An Invitation to Mathematics
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UNIT - I
Polynomials
Lecture - 07
The Chebyshev Polynomials

Welcome back, this time we are going to talk about some nice examples of Polynomials. So, this whole theory of polynomials is a, you know the general theory is of course, very nice, but what makes it especially nice is the existence of many examples with some very special properties and so on. So, there are very large number of examples of nice classes of polynomials, I am just going to talk about two of them just give you the flavor of what should be there.

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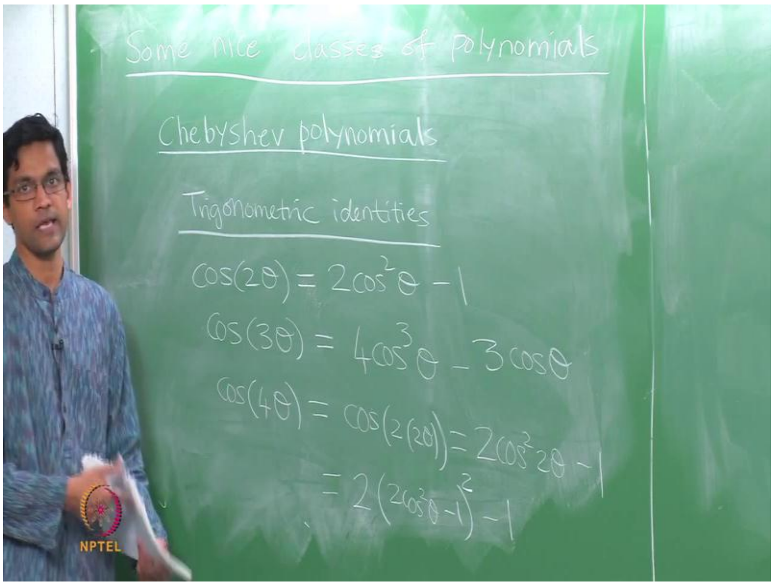

So, the first things I will talk about are, what are called Chebyshev polynomials, it is sometimes called the Chebyshev polynomials of the first kind, this also a second kind. Now, way to these arrives from... So, two sort of better appreciate this what one needs to recall is a little bit of identities from trigonometry. So, here is one of them the double angular formulas, so if you are trying to find cosine of 2 theta, then here is the well known formula for it, it is 2 cos square theta minus 1.

Now of course, there is more if you try to find cosine of 3 theta, so theta of some angle, this is well again a formula that is probably also rather well known 4 cos cube theta minus 3 cos theta. Similarly, cos of 4 theta if you wish can be computed from cos 2 theta by thinking of it as cos of 2 times 2 theta. So, that is going to be I guess 2 cos square 2 theta minus 1. But, I am also going to further write this out cos 2 theta is again some expression in terms of cos theta. So, this is 2 times 2 cos square theta minus 1 the whole square minus of 1.

So, the point being that of course, I am in I could expand this out a little bit more, the final answer is going to be some expression which is a combination of powers of cos theta and so on. So, you keep going, the key thing is that if you write cos of n theta for any n 5 theta, 6 theta, 7 theta and so on they can all be written as some polynomial in cos theta.

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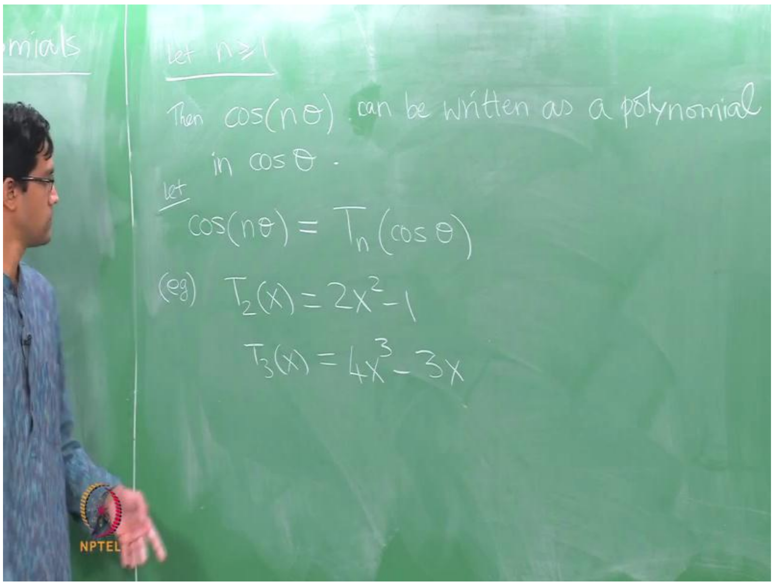

So, the key point here is let n be any natural number, then cosine of n theta is in fact, some polynomial can be written as a polynomial, so polynomial in cos theta. So, observe what we mean by that is the following cosine of n theta can be written as some polynomial called T n evaluated at cos theta. So, T n of x is some polynomial in place of x you plug in cos theta, what you will get is exactly the value of cosine n theta.

So, let see what we mean in here in our examples, if you put at n equals 2 consider the polynomial T 2 of x which is 2 x square minus 1. Now, if you plug in x equals cos theta then that is going to give you 2 cos square theta minus 1 which is exactly cosine of 2 theta. Similarly, you take the second polynomial that I wrote down there 4 x cube minus 3 x, you plug in x equals cos theta it is 4 cos cube minus 3 cos and that exactly cosine of 3 theta.

So, this is what we mean in general, there is a sequence of polynomials T 0, T 1, T 2, T 3, T 4 and so on such that, in place of x you put cos theta what you get is exactly the value of cosine n theta. So, first let see why is this statement even true, why is a true that cosine of n theta can be written as a polynomial in cos theta and also simultaneously what these polynomials T n look like.

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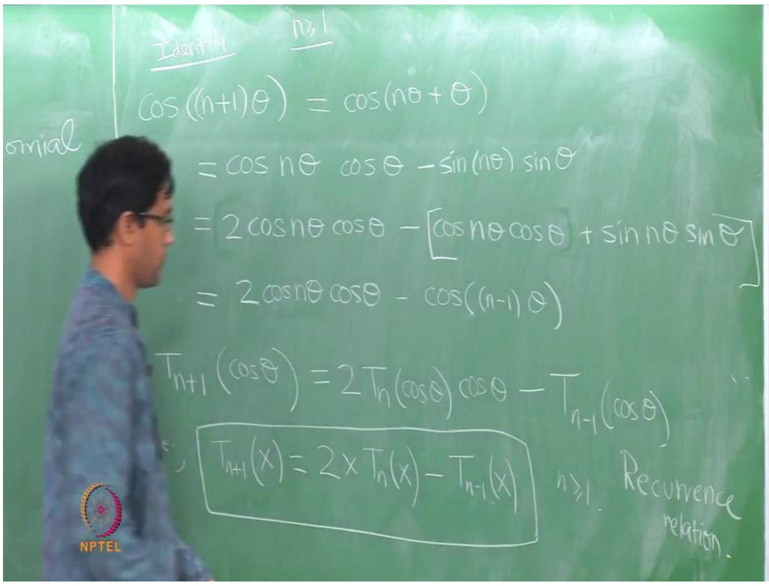

So, to do this, so let us recall the identity for addition of angle, so I am going to write the following identity down, let us look at cosine of n plus 1 theta. So, that is just cosine of n theta plus theta, so it is like cos of a plus b. So, the addition formulas says, it is cos of a times cos of b minus sin of a sin of b and what we will do with this is the following, so just small trick to manipulate this. So, let us add and subtract cos n theta cos theta the first term. So, I will do the following I will rewrite the first term, I will add another copy of the first term and subtract out what I have add.

So, what is this manipulation give us? Well, if you observe what it does to the last two terms now. So, this is cos of n theta cos theta and this is becomes minus of cos n theta cos theta plus sin n theta sin theta. So, let me just do it write here, so this is a minus and that is a minus, so I can put them together with the plus, but that is again the identity, but now for the difference of the angles.

So, that is now cos of a minus b, so this becomes minus cos of n theta minus theta, so that is n minus 1 theta. So, what is this tell you, well it says that if you look at this polynomial T n plus 1 x. So, let us look at the left hand side if we grand that it is a polynomial, T n plus 1 evaluated at cos theta that is the left hand side, the right hand side is well what is it again I am going to write cos n theta as a polynomial in cos theta. So, that is 2 times T n evaluated at cos theta times cos theta minus what this again cos of n minus 1 theta. So, this just T n minus 1 evaluated at cos theta.

So, all I am doing is... So, for the moment just take this for granted that cos of n theta can be written as some polynomial evaluated at cos theta. I am trying to find out what property those polynomial satisfy, because we have this identity and because of all these manipulations, this polynomial T n plus 1 must be equal to 2 times T n of cos theta times cos theta minus T n minus 1 evaluated at cos theta. So, these two things are in fact, the same. So, what is this tell you about, so observe that we have really plugged in cos theta and place of x.

So, replace all the cos thetas by x's, in other words it says that this polynomial T n plus 1 evaluated at x is in fact, 2 x. So, the cos theta here is an x minus T n minus 1 for this and this is valid for what values of n. Well, here I need at least n equals 1 in order for all these to make sense, because I am sort of looking at least cos of 0 over here. So, T n plus 1 of x is just 2 x T n x minus T n minus 1 x, this is valid for all n at least 1.

So, in fact this polynomials T n's that we are trying to understand here are given by this very simple formula here. Now, this is not quite like a usual formula, this is what is called a recurrence relation. So, we would call typically a thing like this a recurrence, because it gives you a formula for T n plus 1 in terms of lower T n's, in terms of T n and T n minus 1. It does not give an explicit formula for the T n itself, but just in terms of the lower T n's. But that still quite a valuable piece of information, because it allow us to quickly compute the first few values. So, observe... So, let just use that to write out this table here.

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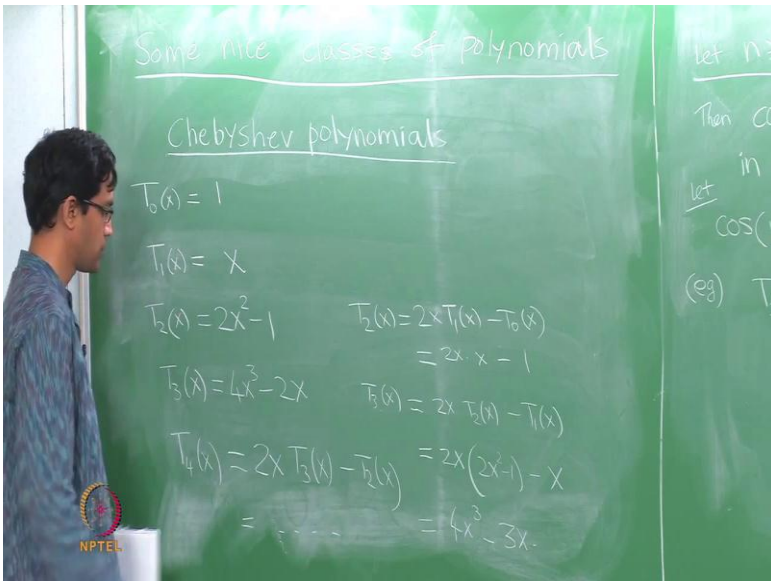

So, observe what are these. Well, I should also have written the earlier polynomials. So, here I had written T 2 for a q minus 2 x. So, observe in fact, we also know T 1 and T 0 are. T 1 is just, T 1 is suppose to be what, if you plug in cos theta as a polynomial in cos theta. So, that is just the polynomial x itself, so T 1 of x is just x, T 0 of x is I plug in, in place of n I plug in a 0. So, it is cos of 0 that is just the constant 1, so in fact, T 0 of x is the constant polynomial 1, T 1 of x is the polynomial x, the next guys 2 x square minus 1 4 x cube minus 2 x and so on.

So, now, let just calculate... So, ((Refer Time: 12:05)) let just see whether this recurrence relation actually holds true. So, what is the recurrence relations say T 2 we should already be able to figure out T 2 from there. So, remember this is the recurrence relation ((Refer Time: 12:21)) T n plus 1 of x is 2 x T n x minus T n minus 1 x. So, I plug in n equals 0 here or rather n equals 1 here I will get that T 2 of x is 2 x T 1 of x minus T 0 of x, so let us do it there. So, T 2 should actually have been 2 x times T 1 minus T 0 and this is what 2 x times x minus T 0 is a 1, that is exactly 2 x square minus 1.

Let us compute T 3 in a similar fraction, T 3 according to the recurrence relation should have been 2 x times T 2 of x minus T 1 of x. Let us use this to calculate, it is 2 x times T 2 of x is 2 x square minus 1, T 1 of x is x. So, that should be 4 x cubed minus 2 x minus x which is minus 3 x. So, observe that the well known values of T 2 and T 3 can actually been obtain in terms of the preceding values by using the recurrence relation.

So, similarly if we did T 4 it would just be you take 4 x cube minus 3 x, you multiply it by a 2 x and you subtract out the previous term. So, T 4 should actually be 2 x times T 3 minus T 2 can of course, one can just work that out in for, see if it matches up with the thing that we already had. So, that is a these polynomials here are what are called the Chebyshev polynomials.

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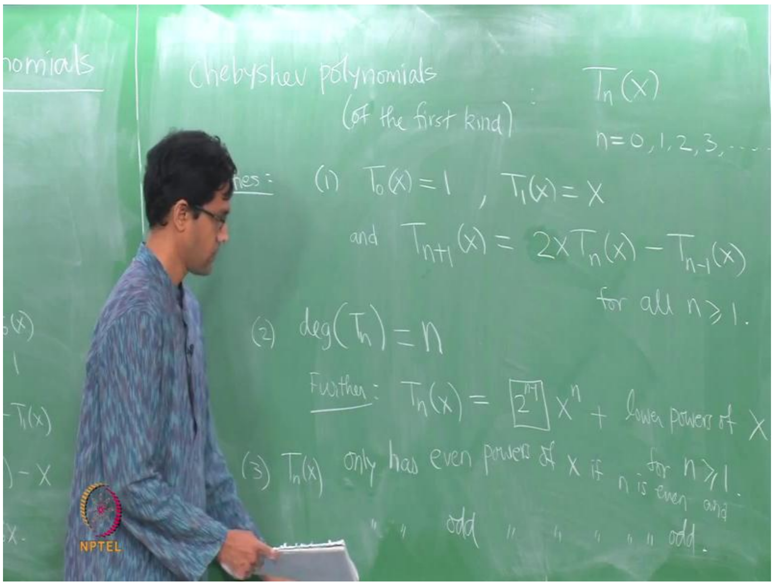

So, Chebyshev these polynomials for n equal to 0, 1, 2, 3 it is really a sequence of polynomials. Now, what are the main properties that we have sort of encountered so far, well the first thing is that it satisfy a recurrence relation, that is one of the most important properties. T 0 is 1, T 1 is x and all higher values and you wanted to know what T n plus 1 was, you could get it in terms of the two preceding values of T. So, you multiply T n by 2 x and you subtract T n minus 1 which is for all n at least 1.

In other words T 2, T 3 and so on can or the ones for which you can apply this recurrence relation. So, as the first property that they given by the recurrence relation, the second interesting property is sort of obvious when you look at the table they are one, the first few polynomials are 1 x 2 x square minus 1 4 x cube minus 2 x and so on. In fact, the degree is of these polynomials sort of grow by 1 at each time.

So, in general T 0 is a degree 0, T 1 is of degree 1 and so on. T n is of degree n. In fact, further it sort of look at the table again, notice that T n of x looks likes the following. It sum multiple of x power n plus of course, lower powers of x plus linear combinations of lower powers. But, this leading co efficient, the co efficient of x power n is in fact, the power of 2, it is 2 power n, n minus 1.

So, here for instance I have a 2, 2 power 1 this guy is 2 square, this is 2 cubed and so on. So, this is true for let see what values when at least 1, it looks like this for n at least 1. So, even the leading power is very easy to determine in this case and what are the other properties that one notices from just the table is, that T n of x only contains. So, for instance T 2 only has x square and the constant term, it does not have an x term.

Similarly, T 3 has x cube and x it does not contain x squared or a constant. In other words, if you take T n and if n is even it will only contain even powers of x and if n is odd, it only contains odd powers of x. So, I am sort of stating various interesting properties without proof really, but each of them can be proved rather easily. So, let see T n of x is only has even powers of x, if n is even and similarly only has odd powers of x, if n is odd. So, it has the very nice property, now here is the rather interesting property of the Chebyshev polynomials.

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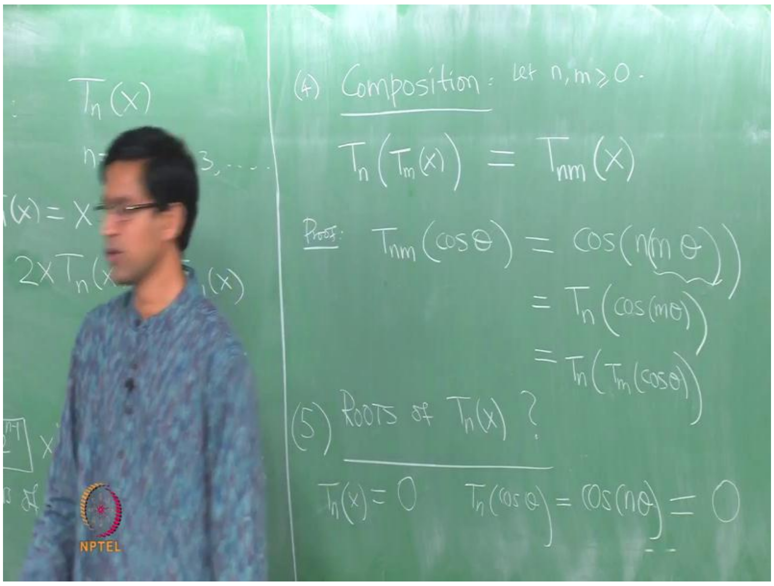

So, they have a really interesting behavior under composition. So, notice that I mention one of the earlier lectures that here is an operation you can perform on polynomials. You take f and g two polynomials you compose them, you compute f of g of x what you get as another polynomial, whose degree is the product of the degrees. But, the thing with the Chebyshev polynomials is, if you take the Chebyshev polynomial T n and you compose it with the Chebyshev polynomial T m.

So, this is degree n polynomial which you are sort of composing with a degree m polynomial. So, what are n and m here, so they are just greater than equal to 0 what this gives you is well in general it is a polynomial of degree n times m. So, that is a best you can say, but in fact, it terms out to be the Chebyshev polynomial itself of degree n m. So, it is rather interesting that, when we compose to Chebyshev polynomials the answer is again the Chebyshev polynomial.

And so let just prove this, because the proof actually very, very easy and just to direct consequence of the trigonometric definition. So, observe that on the right hand side, so let us calculate what is T and m of instead of x I will put cos theta. So, that it is, it leads back to the definition, if I plug in x equals cos theta on the right hand side T n m of cos theta is by definition just whatever you get when you compute cos of n m theta. So, you take cos of this multiple of theta, you write it in terms of cos theta that is exactly this guy T n m of cos theta.

But, this thing here can actually be thought of as cos of n times n theta and notice that in fact, we did this initially when we trying to compute cosine of 4 theta, we said 4 theta is 2 times 2 theta. So, if you know the answer for cosine 2 theta, you can use that to do the next calculation. So, this similar if you want to compute cosine of n m theta, it is cosine of n times m theta and think of this as your new angle, m theta is your new angle. So, it is like trying to find cosine of n times an angle.

But, that is again given by the Chebyshev polynomial to find cosine of n times alpha, you just have to compute T n of cosine alpha, where alpha now here is n theta. And of course, again cosine of m theta by definition is just what you get when you plug in T m of cos theta. So, just this cosine of a multiple of theta the definition is what short of gives you this property when T n m evaluated at cos theta is same as T n of T m of cos theta.

And of course, from that you conclude that these two things are really the same T n m of x is same as T n of T m of x. So, that is a really proof of this composition property of Chebyshev polynomials and he is the final property that I want to talk about what are the roots of T n. So, if you want to figure out what the roots of this polynomial are well is what you have to do, you must ask what are the values of x which will make T n of x 0.

So, we are trying to ask if I want to find x for which T n of x is 0, what is that tell me about x. So, as usual let us plug in x equals cos theta, so that is short gives as a natural handle on the problem. So, you put thing of x as cos theta T n of cos theta let us compute it, it just well we know by definition is cosine of n theta and so the question really becomes for what values of cos theta would cos of n theta b 0. So, to find the root it is really the same as trying to solve this problem, we want to figure out values of theta for which cosine of n theta is 0 and of course, that is a very easy problem. So, we know exactly when cosine become 0.

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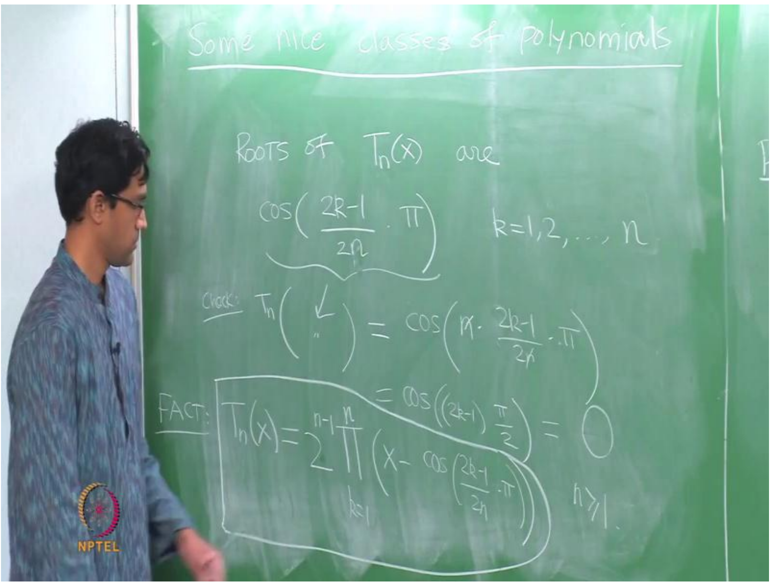

So, what are the roots these are exactly, so what are the values of theta, so we are now looking at... So, the roots... So, let me just write down this statements, so the roots of T n of x are the following numbers cosine of theta, where theta is basically of the form 2 k minus 1 divided by 2 k times pi, k is a number between 1 and n. So, the claim is these are the values of x for which T n of x should be 0. So, let us see if you plug in let us check that this is true, if you take x to be this value. So, let just check suppose I plug in, so what is T n evaluated at this number for a instance.

So, you take k to the in anything between 1 and n let us try to evaluate T n at cos of this number. So, again by definition this is nothing but, this is 2 n, so by definition T n of cos of something is just cos of n times that angle. So, it is n times this angle, so 2 k minus 1 by 2 n times pi, but now the n cancels the n, what this means is this is cosine of 2 k minus 1 which is an odd number times pi by 2, but cosine of an odd multiple of pi by 2. So, it is either pi by 2, 3 pi by 2, 5 pi by 2 and so on.

Cosine of odd multiples of pi by 2 is exactly a 0, so this is in fact, 0. So, each of these numbers if you take x to be this number and evaluate T n on this number what you are guarantee to get is in fact, is 0. So, here are the roots of a the Chebyshev polynomials, so what; that means, is in fact, it implies the following fact that you can actually since you know all the roots, remember we have done this again once before if you know the roots you can factorized the polynomial.

You can write T n of x as what is it each of these roots will contribute of factor. So, this is just product of x minus cosine of 2 k minus 1 by 2 n times pi. So, it is x minus this where k runs from 1 to n, so these are the n roots n distinct roots of this polynomial T n. So, each of these will in fact, the factor of T n, so you have this, but of course, there could be a constant in front. So, these will not capture the constant in front.

But, again remember that was one of the things I said T n of x looks like 2 the n minus 1 times x power n. So, this should really be the constant in front is exactly the leading co efficient 2 the n minus 1. So, here is the final fact which you more and less can reduce from the knowledge of the roots. So, this statement is go for n at least 1 here is an explicitly expression and it is a rather remarkable that this expression holes, because just looking at the sequence of polynomials in over more over 2 x square minus 1, 4 x cube minus 3 x and so on it seems rather remarkable that when you factorized it what you will get will be cosines of some strange angles. So, it is a rather remarkable thing.

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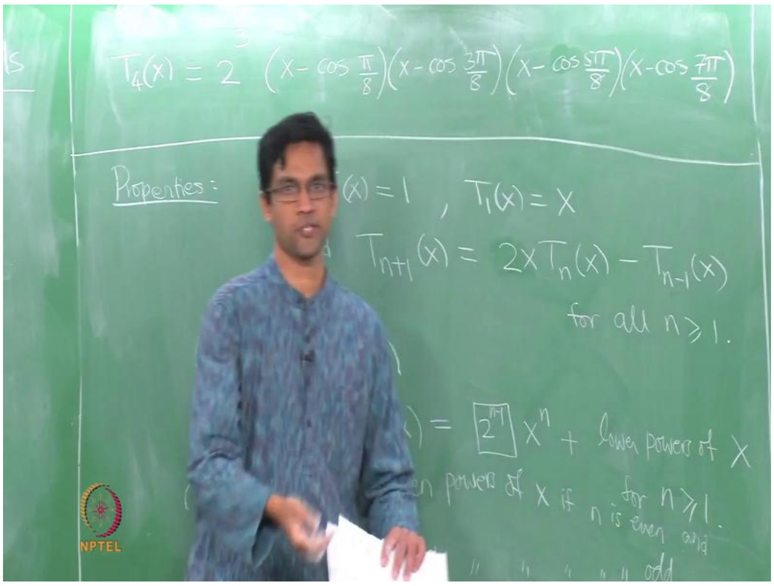

So, let me just end this we just observing, if you look at T 4 of x, so just plug in this formula here is the example, if you take T 4 of x it is in fact, 2 q let us n minus 1 times x minus what are we you are going to get cosine of pi by 8 x minus cosine 3 pi by 8, 5 pi by 8 and 7 pi by 8. So, that is the formula for T 4 of x. And of course, you can also explicitly compute it from that recurrence relation and so on, and you will get some strange formula for it mean some degree 4 polynomial and it is rather remarkable that polynomial actually factorizes in this way.