An Invitation to Mathematics
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Unit
Functions
Lecture - 24
More examples of functions on the plane, dilations

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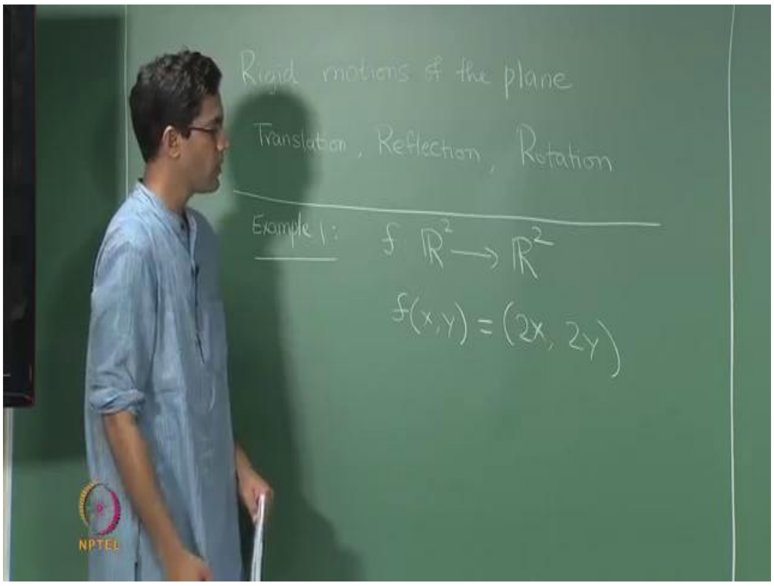

Last time we talked about certain kinds of maps from R 2 to R 2, and we called these the Rigid motions of the plane. And we studied a few examples, so the kinds we looked where maps which we called translations, then there where reflections, we looked at rotations. So, these were all examples of maps which had the following property that they did not change either lengths or angles. So, they sort of just move the plane in some rigid fashion and we also studied things like an invariant figures under these types of maps and so on.

So, now, what we want to do next is to talk about non rigid motions, so let us again start doing these by examples. So, now, we look at things which do not preserve lengths or angles or things like that in general. So, the first example will be the following map, so remember we are looking at the maps from the plane to the plane. So, let us define f to be the function which takes the point x comma y on the plane to the point, let say 3 x comma 2 y or, so for let us start with 2 x comma 2 y which is again a point on the plane. And let us try and study what this function does, so at the outside all it does is maps multiply the x co ordinate by 2 and multiply the y co ordinate by 2.

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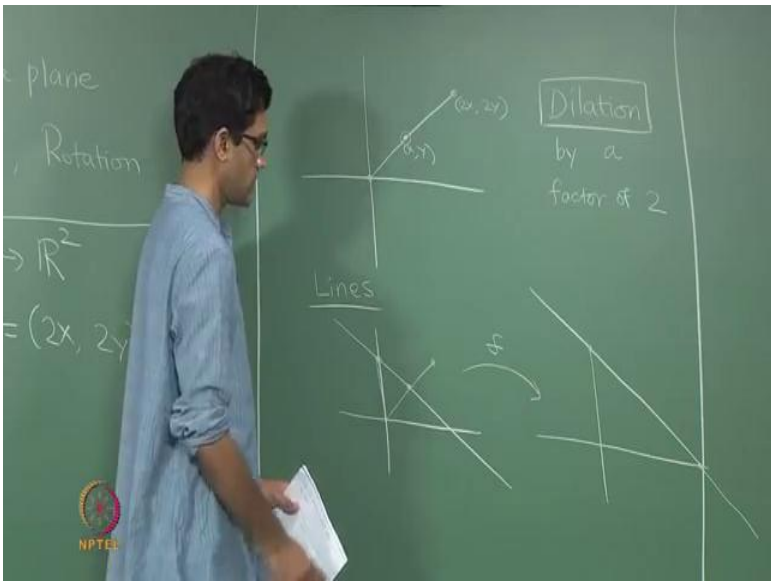

Now, what this does geometrically speaking is the following, it takes this point x comma y. So, I just joined it by a line segment to the origin and it maps this point to the point along the same line, but twice as far. So, it maps it to the point 2 x comma 2 y, so this I should think of the functions. It takes it is point x y and sort of blows it up by a factor of 2 or stretches that line segment by a factor of 2 and maps it to the end point of the resulting line segment.

So, this is sometimes called dilation or a stretch by a factor of 2, dilation by a factor of 2. So, dilation here is word which is used to mean either a sort of a stretch or a shrink by some factor. So, let us do the same sort of analysis that we did for the rigid motions, which is to try and understand what this map does to other kinds of figures on the plane. So, for instants let us study the effect of this map on lines, so I will take a typical line on the plane.

So, let say here is the line on the plane and what we want to do is to apply this function f to each point of this line. So, you take each point of this line apply the function f, you get a new point and you join all the resulting new points and see what sort of figure it becomes. So, for instants from here, it appears if you take this point here on the y axis, the function doubles this distance, it maps in to something again on the y axis, but twice as far.

Similarly, this point on the x axis will map to something along the x axis, but which is twice the distance from here and the same sort of wholes for every point in the middle. You just join it to the origin and sort of blow it by a factor of 2, so this point maps to this point and so on. So, if you sort of try and join these, what you will find is that it is again a line. It is a line, but sort of twice as far from the origin as the original line.

Of course, this is just a pictorial thing, you know I am not really justified this, you could try and convince yourself of this by plotting of few points and so on. But, of course, the most conclusive proof is really by you know writing down actually equations and checking that this function does indeed map a line to a line. So, how does one go about checking things like this, so here is the well let us do the following.

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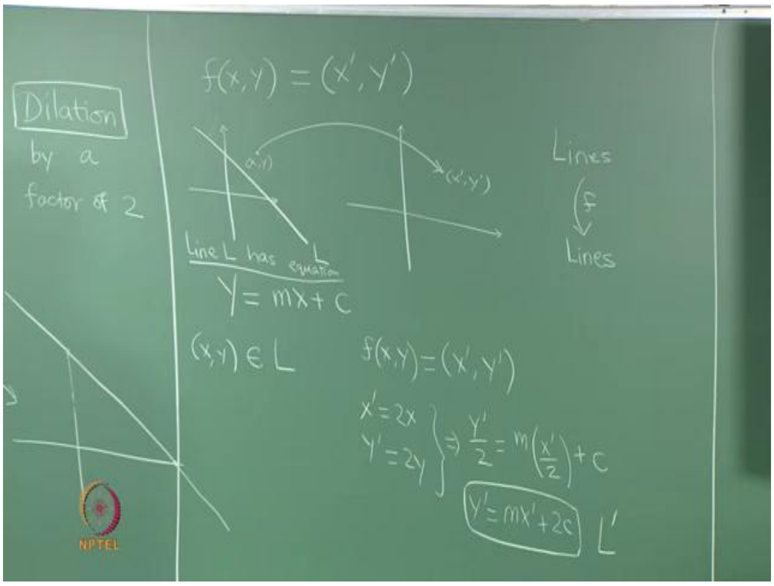

Let us call this function x y, what it maps to the point x y, the image of the point x y under this map, let us call it x prime comma y prime. So, you want to think of this function as sort of sending points here to points. So, it maps it to a point whose co ordinates are now x dash y dash, this point here is and so on. So, so it is removed from here and here to avoid confusion that is what the function does.

So, now, let us do the following let us figure out what this becomes on the new axis. So, first let me take a line here, so I will take a line L. So, what is the equation typical equation of a line, well there are many different ways of writing it, but so here is one familiar form the equation of a line, let us call it y equals m x plus c. So, this is the equation of the line, so the line L satisfies this equation as equation y equals m x plus c.

In other words, a typical point x comma y which lies on this line will satisfy this equation, so that is what it means to say the line has this equation. So, let us take a point x comma y on this line L. So, in other words x y satisfy this equation and apply the function f to it, so I will apply f to this point x y, there by obtaining another point x prime y prime on the plane and the question out really is the following. As you let x comma y vary along this line as you move them, as you move this point along the line, you want to know how x prime y prime changes, how does that move.

In other words, if you know the equation that x comma y satisfies, can you somehow reduce an equation that x prime y prime satisfies that is really the question there, so let us try and reduce this. So, what is x prime by definition it is 2 x, y prime by definition it is 2 y, now we know x and y satisfy this relation, so that automatically gives you relation for x prime y prime, because x is nothing but, x prime divided by 2 and y is nothing but, y prime divided by 2.

So, from this you conclude the following y prime divided by 2 which is basically y must be the same as m times x which is x prime divided by 2 plus c. So, observe that the point x prime comma y prime satisfies that new equation here which of course, you can simplify to y prime equals m times x prime plus 2 c. So, this new equation is again the equation of a line which we now call L dash, the line L dash and this new line observe is, well it is the equation looks like more or less like the old line L, except that c is replaced by 2 c.

So, remembering that c is sort of like the y intersect, it is a line whose y intersect is twice the y intersect of the original line. So, lines maps to lines that sort of the moral of this analysis, lines map under this function f back to lines. So, that is one aspect of what this map to us, the dilation map.

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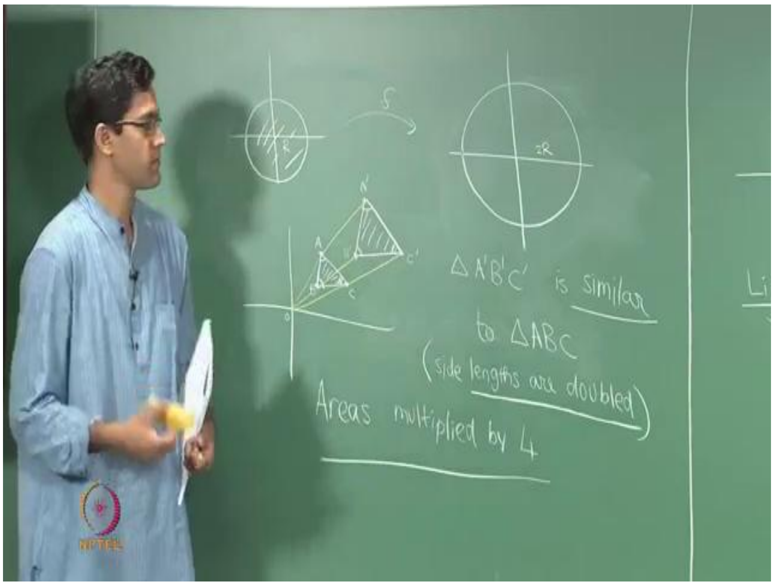

And of course, so it sort of also easy to see what other sorts of figures would transform to under this map. For instants, if you take a circle say of radius 1 or say any arbitrary radius R and apply this function f to it at the circle has radius R, then what the function is going to do to each point of the circle is, it is going to map it to the point which is at a distance 2 R from the origin. So, all you going to get is a circle whose radius is twice the original radius and so on.

So, this is now a circle whose radius is 2 R, so let us going to be the image of this circle under this map f and so on. So, somewhat better circle, so it is call it circle of radius 2 R and of course, well other kinds of figures sort of triangles and so on. So, if say this where a triangle on the plane, then let us do the following, let us think of what this function does. So, imagine joining the three vertices to the origin in this way and what this function does, of course is to dilate it by a factor of 2 which sort of pushes everything out by a factor of 2.

So, imagine now extending these lines all of them to the length and the resulting three points are the images of the vertices of the original triangle. Now, observe whenever you want to figure out the image of some sort of polygonal region, so a region which is bounded by lines. So, for instants here it is a triangle, if I want to know how what happens to this triangle under this map f, it is actually enough to just figure out what happens to these three vertices.

So, if I have the three vertices A, B and C, they map to let us call them A prime, B prime and C prime. Now, A maps to A prime, B maps to B prime and we now know the one sort of important fact that this map sense lines to lines. So, the map… So, the line joining A B must therefore, go again to some line and it must therefore, be the line joining A prime and B prime, because a line is uniquely determined as soon as you know two points on the line.

So, this line had better map to this line, there are no choices. Similarly, B maps to B prime, C maps to C prime, the line B C must map to some line that is the other fact we know. Therefore, it first map to the line joining the B prime and C prime, similarly. So, at least it is somewhat convincing argument I hope, which tells you that this triangle here will just map to this somewhat bigger angle and notice that, this bigger triangle here has all side lengths being twice the original side lengths.

So, you can see this by using similar triangles for instants if that is familiar, the angles remain the same. So, these angles are all preserved, the same angles as in the original triangle, but all side lengths are no twice the original side lengths and to proof this for instants, you need to use some elementary similar triangles, because the side OA is half the length of the side OA dash. So, you will sort of have to play with some triangles here, so I will leave that as an exercise for you.

But, observe that this triangle A dash, B dash, C dash has well, it is just writing like this is similar to the original ABC and all side lengths are double. The angles of course, remain the same, because that sort of one of the properties on similar triangles. So, lengths double, angles remain the same and notice that what happens to the areas, so what is this transformation do to areas, so that is the sort of the final point to note, what happens to areas.

So, observe that if I have circle of radius R, it is area is pi R square and the circle of radius 2 R has area, which is 4 times pi R squared, because I need to do 5 times 2 R whole square. So, this resulting bigger circle here has 4 times the area, similarly in the case of the triangle all side lengths are double. So, if you wish the altitude of the triangle will also be double the original altitude and so the area of which is sort of half base times altitude is also multiplied by a factor of 4.

So, areas get multiplied by a factor of 4, so lengths get multiplied by a factor of 2, areas get multiplied by a factor of 4, angles remains unchanged. So, thus those are the various features of this particular map, it senses everything to twice itself. So, it is dilation, sometimes called uniform dilation by a factor of 2. Now, let us take another, so this is sometimes called inhomogeneous dilation, so let quit this slightly.

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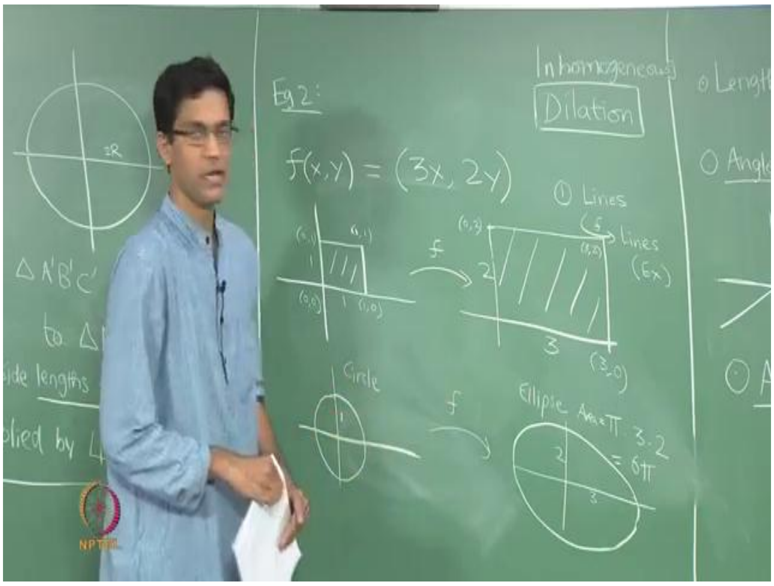

This is example 2, it is called as inhomogeneous dilation, so what is this do. Well, f of x comma y is for instants 3 x comma 2 y, so here is an example of a map which sense x y to 3 x comma 2 y. So, to really understand, why I called it an inhomogeneous dilation or a non uniform dilation, so imagine what this map does to say a square of side length 1.

So, imagine I have a square here 1 0, 0 1, 1 1 that is the origin, so the first fact here is again that lines map to lines.

So, this dilation, this inhomogeneous dilation has the same property that it maps lines to lines, so lines map to lines. So, that I leave as an exercise for you to check, pretty much by using the same calculation which is, you write down equation of a line. So, you take a point x y on that line, and then you figure out what equation x prime and y prime satisfies, x prime is 3 x and y prime is 2 y. So, maybe we will in fact do this in a minute, because we also want to see what happens to triangles. So, the key thing here is if you take a square of a side length 1 and you map, you figure out what happens to it under this map.

So, here is what you will notice, that the point 1 comma 0 maps to the point 3 comma 0 and the point 0 1 maps to 0 2. So, what you going to get now is no longer a square, but rather a rectangle, so and the point 1 comma 1 if you sort of figure out what it does, it goes to 3 comma 2. So, the square of side 1 as now become a rectangle of sides 3 and 2, this was originally 1 and 1. So, this is a reason why I am calling it inhomogeneous dilation, meaning it is sort of like dilation, but it is dilation by different amounts along different directions.

So, the x axis suffers dilation by a factor of 3, whereas the y axis is only stretched by a factor of 2, so these stretching factors are different along different directions. And of course, so the shapes of the original figures are very much going to be distorted, so square here for instants, becomes a rectangle. Similarly, if you try to figure out what happens to a circle of this radius 1 like we did in the earlier example.

So, again along the x axis it gets stretched by a factor of 3 along the y axis by a factor of 2 and what you therefore, get is in fact, an ellipse with major axis 3 and minor axis 2. So, this map actually sends a circle to an ellipse. So, again the result of the fact that it is stretching differently along different actions and the other important thing is that angles. So, now, what can we say about lengths in this case as we said, lengths do get stretch, but by different amounts along different directions. So, we cannot say better much more than that.

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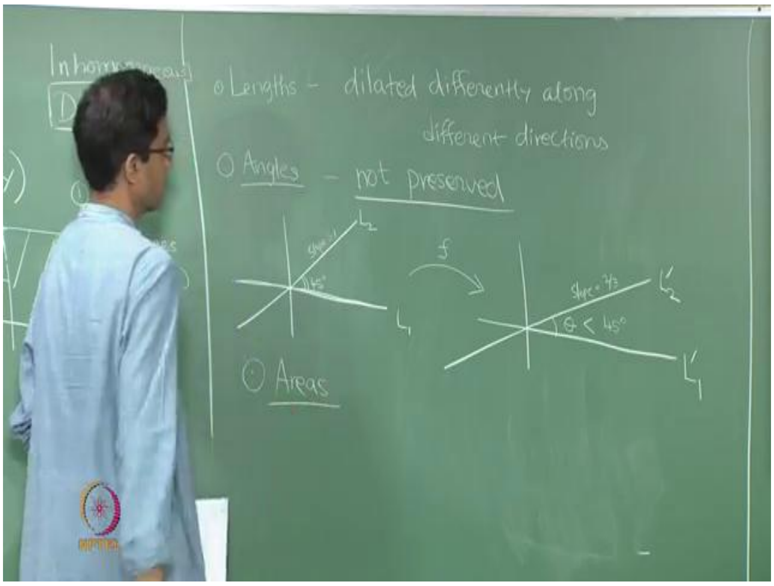

So, various facts lengths get dilated by different amounts along different directions, so dilated, differently along different directions. Angles, so let us consider angles, so here is an important feature of this, angles are not preserved any more. So, we looked at the earlier case of dilation, uniform dilation by a factor of 2 that mapped any figure to a similar figure, where all the angles ended up by preserve.

If you try doing this to the inhomogeneous dilation, you find that angles are not preserved. So, here is an example, let us take the line just the x axis that is one of the lines and let say, let us take the line y equals x as the line L 2, so that makes of 45 degree angle. So, here is a 45 degree angle between the lines L 1 and L 2 and now, if you figure out what happens to these two lines under this inhomogeneous dilation, so L 1 just maps back to L 1.

So, I am just going to tell you the answer, check that this is in fact correct, L 1 the x axis maps back to the x axis, whereas the line L 2 whose slope is 1 y equals x, now maps to something of slope 2 by 3. So, it will sort of move a little closer to the x axis that this is what happens to the line L 2, so check that L 2 becomes a line whose slope is 2 by 3. The original slope here is y equals x, so that has slope 1, so this angle here theta is strictly smaller than 45 degree.

So, what it is done is, because of this sort of stretching differently along the two different directions, it has ended up making the angle different. So, here is an example where angles are not preserved, so pretty much our first example, because of course the rigid motions that we talked about last time preserve everything, it preserve lengths, angles, areas, pretty much everything you can think of, they in the preserve length.

Dilations, they do not preserve lengths of course, they do not preserve areas, but they do preserve angles. Now, inhomogeneous dilations or the first examples of things which also do not preserve angles, they distract things sufficiently, but even angles change. And of course, we talk about lengths, angles, so what about areas, so let us just look at the two examples of figures which get transformed under this map. So, one we said we take a square of side 1, it becomes a rectangle of sides 3 and 2.

So, the area here is a 1 whereas there it is a 6, similarly a circle here of radius 1 has area pi R squared, it is pi whereas ellipse has area. Well, it may not have a formula which is very well known, but the area of an ellipse, if you know the minor and major axis, so the area in the ellipse terms out to be pi times, instead of R square it is A into B, where A and B are the minor and major axis. So, in this case area of this ellipse is pi times 3 times 2, so that is 6 pi.

So, observe again that the ellipse has area which is 6 times the area of the original circle. So, both these, the square becomes a rectangle of area which is 6 times, the circle becomes an ellipse again whose area is 6 times.

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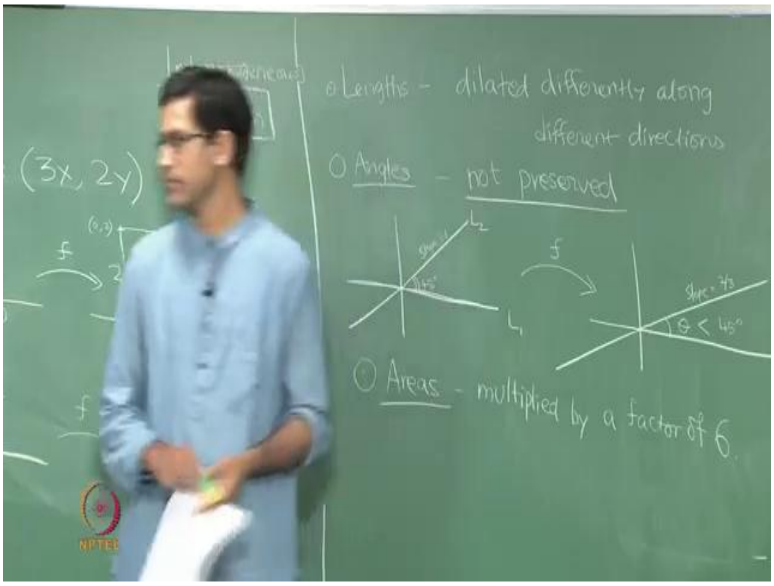

So, the areas in fact, get multiplied by a factor of 6, so areas at least looking at are two examples. So, I am not really proving anything here, it seems at least from this example that the area is multiplied by a factor of 6. So, those are the features again of the inhomogeneous dilation, now let us do one more example.

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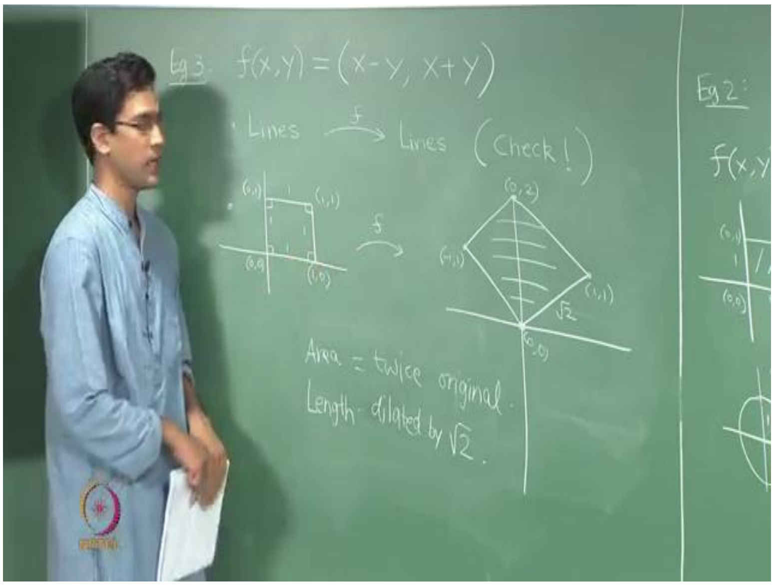

Example 3, the function f of x y just define to be x minus y comma x plus y, so here is a formula for what the function does. Again, let us study this pretty much using the same sets of points, so for a start what does this function do to line on the plane and again, I claim that lines map to lines. So, again exercise check that this is in fact, correct. So, it is very similar to everything we have seen until now in the sense that, lines certainly map to lines.

So, that of course, makes it easier to figure out what it does to polygonal region, regions which are bounded by lines, so let us do that next. So, again I will take a square of side 1, let us take. So, here are the four vertices of the square of side 1 and let us figure out what happens to each of these four points under this transformation f, so let us apply the transformation f to this. So, for a start let us apply, so observe that the origin 0 comma 0, if I apply f 2 it, I will just get 0 minus 0, 0 plus 0, so the origin goes to the origin.

So, the origin just remains where it is, the point 1 comma 0 maps to 1 minus 0 and 1 plus 0, so that is just 1 comma 1. So, the point 1 1 is one of the vertices, now the point 1 comma 1 under this map maps to 0 comma 2, so that is now the point on the y axis. So, so far here is what I have gotten, one on the side is like this, the other side is like that and if you look at what happens to the third point 0 comma 1, that maps to minus 1 comma 1, so that is this guy.

And again, as we said if you figure out what happens to the four vertices, you are more or less done, because the line segment joining them has no choice, but to map to the line segment joining the resulting points. So, what we have really is this maps to the square here and observe that, what is this, this is again a square, but the square has side which is well, what is the length of the side, it is square root 2 times the length of this side.

So, this is a square of side 1, whereas this is a square of side root 2 and well, what else is there it the area therefore, is… So, what happens to areas? The area of this square becomes twice the area equals twice the original area and what else seems to be happening here. So, the length of this line segment, this was a line segment of length 1, it map to a line segment whose length is square root 2 times.

So, length is dilated by a factor of square root, so at least for this square all of these sides are of length 1 and the resulting sides there have length square root 2. So, it seems at least that length of line segment is multiplied by a factor of square root 2, therefore the area is multiplied by a factor of 2, but one nice thing here seems to be the angles are unchanged. So, the angles this was a square and well that seems to remain a square in the other case of square.

So, at least in this example the angles are unchanged and but in the same time observe, this has you know this is sort of the kind of thing that happen if you, when you had dilations. When you dilated something by a factor of square root 2, you would pretty much have all these things happening, lengths go up by square root 2, angles do not change, area becomes twice and so on. But, this map is clearly not just a dilation, the dilation would have multiplied this, would have map this square just to a square, you know along the x and y axis of side length root 2.

Instead, this is mapping it to some sort of a rotated square. So, you still have a square, but it is rotated by an angle 45 degree, so observe this is a 45 degree angle. So, what it seems to be really is the following and we look at this next time, this map f really does two things at once. The first thing it does is, it takes the original square here and rotates it is by an angle of 45 degrees, and then it dilates lengths by a factor of square root 2.

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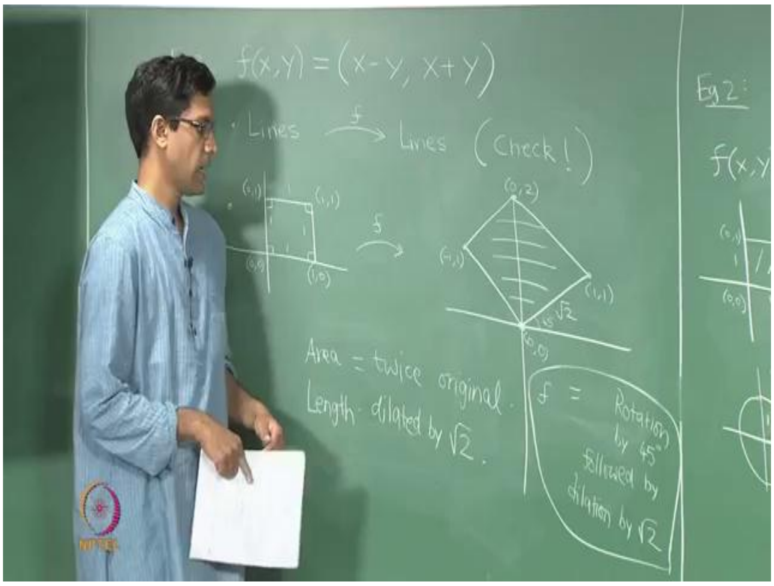

So, at least here is what it seems like f is doing, it is doing two things. First, it is a rotation, so f seems to at the following description. It seems like a rotation by 45 degrees at least for the unit square followed by dilation, followed by dilation by a factor of root 2. So, it is some sort of a composite map it, it something which achieves two things at once, and so we will see that the natural way to think about this is in terms of compositions. So, we will in fact revisit what compositions mean and rewrite f in terms of a composition of two maps, so this is something we will do next time.