An Invitation to Mathematics
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Unit
Matrices
Lecture - 31
Matrices in Combinatorics

This time we will talk about Matrices as apply to Combinatorics. So, recall combinatorics is really a study of counting or sort of discrete structures and so on.

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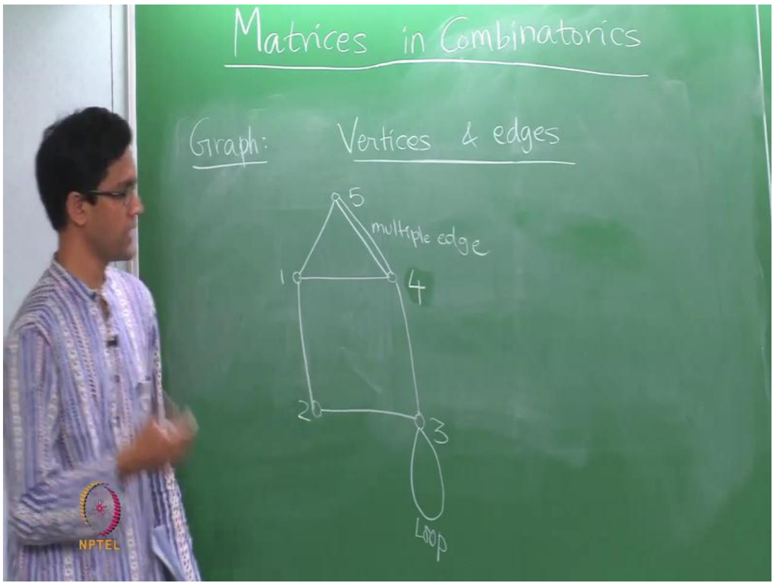

So, what we want to do now is to talk about one such discrete structure which is the notion of a graph. So, this use of the word graph is different from the usually use set of the word graph in the contacts of functions. This graph refers to the following pieces of information, there is a set of vertices which are connected by you know lines which we call edge. So, vertices and edges these together make up the notion of the graph.

So, let us just do it by example where it is a clear, so when I say vertex, so here is the graph which has five vertices. So, let just call the five vertices as 1, 2, so they are labeled, call this 2, 3, 4 and 5. So, there are five vertices which I just, I think of as some five points in the plane and what we have is between every pair I draw a line or I do not draw line or I draw several lines and so on. So, I think of for instance say between 1 and 2 there is an edge.

So, I think of it as an path which allows you to go from 1 to 2, there is an edge between 2 and 3 let say there is an edge between 3 and 4 and an edge between 4 and 1 and let say an edge between 1 and 5. So, this for instance would be an example of a graph, where you have 5 vertices and certain edges which connects some vertices. So, for instance there is no edge between 1 and 3, there is no edge between 4 and 5 and so on 2 and 5, these are all things which do not have edges.

Now, what we also allow is an edge which connects a vertex with itself, so this sometimes called a loop. So, you think of it as a path which starts and ends at the same point and the other thing that you can allow is to have multiple edges, say between 4 and 5 let say there are two paths. So, such a thing is sometimes called a multiple edge, so here is an example of a graph which also has loops and multiple edges. So, we will pretty much allow this as the definition of graphs.

Now, how it is draw as a material, this is just some one pictorial representation of this graph, there may be many other ways, you can just move the vertices around, draw the edges in which way you want, all that matters is really which vertices is connected to which other vertex and by how many edges. So, pretty much that is all the, that pretty much what determines a graph, the drawing itself is does not matter. So, what is another way of saying it, let say encode this graph, the basic data of this graph.

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So, here is the encoding, we just keep track of how many edges there are between every pair of edges. So, I will draw, well I write down a matrix let us call this matrix A, so this encoding this in a form of a matrix. In this case, it is a 5 cross 5 matrix, so the number of the size of matrices is just the number of vertices and what are the entries of this matrix. So, let us call it A, the i comma jth entry of this matrix is just the number of edges between i and j, so vertex i and vertex j, so it is between i and j.

So, in this example the matrix A for this graph would be the following, so think of... So, let us just say, so I have this is the 1st row, here is the 2nd row, 3rd row, 4th row and 5th row and I also have 1st column, 2nd column, 3rd column, 4th column and 5th column. And let us says write this down between vertex 1 and vertex 1 there are no paths, because there is no loops there. In fact, that also tells us between 2 and 2 there are no edges, between 3 and 3 there is a loop, which means that is in fact, one edge connecting 3 with 3, there are no edges connecting 4 with itself or 5 with itself.

So, you only get entries along the diagonal when there are loops in the graph, so it is one thing to keep in mind. Now, there is an edge connecting 1 and 2, 1 and 4 and 1 and 5, so between 1 and 2, 1 and 4 and 1 and 5, so there is no edge connecting 1 and 3, so the number of edges is 0. Now, similarly connecting 2 to 1 and 3 you have edges, so 2, 1 and 3 I have 1 edge each and 0 elsewhere.

Similarly, connecting 3 to 2 and 4, 2 and 4 I have one edge each, to itself there is a loop and from 4, I have to 1 3. So, from 4 to 1 there is an edge, from 4 to 3 there is an edge and 4 to 5 there are two edges. So, it is a multiple edge there, so there are two paths between 4 and 5. Now, similarly between 5 and 1 there is an edge between 5 and 4 there are two edges. So, that is the final graph that you get, the final matrix that you get corresponding to this graph.

So, this just encodes the number of paths or edges which connect each pair of vertices i and j and observe that this is necessarily a symmetric matrix, meaning the i comma jth entry is the same as the j comma ith entry. Because, both of them are just the number of edges connecting i and j, whether you think of an edge as connecting i to j or connecting j to i, it is a same thing. So, this is what is called a symmetric matrix, things about the diagonal entries belong the diagonal are equal, this and this are equal, these two are equal, these two equal, these two equal and so on, similarly this one and that one are the same.

So, this is often called an adjacency matrix of the graph g that you started out of let say of the graph. So, adjacency just means tells you how many edges connected to vertices. So, now what do we want to do, we want to see how this matrix allows us to perform certain kinds of counting on the graph. So, we finally, want to do combinatorics we want to count things. So, what is the natural think that we might want to count, it is paths.

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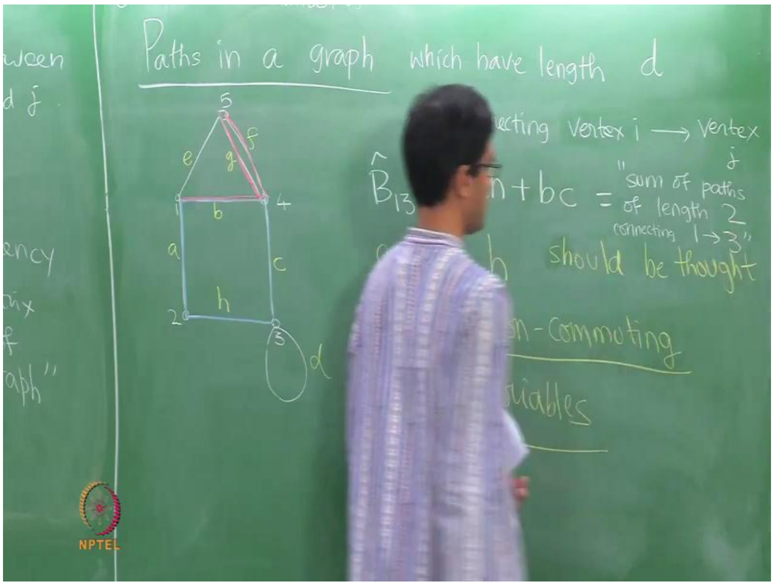

So, observe the edge, so I have say two vertices one and so we had 1 and 2 having an edge between 2 vertices is just you can think of this as an paths of length 1. Now, you can extend that notion to for instance paths of length 2. So, I can go from 1 to 3 by following the path of length 2, by the thing which first goes to 2 and then goes to 3 or I could say consider a path of length 3 as follows I go down to 2 go to 3 and then follow the loop at 3, so here is the path which will get me from 1 to 3 in 3 steps.

So, the notion of the path itself is a rather inductive and easy thing, now what we want to do that therefore, is to count paths. So, what we would like is to find the number of paths in our graph which connect any one vertex to any other vertex. So, here is the problem, question find the number of paths, the graph which connect vertex i to vertex j let say which connect which have length. So, the length of the path is also important, which have some given length say length d and which connect two given vertices and connecting vertex i.

So, that needs to be a starting vertex, so connecting vertex i to the vertex j going from this vertex to the other vertex. Now, how does one do this, how does one compute the number of paths? So, in order to do this let us sort of redraw the same graph that we had before, but with some decoration. So, I am going to draw the same graph 4 connecting 5, say with multiple edges, but now in addition to having just the edges themselves, let us also label each of these edges.

So, that you know which edge you take, so let me call this as a that is the edge label for edge connecting 1 to 2, I have b which connects say 1 and 4, c which connects 3 and 4, d is the loop c, so I also have e, f, g and h, so arbitrary labels. So, there are eight edges, so I am just calling them you know a to h edge in pretty much any order, it does not matter how I label them. Now, here is what we will do, let us consider the adjacency matrix, but with this further decoration.

So, let us go back to our adjacency matrix, so same think that we drew before, so ((Refer Time: 11:01)). So, here is the adjacency matrix, but let us also incorporate edge labels. So, I am going to just do it right here.

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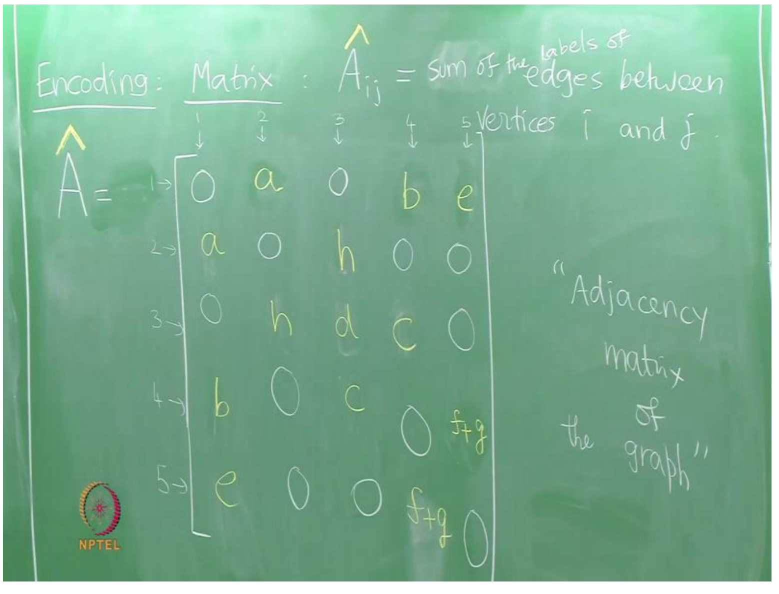

So, let me call this new matrix as A hat, the labeled adjacency matrix if you wish, so the adjacency matrix with labels. So, what is this going to be now, this A hat i j is not just going to count the total number of edges between the vertices i and j, but it will sum up the edge labels. So, think of A hat i j has being the sum of the labels of the edges between i and j. So, it is slightly more information, sum of the labels of edges between vertices i and j.

What is that mean in this particular example, let us do this. So, I put a 1 here in the adjacency matrix, because there is an edge connecting 1 with 2, but that edge is actually the edge with label a. So, I should really call this guy as an a, similarly the thing connecting 1 and 4 is a b. So, the thing connecting 1 and 5 now is an e and so let us do this to all the other ones. So, this is an a an edge and this is h, d, c, now here is the interesting part, so this is b and c. Now, observe there are two edges which connect vertices 4 and 5 and so what I am supposed to do here is to write down the sum of the edge label.

So, I should now call this entry as f plus g and so now the rest of the entries are easy. So, this is again f plus g, this is e, so this is e, this is f plus g. So, what we have now done is taken the adjacency matrix, but also now kept track of the labels which edge is which, that is now clear from here. So, now let us do the following, let us think for our premises let us think of a through g as sort of being like variables. So, if you think of them as variables, then here is the relation between A hat and A if you wish.

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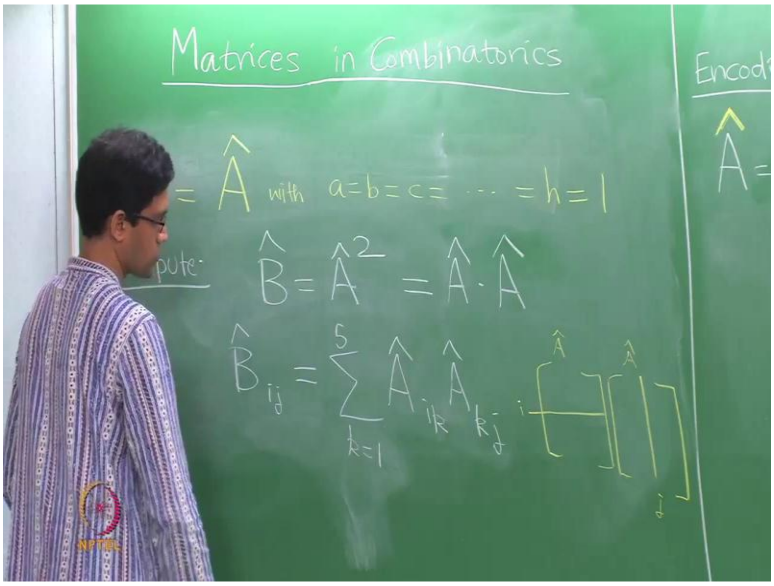

So, I had the matrix A to start with, what is the different between A and A hat, well A is the following it is just this label matrix A hat, in which you substitute all variables to be equal to 1. A hat with the following substitution a, b, c etcetera all the way to edge are given the value 1, so that is easy to see, if you just make them all 1, what you get back is of course, the original matrix say. Because, now you do not keep track of the labels themselves, but you only want that total number. So, you pertain each label is really just the number 1 and then what you will count is just the number of edges.

So, this is the relationship between the labeled and unlabeled adjacency matrices, but this labeled adjacency matrix has the following nice property. So, let us compute the following, so let us calculate the square of this matrix, so let me call it something I will call it B hat which is A hat squared. Now, what is square means, square just means it is A hat times A hat, I need to multiply the matrix with itself.

Now, multiplying this matrix with itself of course, means you need to go back to the definition of matrix multiplication. So, recall that if I want to find B hat say I want to find the i jth entry of B hat, this is just nothing but it is given by summations it is the following k going from 1 to 5 A hat i k times A hat k j. So, what is that of course, pictorially it is just you take the matrix A, A hat rather and write down A hat next to itself, look at the ith. So, this is the ith row of A hat and you take varying entries in the ith row.

And similarly you pair it up with the jth column of A hat and look at various entries there and now you sort of pair entries in the ith row here with the jth column there and sort of multiply them out in pair, so add all the answer right. So, that is basically the same thing written out in this fashion. So, for instance what would this give you for you know certain special values of i and j. So, let us compute in this case, so let us go here let us calculate a few entries.

So, if I compute B hat 1 3 using this definition, here is what I will find. So, again exercise to check that what I writing down is in fact, correct. So, this is just going to be a h plus b c, so just write down the definition of the product and then you will find that the 1 comma 3th element is just a h plus b c. Now, here is something again that I should sort of n for size, I said think of a b c and so on as being like variable that is really why we are now thinking of you can add them or you can multiple them and so on.

But, I want to also keep in mind the following the variables a through h are to be thought of has being non commuting variables. So, think of these as do not commute which each other should be thought of as non commuting variables that is sort of important you will see in a little while why, when we say non commuting we mean when you multiply a with h that should be thought of as being different from multiply h with a, so a h is not same thing as h a.

Now, so we will come back this in minute when you do the multiplication you need to keep this in mind all the time. The variables occur to the left must be retain on the left, the variables in occur on the right must be retain on the right should in change the orders. So, now, let see what this mean. So, when I say b hat 1th comma 3 is the h plus b c.

Now, the word a h if you think of it as somehow encoding the path with first goes along with a and then h and d c as encoding the path with first goes along b and then along c, you will observe that a h can b c just refer to paths of length 2 which connect 1 with 3. So, both of these guys here a h and b c are paths of length 2 which connect the vertex 1 with the vertex 3.

So, similarly if you compute this in the definition of the matrix product you compute B hat 1 5 here is what you will find it is b f plus b g and if you just see what b f and b g mean b is this, f will be moving along the first of the 2 multiply just b g will still mean go along b, but then follow g instead. So, observe these are again paths of length 2 which connect the vertex 1 with the vertex 5. So, B hat 1 5 is now well what is it, it really counts the sum of all edge labels or words which encode paths of length 2 between 1 and 5.

So, this is really should think of it as sum of ((Refer Time: 19:56)). So, sum of paths if you wish are rather labels of paths of length 2 connecting vertex 1 to vertex 3 and similarly. So, here is the interesting relationship that if you compute B hat i j what it really does is. So, let us comeback here B hat i j, if just B hat i j.

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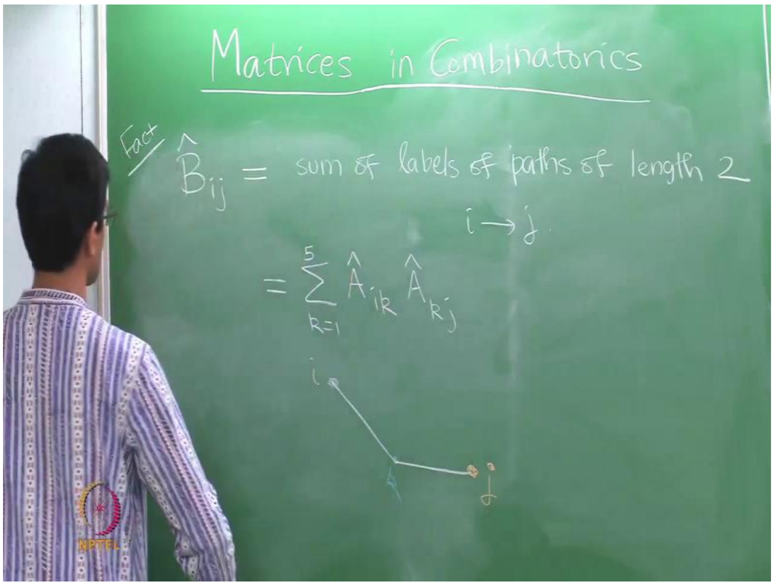

So, this is the fact which again you should check at least by examples to convince yourself that this is really correct B hat i j is just the sum of all paths of lengths 2 between i and j sum of labels for wish of paths of lengths 2 well let say just connecting i to j. Now, and it is not to hat to see why is this really happen, because recall the definition of the matrix product that is the following it is A hat i k with A hat k j and where k is suppose to un over all possibility, so our all possibly vertices.

So, you should think of these as saying something like the following, I have the vertex i to start with. So, here is the vertex i and I have the vertex j. So, I am trying to get from the vertex i to the vertex j in two steps. So, how could I possible do that well I need to have an intermediary vertex k that is where I will jump after one step. So, I first A hat i k tell me how to go from i to k which edge I should follow to go from i to k and A hat k j tells me which edge I need to follow when going from k to j and you sort of concordinate these two things together that is the product and you sum over all possibilities for this stop over vertex k.

So, it is kind of clear that this exactly does what we want to do which is it gives you all possible ways of going from i to j in two steps and more or less repeating this same argument if you try you know taking A hat power 3. So, this is not a special think just for 2, if you computed A hat power b for instance, so let us just to here.

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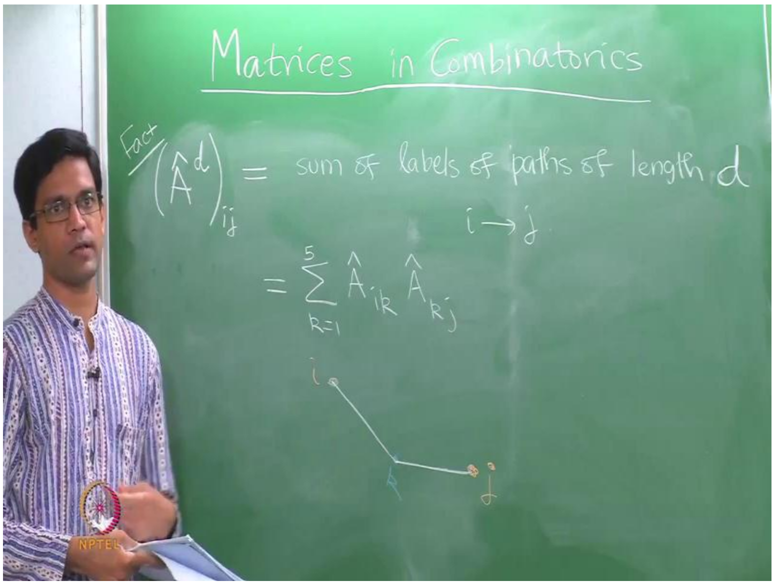

So, here is a more general fact, if I take the label adjacency matrix A hat and I multiply with itself d times not necessarily two times, but I could do any number times d times and it this is just A hat multiplied with A hat multiplied with A hat d times. And I try to find the i jth entry of this matrix, what I will get is some of labels of all paths of length d which connect i and j. So, I sort of i j to play with this and see how this really works out in various example of parts of lengths 3 for instance.

Now, so here couples of things to keep in mind here which is as I said if we do not really want... So, the initial problem that we started out wanting to do is to find the total number parts, suppose we do not care about knowing what a parts themselves are.

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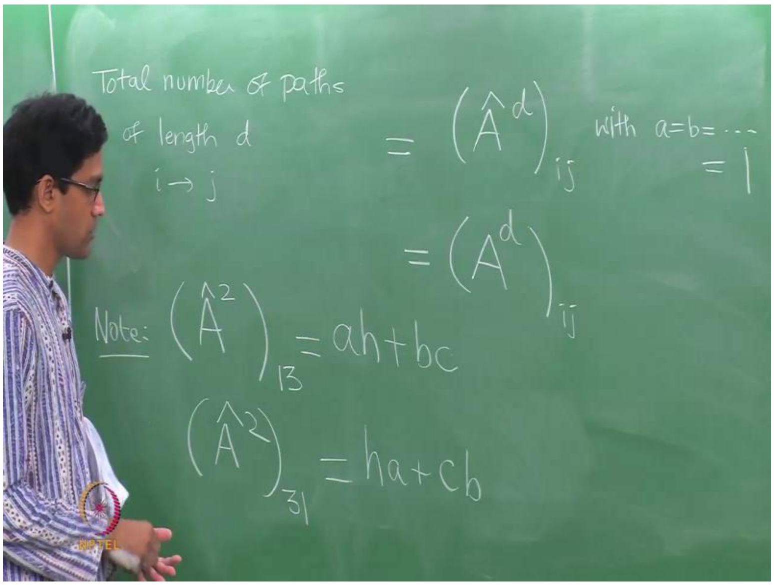

So, total number of paths of length b say from i to j, if that is all you care for well, then what would you do, you would do the following you first take all parts with labels. So, you first take A had power d as I said you take the i jth entry that will actually list out all the paths for you with labels. And then you just ignore the labels in other words wherever you find a, b, c and so on you said them all equal to 1 think of them as being substituted as one with all the variables assign the value 1.

But, recall when you do that what you really are doing is replacing the matrix A hat we just the unlabeled adjacency matrix A. So, in other way of saying this is to say well forget the labels in the first place, consider just the unlabeled adjacency matrix raise it to power d and then you take the i jth entry. So, this is really the formula that this is the think I mention the beginning and application of matrices two counting problem involving the graphs which is to find the number of paths of lengths d and the answer this, you take this matrix take the dth power of this matrix and just take the i j entry, so that is the very nice connection.

The other point I want to make here is about the non commuting nature of the variables has the following consequence. So, suppose I take A hat squared nothing that we trying to do and so this what I called B hat and if I compute the 1 comma 3th element, then what you would get as we mention earlier is a h plus b c. But, if you know compute it A hat squared, but computed 3 1th element, not the 1 3th element then here is what you will find, it will be h a plus c b.

So, please check both of these relation, the first one again is something we are already saw. But, here is the somewhat surprising think if you did not the other order and since I am sort of not allowing the variables to commute the order is important, here h will occur first a next, c first and d next and this not to hat to say why. So, it seems as if this matrix is not symmetric any more, if I do this square q and so on I will not get symmetric matrix is because my variables do not commute with each other.

But, this is really how I would like to have them, because as we said earlier ((Refer Time: 26:45)) the, word a h refers to the path which first goes along a and then along h. So, it is a path which connects 1 with 3 and b c is a path which sort of goes from 1 to 4 and then down to 3, so those are the two paths. But, if I want to compute B hat 3 1 instead of B hat 1 3, then well the answer I get is first edge and then a, which is really how I would like it to b, because it first tells me I need to go from 3 to 2 and then from 2 to 1.

Similarly, I will get c b rather than b c, because c is the path which first starts from 3 and then following b is what will get me to 1. So, I would in fact it is most appropriate for them not to be thought of commuting variables, because only then is correctly capture the actual labels of paths going from the given vertex to another vertex. So, various you... So, there is really inter play of several different things here, one the fact that we construct the matrix and used sort of matrix operations, pretty much just matrix product in order to capture this notion of paths that is one.

The other thing is the some of the use of these variables, the labels is a, b, c, d and the non commuting nature to very accurately capture the actual path labels themselves connecting any start vertex with an end vertex. So, again I will left several things here for you to check all of them are easy verifications, but it something that you should certainly do yourself in order to get very good understanding of what is going on here.