet another way of thinking about or depicting this permutation, it just say something which sense 1 to 2 to 3 to 4 back to 1. So, that is a reason why it is called a cycle, it is you can think of it most ideally as well numbers 1 2 3 4 arranged in a circle with sort of arrows telling you. So, let us assume we have to go clock wise.

So, it tells you, you know which maps to which and they sort of map to each other in a cyclical fashion. So, this is why such a permutation is called a cycle, sometimes it is called a 4 cycle to emphasize the fact that there are 4 numbers.

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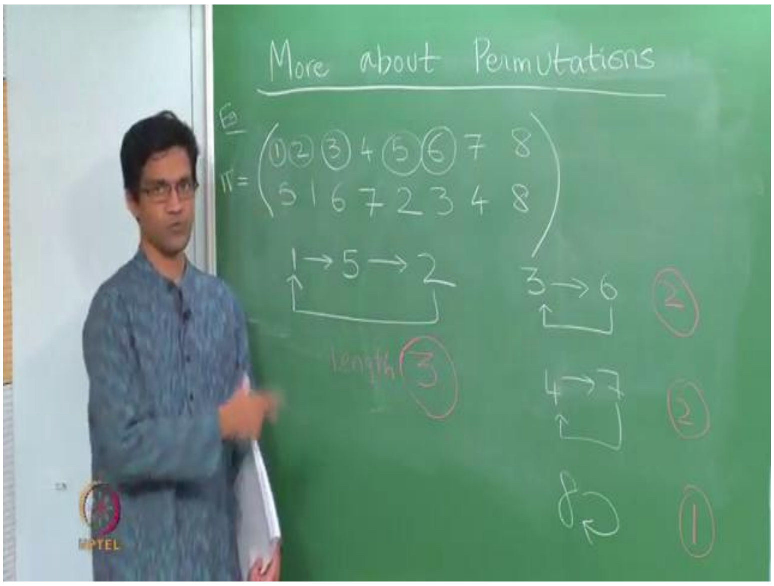

Now, so cycles themselves are a very important special case of permutations. Now, let us do a more general example, so let us take the permutation which we had written out at first, so 1 in two line notation 3 4. So, here is the original example 5 1 6 7 2 3 4 8 and so this was the two line notation. So, let us give this permutation and name let us call it pi, the permutation pi does this.

Now, let us try and think of this in terms of cycles, so what is that mean. Let start with the number 1, so we start with the number 1, so it is a pi and look at what it goes to, so 1 maps to 5, so I write it in the same way I have 1 mapping to 5. Now, well what is next now the time sort of at 5, I look at what 5 maps to under this permutation. So, 5 maps to 2 under this permutation, so I note that down. Now, I am a 2, so I go back to 2 and look at what that maps to let seems to map back to 1. So, it is look like 1 5 and 2 they form a cycle in this permutation 1 maps to 5 maps 2 it maps to 1.

So, that takes care of these three numbers 1 2 and 5 are taken care of, they form part of this cycle. So, let us keep going, let us look at the next number that is not in the list is a 3. So, I write down the 3 and see what a 3 goes to, 3 goes to a 6, but then the 6 maps back to a 3. So, it seems as if I have the following 3 goes to a 6, 6 maps back to 3, so I am also done with these two numbers 3 and 6. Now, I look at 4 7 and 8 I look at 4 which maps to 7, 7 maps back to 4.

So, I again have 4 maps to 7 maps to 4 and 8, so last guy here since to map back to itself. Now, this picture that I have drawn here is not that of a single cycle, but rather of 4 different cycles. So, I have here a cycle of length 3, so let us keep track of the lengths of the cycles, here I have a cycle which has three numbers. So, I call that cycle of length 3, this guy here is a cycle of length 2, this guy here is a cycle of length 2 and the last guy the cycle of length 1. So, cycle of length 1 just means it something which maps back to itself and the permutation.

So, now what we have done is a following we looked at these eight numbers, we have looked at a permutation of these eight numbers and we have what we would call decomposed it into a cycles. So, this permutation really is composed of well there is a 3 cycle, there is a 2 cycle, another 2 cycle and a 1 cycle, these 4 cycle put to gather give you back this permutation. So, some times this way of decomposing is called the cycle decomposition and so we often have a yet another notation for this, so this is call the cycle decomposition.

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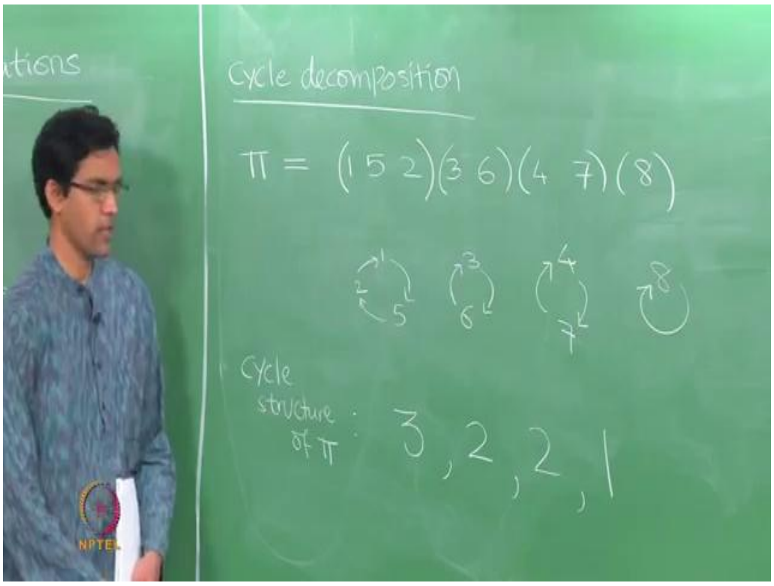

So, the notation for this the following with given permutation pi that we had the beginning we are notice that it has the following cycles there is 1 goes to 5 goes to 2 these form a 3 cycle. So, we write this with in brackets, this is to be interpreted as well this is just 1 maps to 5 maps to 2, this is just another way of writing it without having to write circles and so on. Now, I have 3 6 which is short hand for 3 maps to 6 maps to 3 and I have 4 7 and 8.

So, this is again short hand for 4 maps to 7 and this is goes to 8 maps to itself. So, this notation up as is just what is called the cycle decomposition of the given permutation and the cycle structure or cycle type. So, some times to words for this, sometimes we call it the cycle structure of pi or the cycle type of pi is the following it is the you just keep track of the lengths of the cycles. So, for a example here it there is a 3 cycle and there is a 2 cycle and another 2 cycle and 1 cycle.

So, we would often say that the cycle structure of pi is just the list, so what is the cycle structure, it is a list of numbers arranged in descending order. So, there is we do not really want to know the order of the... So, we would in want to say the cycle structure is say 2 plus 3 plus 2 plus 1 and so on, we are only interested in knowing what are the various lengths which appear. So, we just arrange them in descending order, so we have 3 3 2 1.

And notice that the total the sum of all these numbers must always be 8, because that is the numbers in all the total number of numbers is 8. So, of course, the sum lengths of all the cycle that you get must at better be 8. So, 3 2 2 and 1 would have to add up to 8.

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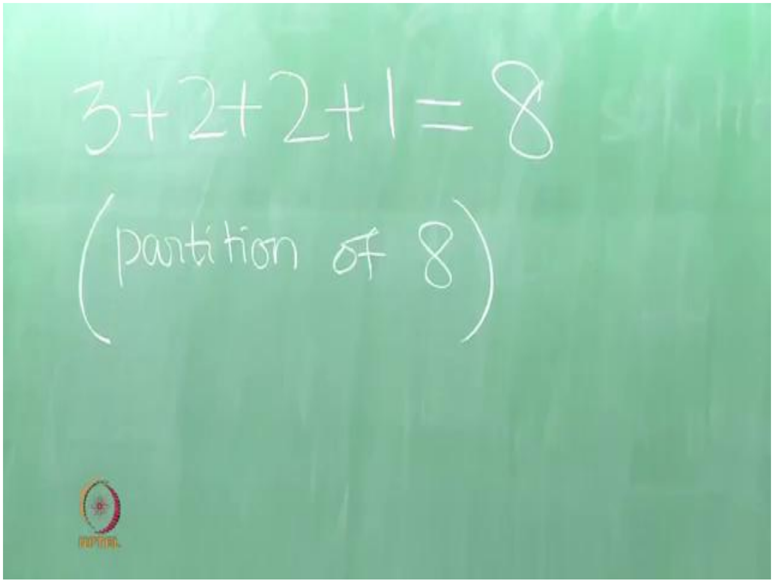

So, observe that the cycle structure always has this property has single type and so sometimes we call this is usually what is called the partition of 8 a partition definition is it is a way of writing the given number as a sum of natural numbers. So, what are other possible partitions of 8 instead of 3 2 2 1, you could say for a example 7 plus 1 would be another partition or 6 plus 1 plus 1 would be another partition and so on.

So, there is a very long list of partitions of 8, this particular privation has this cycle structure which is you know which gives you this particular partition of 8. So, let us sort of look at counting problem which again arises naturally in this context. So, it is a finer counting problem then just counting all permutations.

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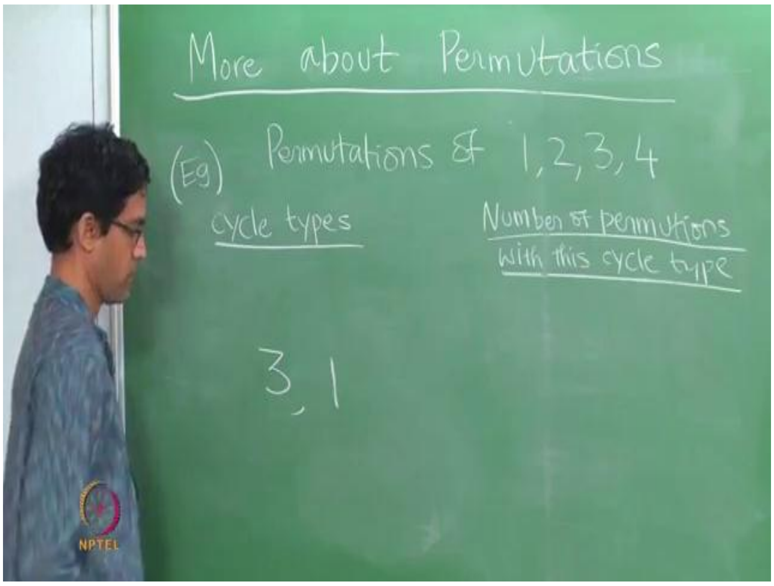

So, again we will do it by example, so 8 is somewhat large for the purposes let us take a just permutations of 4. So, I am going to look at permutations of the numbers 1 2 3 and 4 observe that the total number of permutations is 4 factorial which is 24, but what I want to do is we have just talked about cycle types. So, I want to do the following, I am going to look at I am going to categories them by cycle types.

So, I am going to make a table which has cycle types on one hand and the number of permutations with that cycle type on the other column. So, for a example what do by mean, so let me look at the following let me look at permutations who cycle type is 3 1. So, I am going to the fill is, but let me start with this I am going to look at permutations who cycle type is 3 comma 1. So, what do we mean by that.

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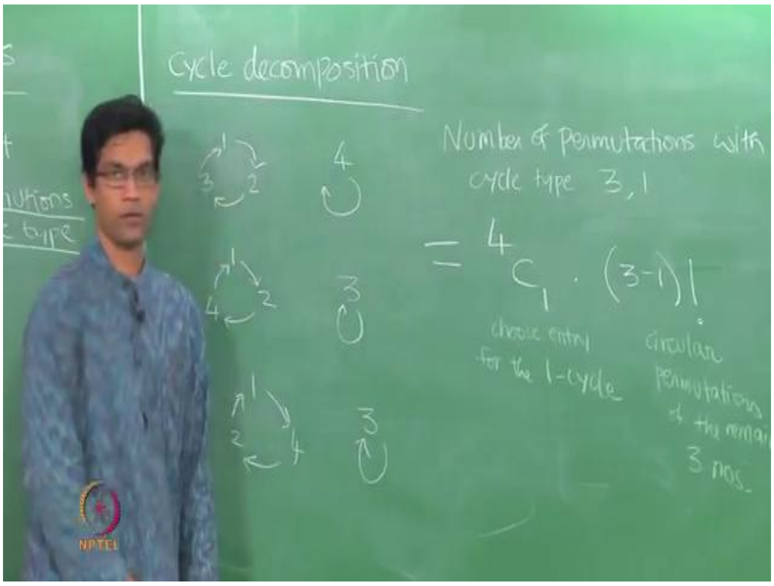

So, notice that when you say cycle type is 3 and 1 what it means is I am looking for a permutation of 1 2 3 and 4 in which there is a 3 cycle for instants 1 maps to 2 maps to 3 back to 1 and 4 maps to itself, here is an example of a permutation who's cycle structure is 3 comma 1 and well what are other examples. So, I well ((Refer Time: 17:01)) I have a 3 cycle and I need to have a 1 cycle.

So, how do I write down other examples well first let choose the what goes into the 1 cycle that is the easiest thing to do. So, instead of 4 I could pick some other number let we need 3 as the element which is in the one cycle, then the numbers in the 3 cycle are determine. So, 3 is already done, so I need to look at the numbers 1 2 and 4. So, I have the numbers 1 2 and 4 and how many different 3 cycles can I form with those.

So, for instance I could arrange them as 1 going to 2 going to 4 or I could arrange them as 1 going to 4 going to 2. So, at the moment what I have done is just write down the bunch of examples of permutations of the numbers 1 2 3 and 4 for which the cycle structure is 3 comma 1. Now, what we want of course is to count the total number of such permutations. So, let us try and do this, so the number of permutations with cycle type 3 1 how do we trying count this.

So, here is the idea we have already used at when trying to write down the examples of such permutations, first we pick the number which goes into the 1 cycle. So, first meet out of these four numbers, we choose one number. So, that is 4 C 1 ways of choosing one number and what is this 4 C 1 represent, this is the number of choices for the entry which goes into the 1 cycle. So, here we just chosen choose entry for the 1 cycle. So, that is 4 C 1 choices having done that what are we left with, we are left with three numbers, what are that one number is gone and we have three numbers. For instance, here we have the numbers 1 2 and 4, now there is still remains the issue of arranging those three numbers into a circle.

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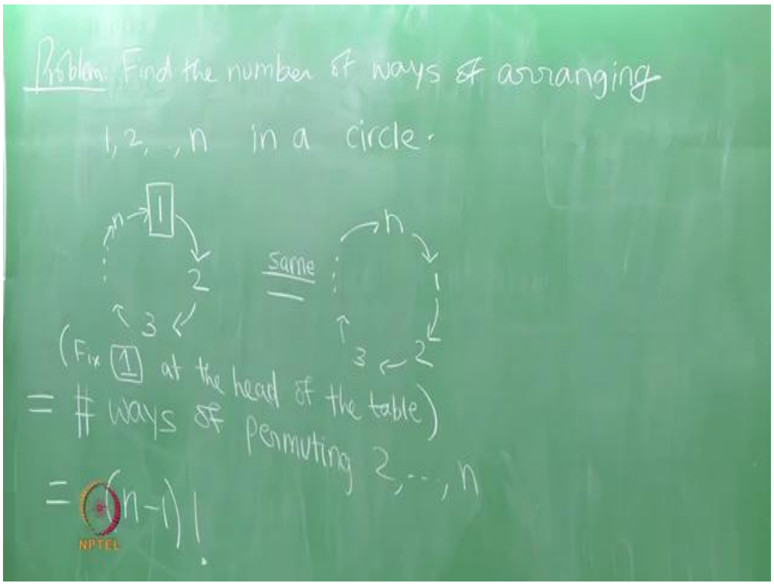

So, what we still need to do is the following we need to figure of how many these are there of arranging say these three numbers in a circle are more generally here is a problem that will come up when you want to do this find the number of ways of arranging n numbers in a circle, let say the numbers 1 to n in a circle. So, observe here is the thing that is a for a instance I have the arrangement 1 going to 2 going to 3 going to n going to 1.

Now, I will thing of this arrangement as being the same as an arrangement in which I move the numbers clock wise. So, as say I want to only arrange them in a circle what I mean is that I only want to look at the relative positions, what matters is only now what is to the right of the 1 and what is to the left of the 1 in a not so much where 1 is sitting. So, for instance this arrangement here I will think of as being the same as the arrangement in which everything is rotated say by the 1 units in n comes to the top and 1 the 2 and so on, this arrangement is the same as the arrangement in which the one this short of the head of a table.

So, now, this is the usual thing of circular permutations, so how do we count this number.

So, the key idea is since we do not really mind rotating all of them by some unit, here is what we will do, we will fix once and for all somebody in the head of the table. So, when you have the circular table like this, think of this the head of the table this position as being fix. Because, I can always assume that one for instance, the number 1 occurs at the head of the table and why could I do this.

Because, since I do not care about rotating them all I could easily rotate if one is intent by head I would rotated sufficiently. So, as to bring one to the head of the table, now once you have done that the remaining n minus 1 numbers you permute them in all possible ways. So, the number of ways of arranging the numbers 1 2 n in a circle is in fact equal to the number of ways of... So, we do the following we fix the number 1 at the head at the table, meaning in say this top position.

And just count the number ways of permute the remaining n minus 2 numbers in all possible ways, see permute these every possible way and since there are only n minus 1 of these the total number of ways of doing this is only n minus 1 ((Refer Time: 22:54)). So, this is what usually call circular permutations, so we are only interested in the relative positions and the number of such configurations is in fact n minus 1 factorial ((Refer Time: 23:10)).

So, coming back to this problem here, we have chosen the entries which goes into the 1 cycle, what is left is three numbers and those three numbers must be arranged in a circle and the number of ways of doing it as we just saw is 3 minus 1 factorial. So, it is 4 choose 1 times n minus 1 factorial, where n is 3. So, what is, this is just the number of circular permutations of the remaining 3 numbers. So, we have minus to solve this counting problem, the total number of choices would therefore, b 4 choose 1 is 4 and 3 minus 1 factorial is 2, so it is 4 into 2 which is 8.

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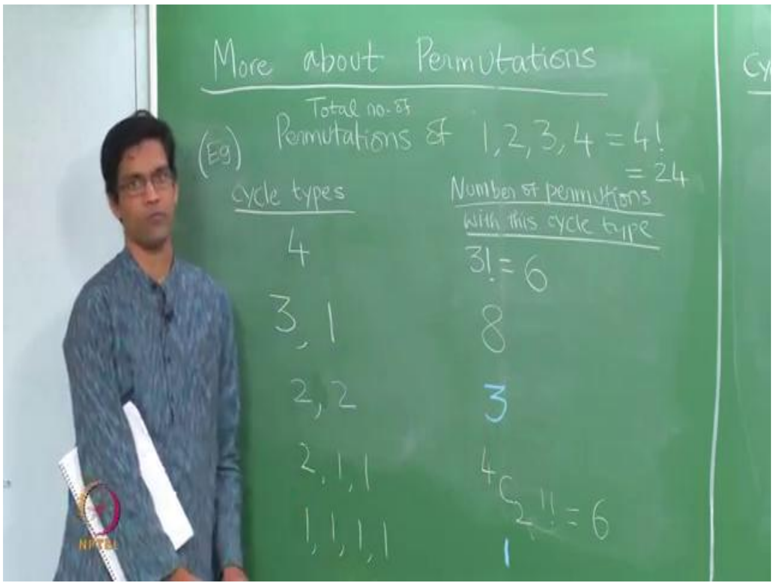

So, the number of permutation of cycle type 3 1 is in fact, 8 now of course, one wants do it for every other cycle type as well. So, observe as we said a cycle type would always be a partition of 4. So, you could have a 3 cycle and 1 cycle or you could just have a 4 cycle or you could have a 2 cycle and a 2 cycle, you could have a 2 cycle and 2 1 cycles or you could just have all 1 cycles. So, you could have these are the various possible cycle types and so I will just partially complete this table. So, if you have...

So, for instance let us count how many permutations would have cycle type 4, to have cycle type 4 just means how many 4 cycles are there, how many ways are there of arranging the numbers 1 2 3 and 4 in a circle and we have already solve the problem it is 4 minus 1 factorial. So, it is 3 factorial which is 6, so similarly if I have 2 1 1 you want to know how many ways are there of how many permutations are there with this particular cycle type, here is what you would do, you would pick well there are 2 1 cycles and 1 2 cycle. So, let us pick the two numbers which form part of the 2 cycle.

So, once you fix the 2 cycle the elements in the 1 cycles are automatically fixed. So, the number of ways of doing this is you first pick the two entries which will form part of the 2 cycle. So, there are 4 choose 2 ways of picking those 2 entries and once you have pick them you still need to arrange them in a circle. So, the number of ways of arranging 2 numbers in a circle is just 2 minus 1 factorial which is just 1 factorial. So, this is again 4 choose 2 which is the 6.

And so I am sort of doing this a little fast, because I really want you to think about this on your own. So, these remaining 2 entries observe that the total sum of all these numbers had batter the 24, because the total number of permutations of 1 2 3 and 4 is exactly 24. So, observe total number of permutations, so before, so here I have 6 8 6 and let me just fill these remaining two numbers and you should work these out to see that you get exactly these numbers, this is the 3 in the last minus 1 is as 6 8 3 6 1 let us check it is 24 let us 14 plus 317 23 24.

So, these are the various numbers of permutations of differ various cycle types. So, observe this is the somewhat finer counting of the set of permutations, the total number is just in a counted very easily it is 4 factorial. But, when you want to get a finer idea of how many have cycle type 4, how many have cycle type 3 and 1 or 2 2 and so on. Then you need to short of use these ideas of circular permutations and first choosing the entries something that. So, it is a slightly most sophisticated counting problem which brings in many of the techniques we have looked at so for. So, next time will do a little bit more about permutations specifically related to cycle types and the notion of sin of a permutation.