Graph Theory

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Lecture - 09
Part 1
Independent Set and Edge Cover

Welcome to the 9th lecture on Graph Theory. In this lecture we learn; what is independent set in a graph, and how this is related with vertex cover of a graph. And also we learn; what is edge cover and it is relation with maximum matching. So, let us start with independent set first.

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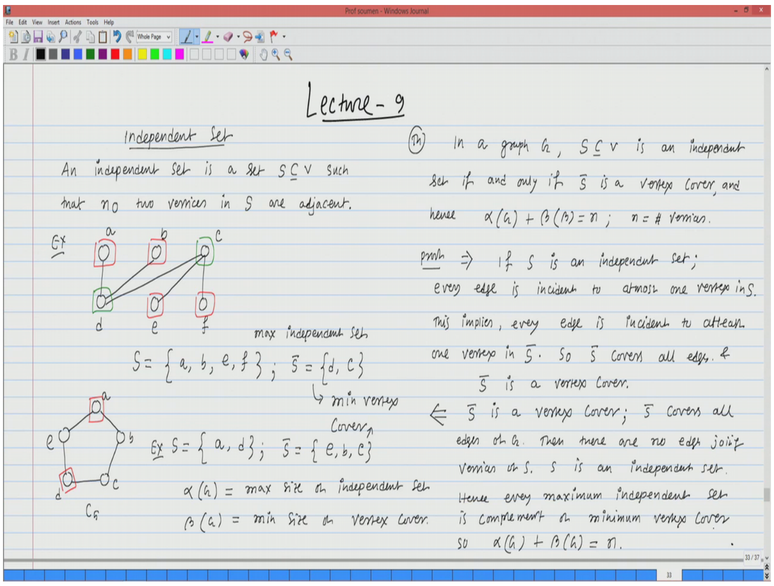

So Independent set, an independent set is a set of vertices such that no 2 vertices in s are adjacent, ok.

So, let me just give an example to explain the definition of independent set. I will consider the same example same graph basically, bipartite graph. So, I label the vertices by a b c d e f. Now the independent set is a set of vertices such that no 2 vertices in s are adjacent. And this is a maximization problem; you have to find maximum independent set. Because finding minimum independent set is a trivial problem you just one vertex is also an independent set.

So, you have to find a maximum set of vertices such that no 2 vertices are adjacent. And in this example you can see that if I consider this vertex a b e and f and then you can see that 4 vertices are not adjacent. There is no 2 of them are adjacent. So, so the s here is equal to a b e and f. So, this is a maximum independent set ok.

So, this independent set definition is of course, not only for bipartite graph. Let me give another example also of So, I will consider 5 cycle. This is cycle of length 5 this is called c 5. Now here I again label them by a b c d and e. So, the independent set could be here I can take a, and then I cannot take b and e because there adjacent with a. So, I can pick either d or c. So, if I pick d then my independent set is consists of 2 vertices, and this is the maximum independent set that you can verify, that you cannot increase this size you can not include any other vertex in the independent set in this independent set.

Now, we use these notations like alpha G, alpha G is maximum size of size of independent set. And beta G is minimum size of vertex cover. Now one thing to observe here that see the complement of s here is d and c. And you can check that this s complement is. In fact, it is a minimum vertex cover. Whereas, for this example also you can see that here in this example s complement is equal to e b and c. And this is also a minimum vertex cover ok.

So, let us proof this theorem and this is in general true that in a graph G s is an independent set if and only if s complement is a vertex cover. And hence alpha G plus beta G I am using the notation here is equal to n. So, n is the number of n is the number of vertices total number of vertices in G ok.

Let us prove this one; this is a important result that complement of vertex cover is independent set. If s is an independent set, then what we can say in terms of every edge that every edge is incident to at most one vertex in s. So, let us look at an edge. So, this is an independent set. So, this a b e f this is an independent set. You take one vertex, sorry one edge and this edge is incident to one vertex from the independent set s. This edge is not incident to any vertex in the in the in independent set.

So, this implies that this implies every edge is incident to at least one vertex in s complement. So, you look at the edges here. So, every edge is incident to So, this is this form the vertex cover c and d this is s complement is c and d. So, every edge is incident to at least one vertex in s complement. So, this vertex is incident to both c and d, which is third definition of vertex cover. So, because vertex cover says that it covers all the edges. So, every edge is incident to a vertex in s complement.

So, you can write that s complement. So, s complement covers all edges and s complement is a vertex cover right. And the other part that So, if we have proved that if s is an independent set this s complement is a vertex cover. Now as in that x complement is a vertex cover, x complement is a vertex cover. So, and then have to prove that s is an independent set. Since this is an vertex cover s, s complement covers all edges of G right, because this is a vertex cover. So, it covers all the edges of G.

Then there are no edges joining, vertices of s which is obvious. So, s is an independent set. Hence every maximum independent set is complement, complement of minimum vertex cover. So, alpha of G which is size of maximum independent set plus beta of G is equal to n ok.

So, we have learned; what is an independent, independent set and we prove that the complement of an independent set is vertex cover. Next we talk about edge cover and we see how edge cover is related with matching, show edge cover.

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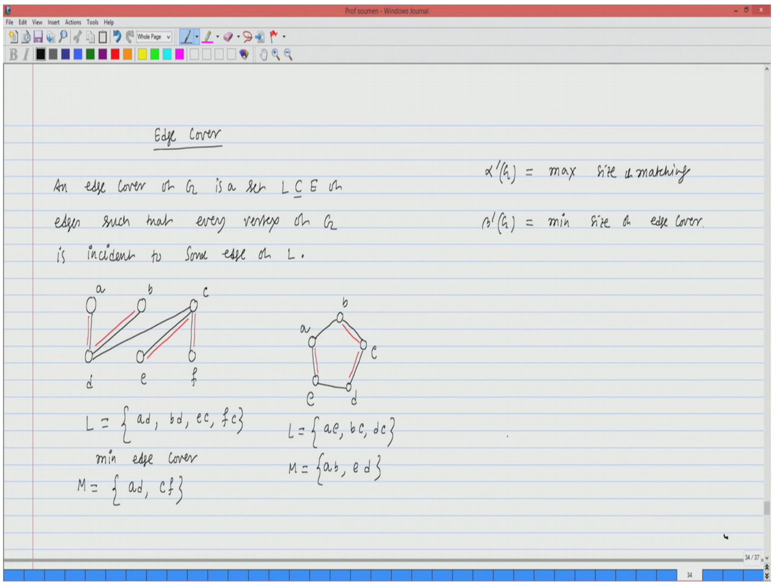

And edge cover of G is a set of edges basically. A set L subset of E set of edges, such that every vertex of G is incident to some edge of L; that means, every vertex is covered by the set of edges in edge cover. Let me give example again, I will consider the same example again or. So, the same graph again.

So, this is a bipartite graph and these are the edges of this graph. Well So, I want to find a set of vertices set up edges that covers all the vertices of the graph. Let me consider this edges for example, a d d b e c and f c. So, I consider my L to be this edges for simplicity I am just writing a d, a d means it is a it is an edge a d a d b d e c and f c. So, this is an edge cover because you can see that every vertex is cover.

So, this 2 vertices are covered by this edge b and d are covered by the edge b d. C is covered by the edge c and e are covered by this thing this edge e c. And f is cover by the edge f c. So, this is this is an edge cover and it is minimization problem. So, this is minimum, minimum edge cover. And we know that for this graph the maximum matching, the maximum m the maximum matching is of size 2. And we know that this is matching say a d and c f is one maximum matching. There could be several, but the maximum size will remain the same the maximum, matching size the size of the maximum matching is 2.

Let me consider say another graph again a pipe cycle; that means, a cycle of length 5 and I label them by a b c d e. And for this one this could be an edge cover, but this 2 edges are not enough see that d is not covered yet. So, I have to either add e d or I have to add c d. So, let me consider this edge. So, my edge cover L consists of these 3 edges a e b c and d c. Well, and the we know that for this one the maximum matching could be say a b maximum matching size is again 2. A d a b and e d is a matching of maximum size 2.

Now, the question is how this the size of minimum edge cover and the size of maximum matching these 2 are related. So, we look at one theorem in this rigor let me use this notation that alpha prime G is the notation for maximum size matching. And beta prime G is minimum size of edge cover. So now, will prove one theorem connecting the these 2 sizes theorem.

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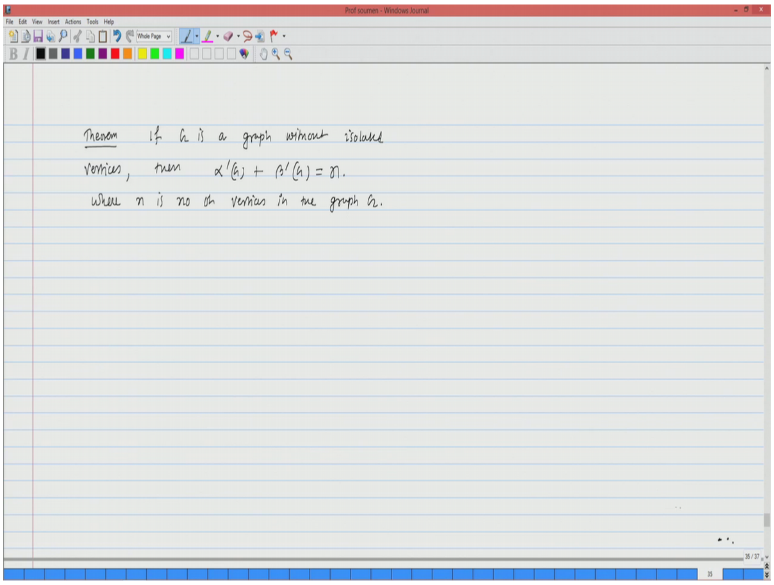

If G is a graph without isolated vertices, then alpha prime G plus beta prime G is equal to n the number of vertices. So, n is the, where n is the number of vertices in the graph.

Let us see whether this is true in our previous example of course, it should be. So, this is the size of maximum matching, maximum matching here. So, the size of maximum matching is 2 and the size of minimum edge cover is 4. So, 2 plus 4 is 6. And 6 is the number of vertices here in this graph. And here you can see that the size of maximum matching is 2, and the size of minimum edge cover is 3. And so, 2 plus 3 is equal to 5, 5 is the number of vertices.

Well, so that is all for this part will prove this theorem in the next second part of lecture 9.

Thank you very much.