Graph Theory
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Lecture - 09
Part 2
Independent Set and Edge Cover

Welcome to the second part of lecture 9. So, in this lecture we have learned; what is an Edge Cover. Now we will prove a theorem stating that the size of minimum edge cover is a plus the size of maximum matching is equal to the number of vertices. So, edge cover you have must have noticed that it is the set of edges which covers all the vertices; whereas, the vertices is vertex cover is that it is set of vertices which cover all the edges. So, some similarity, let us prove this theorem which says that the number of the minim the size of maximum matching plus the size of minimum edge cover is equal to the number of vertices in the graph G.

So, I have stated this theorem in the previous part of this lecture.

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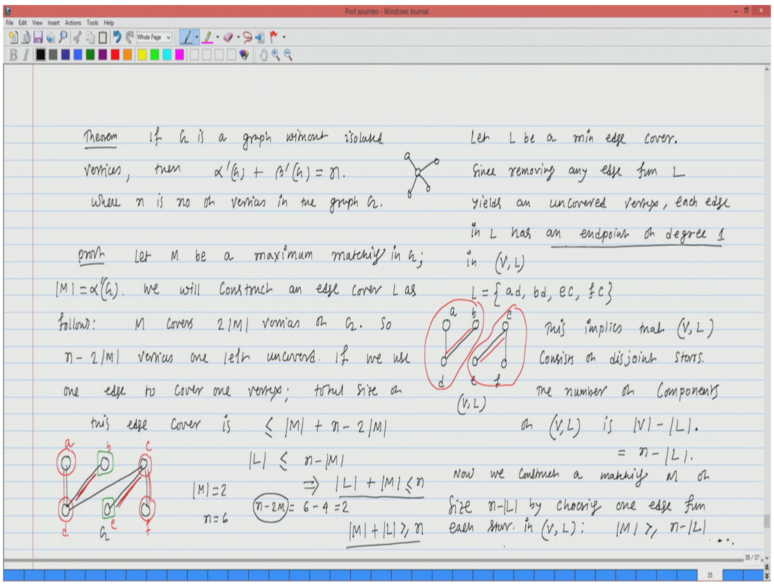

Let M be a maximum matching maximum matching; so using this notation. So, M is basically alpha prime maximum matching in G. So, cardinality of M is equal to alpha prime G. Now starting with this maximum matching we will construct an edge cover. So, we will construct an edge cover L as follows. So, so I just refer the previous example again to explain this theorem the proof of this theorem. Well so, this is the graph G. And sub and we know that this is a maximum matching. So, this is my M, a b c d e f ok.

Now the matching M covers 2 M vertices of G this is true. Because here my matching M is of size 2 and it covers 4 vertices. So, it covers this vertex, this vertex, this vertex and this vertex. This is true because matching is a collection of disjoint edges. So, it will cover 2 M vertices of the graph G. So, N minus 2 M vertices, vertices are left uncovered right. So, here N minus 2 M, means N is N is equal to 6 here. So, N minus 2 M is equal to 6 minus 4. The 2 vertices are left uncovered. So, those 2 vertices are basically this vertex and this vertex. These 2 vertices are left uncovered. So, I have to find. So now we will use one edge to cover one vertex. So, if we use one edge to cover one vertex; that means, I will use this edge to cover this vertex. Then I will use these edges to cover this vertex.

Then that I have to use this many edges in my edge cover to cover this many vertices. Then the total size of this edge cover is less than or equal to M, that is this 2 edges plus I am using this many N minus 2 M edges for N minus 2 M uncovered vertices; so N minus twice of M. So, I got h cover L which is which size is less than or equal to N minus cardinality of M. So, this implies that a cardinality of L plus the cardinality of M is less than equal to n. So, this starting from matching we got out h cover. And from there we concluded that the cardinality of the minimum h cover, plus the cardinality of the maximum matching is less than equal to n.

So, next we start with h cover; let L be a minimum edge covered. Well so, the minimum edge cover for this graph we know that minimum edge cover size is 4. So, the minimum edge cover consist of this edge this edge this edge and this edge. Well so, what it says is that since removing any edge from L yields an uncovered vertex. Each edge in L has an endpoint end point of degree 1 in V L is what is this V L? V L is the basically the sub graph if my let me refer this example again.

So, L is consist of the edges a d b d e c and f c. Now what is what I mean by this V L, V L is this graph simply the graph consists of this edges only. So, a d So, this is a b c d e f this is a d b d e c and f. So, this is the graph V L. So, it says that since removing any edge from L yields an uncovered vertex there is no unnecessary vertex, because this is a minimum edge cover. So, if you remove there is no edge in this graph, such that both the end points are of degree 2 then basically you can remove that edge from the edge cover. So, you can see that every edge in L or in this graph has an end point of degree 1. So, it can not happen that there is an in the V L graph is like this there is an edge here. So, this edge both the end point of degree 2 that is not possible right, that is easy to understand.

So, this implies. So, this property that every edge has an end point of degree 1, this implies this implies that the graph V L consists of disjoint stars. I have not said what is star graph yet. So, so for example, this is a star graph it is just tree, but it is a it is also a tree of course, but tree can have an edge like this also, but that is not allowed. So, star means this property that a each edge has an end point of degree 1. So, this is this is a star. So, this is also a star this is also a star this is also a star. So, this implies that V L consists of disjoint starts. And the number of components of V L is v minus l. So, this is also not difficult to observe. So, this is equal to N minus l.

You think of that that result that we proved in case of tree that a tree with not tree a forest a forest with N components, or say a forest with N vertices and k components has N minus k edges So that will be helpful to prove that the number of components is N minus L here.

So now what we do is that, now we construct we construct a matching M of size N minus L by choosing one edge from each star. So, here I have 2 starts in my example graph. So, this is a this is the this is one star this is one star, the other star is this one. So, once So, once I have the stars in my graph V L this is construction technique. So, given you are given a minimum edge cover from there you are constructing a matching, and this is the construction technique.

So, what you do now is that from each star you just pick one edge you can pick this edge no problem you can pick this edge also from this graph from this star. So, you will get a matching of size 2 right. So, this is the construction technique stars in V L. So, what you got is that you will got it matching. So, the matching size you got a matching of size N minus l, but the maximum matching could be greater than equal to N minus L. And this gives that this gives that the cardinality of M plus the cardinality of L is a greater than equal to n. So, before we got that this is the cardinality of L plus cardinality of M is less than equal to M, now we prove that the cardinality of L plus the cardinality of L is greater than equal to n. So, finally, combining these 2 we get that the cardinality of M plus the cardinality of L is equal to n.

So, what we have done is that we have learned the definition of edge cover and then we have realized that finding edge cover it is a minimization problem, because if you just include all the edges in the graph it will be an edge cover. So, finding minimum edge cover is the optimization problem here, and we re and we observed that that the size of minimum edge cover plus the size of maximum matching is equal to the number of vertices.

So, next we solve a problem which combines sort of matching Vertex cover slightly complicated problem may be.

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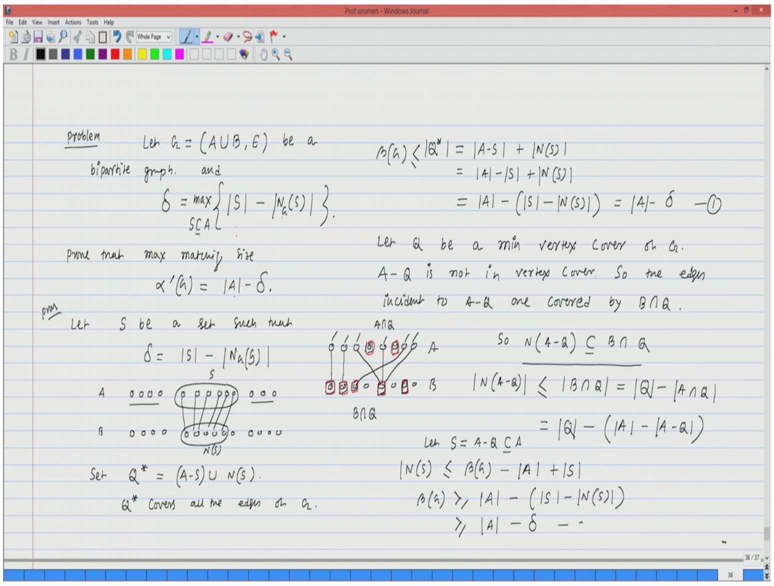

Let G be a bipartite graph A union B be a bipartite graph. And delta G or just delta is equal to this is c this is s is a subset of A, and neighbour of s we understand all these things in G we look at their diff difference. What is the halls theorem once is that this has to be greater than this one, but if it is not greater than if it is less than suppose this is greater than this one then what size of matching you can get this is all about this problem basically. So, and you compute all these differences and take the maximum overall is in A.

Prove that, prove that maximum matching size the notation that we have used that alpha prime G is the notation for maximum matching size of G is equal to the cardinality of A minus delta. So, to relate this problem with the first problem that we started that there is set of 4 girls and the set of 5 boys. And there is some reference list for the girls and the boys. And their if you remember suppose now I denote the boys side as A and the girls side as B, as far as this example this problem is concerned. If you see that the 5 boys suppose that is my set s.

And then they collectively prefer 4 girls right. So, there this delta will be one because you are considering this s to be the set of all boys. And N s is the set of all girls that they prefer collectively. So, there it is this is 5 this is one. So, you will get a matching of size I said a is the set of boys for me now, ok.

Now, we prove this theorem or this problem we solve this problem. Let s be a set such that for which this maximum is attained, such that delta is equal to cardinality of s minus N G s. We said you take maximum overall s and c. And so, where we are assuming that s is that set for which this is maximum. So, so proofing the maximum size matching is same as you prove that the minimum vertex cover is A minus delta. And since, in case of bipartite graph we know that maximum size of a matching is equal to the minimum vertex cover. So, if you can prove that the minimum vertex cover in this case is A minus delta. So, you are done basically.

So, this is my set A this is my set B and there are some vertices here in a some vertices there in B. And this set this is my set s and this is my set say N s no I want to mean that my s set is bigger. So, so this is my set s and this is my set N s. Now you said this quantity this is say Q star I am trying to construct vertex cover Q star is equal to A minus s A minus is this vertices this plus this union N s right.

So now it is. So, A minus s is this plus this and this one. It is not difficult to observe that Q star covers all the edges of G. Because the So, the neighbour of s is here. So, all these edges will be covered by N s. And the remaining edges will be covered by this part. This part right basically it is taking the whole A, but this slice we take for this instead of s we take the N s part here because is smaller than s. So, this Q star what is the size of this Q star Q star is A minus s cardinality of A s plus N s and this can be written as A minus s plus N s right. And this also you can write as A minus s minus N s. And we know that this is delta. So, this is equal to A minus delta.

So, beta G a beta G is the notation for minimum vertex cover. That size that we have used. So, we have constructed arbitrarily vertex cover. So, the minimum vertex cover size is less than this one. So, a all agree with this one. So, we have to prove that beta G is equal to this one, which is same as proving that beta G is the minimum vertex cover size which is same as the maximum matching size. So, proving that beta G is A minus delta is same as proving the maximum matching size is equal to A minus delta.

Now let Q be a minimum vertex cover of G. Then if Q is the minimum vertex cover then of course, A minus Q is the vertices in A minus Q is not in vertex cover. Well, Q has a some part this is my A, this is my A, this is my B, and some of the some of the vertices here are part of Q. So, A minus Q are the remaining other vertices. So, this is not part of Q.

Suppose Q has these 2 vertices from A and some vertices from B right. Well so, so the edges incident to A minus Q; that means, the edges from these vertices. They must be incident to A minus Q are covered by B intersection Q. So, these 2 vertices are basically A intersection Q and these are part of Q the square vertices. So, these are basically B intersection Q and the edges from A minus Q, they must be covered by B intersection Q because Q is a vertex cover ok.

So, the neighbour of A minus Q slightly complicate problem, but you should be able to understand that the neighbour of A minus Q which all the neighbors must be a subset of b minus Q, because whatever edges are coming from A minus Q they are covered by B intersection Q. So, all the neighbors of A minus Q must be a subset of B intersection Q. And this is a very good observation and from there the result will follow now.

Now, it is mostly an algebra you compute the cardinality of N A minus Q. This is less than equal to the cardinality of B intersection Q right. Which is equal to Q minus Q is the total vertex cover and B intersection Q is this part, and this part is let me N make them red. So, this is A intersection Q and these are the vertices of the vertex cover in the b part. So, B intersection Q is Q minus A intersection Q, no problem. And from here this is equal to I can write Q minus. Now A intersection Q I write in this form. A minus A minus Q this is also easy observation right.

Now, let me denote this see A minus Q is just a subset of A. Let me call it s equal to A minus Q which is a subset of A. Now from here what I can say that now just from here I just replace A minus Q by s. So, the cardinality of N s is less than equal to, this is the this is the cardinality of minimum vertex cover. So, I will use the notation B G for that this is B G right not b beta G minus A plus s right. So, from here what I can write is that beta G is greater than or equal to cardinality of A minus s minus N s. And this implies that this is cardinality of A minus, and you note that this quantity we have assumed this is equal to small delta.

Thus what we got here is that we got at the beginning here we got beta G is less than equal to A minus delta. Now we got beta z is greater than equal to A minus delta. So, this is say equation 2, and combining these 2 we get thus beta G is equal to A minus delta. Now beta G is same as alpha prime alpha prime G, because alpha prime G is the notation for maximum matching. So, proving that the size of minimum vertex cover is cardinality of A minus delta is same as proving that the size of maximum matching is equal to A minus delta. So, we have proved that the size of maximum matching in this case possible is cardinality of A minus delta, ok.

Thank you very much.