Graph Theory
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Lecture – 11
Part – 2
Gale-Shapley Algorithm

Welcome to the second part of lecture 11 on Graph Theory. In the first part we have learned what is stable matching for a given instance. And now we learn how to find a stable matching given an instance to the problem. Before I talk about the algorithm to find stable matching, I just want to mention that this stable matching problem has many applications mostly on admission process for example.

Suppose, there are 10 good institutes and student will give their preference list of institutes their first preference second preference third preference. And similarly the institute will have their own preference list mostly the ranking of the students they every institute will prefer higher rank students. So, there also you can use a variant of stable matching to decide student allocation as I said that matching is a one to one correspondence; that means, one student will go to one college that is not possible.

So, or one institute will be matched with only one student that that is not the case. So, what have to use you have to do is that you just replicate the institute several times with the same preference list. And then you apply the stable matching problem to find to allocate the students in different institutes. I hope that you understood the definition of stable matching, a matching is stable if there is no blocking pair with respect to that matching. And the blocking pair is as I said that it is analog to the extramarital appear ok.

So, now we talk about gale Shapley algorithm to find stable matching for a given instance.

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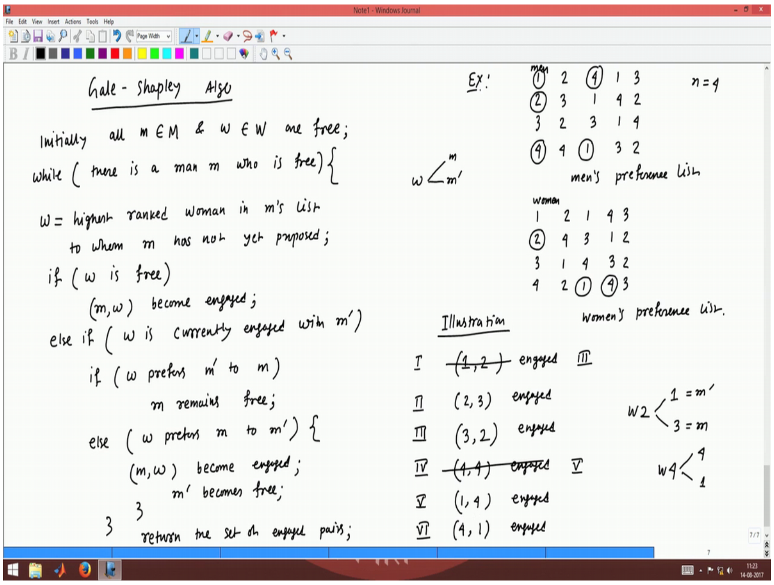

So, here is the algorithm gale Shapley algorithm and I took the same Instance that we considered in the first part. So, here in the gale Shapley algorithm initially, initially all man belongs to M. So, I have used notation m for the matching also, but this is the set up this M is the set of all men. And w belongs to the set of all women are free. Initially no 1 is matched.

And then while there is a man m who is free, you do the following. First you look at the w which is the highest ranked woman in m's list this is. So, w is the highest ranked woman in m's list to whom m, m has not yet proposed. So, you understand the definition of w, w is the highest ranked woman in m's list to whom m has not yet proposed.

Now, if w is free the woman w is free then m w become engaged. Else it might happen that m is already engaged, sorry w is already engaged. Else if w is currently engaged with m prime say, then w will see w is currently matched with m prime. And now she has been proposed from or she has a proposal from m also. So, what she will do is that she will see who is in the higher rank in her preference list.

So if w prefers w prime, sorry w prefers m prime to m; that means, she prefer her current matched partner more compared to the new proposal m, new proposal from m. Then she will not break her engagement she will sort of reject m at this moment. So, m remains free. So, she will reject the proposal from m because she prefer m prime more compared to it is current it is new proposal m. Else if w prefers m to m prime, then m w become engaged and our current must partner m prime becomes free ok.

So, when you see that when you know at some point of time when w is free, you just make m w engaged, but this engagement this engagement is not the final engagement. It might break at some point of time during the execution of the algorithm. So, in this case you do these 2 things right, and here is the end of this while loop this is my closing bracket this closing bracket which corresponds to this opening bracket. And once you outside this while loop; that means, every man is engaged then you return at the end return the set of engaged pairs. And that forms this set of engaged pair is a stable matching basically, before proving that part let me just illustrate this algorithm using this example.

So, we have suppose this is the input to this algorithm where n is equal to 4 therefore, man. So, this is the man list and mans preference list this is the 4 women and their preference list. Well so, initially you see that initially everything is free all men and women are free. So, then the while loop says that if they while there is a man m who is free well man 1 is free at this moment. So, will start with man 1 I am just trying to illustrate this algorithm using this example illustration.

And of course, this is called man optimal stable matching because it is the man who is proposing, you can do it in other way also. So, first we start with men one. So, men 1 is free and men one look at the highest ranked women in man 1 list. So, that highest rank women is 2. So, one proposes to 2 and 2 is free at this moment. So, 1 2 become engaged like following this technique, 1 2 become engaged.

So, this is the first iteration. Now in the second iteration still there is a man who is free. So, we go to second man and second man will look at. So, w will be 3 because in the second man least the highest rank woman whom he has not proposed at is 3. So, man 2 will propose 3 and 3 is free. So, 2 3 will become engaged. Now in the third iteration still there is a man who is free. So, 3 is free. So, 3 will propose to 2 now here the problem occurs 2 is not free.

So, 2 is not free. So, what 2 will do? 2 is currently engaged with 2 is currently engaged with 1. And she is engaged with 1 and 2 has a proposal from 3. Now 2 will look at her preference list. So, look at that second women preference list. And you can see that 3 is a higher ranked man in her preference list. So, so she will break this engagement with one. So, one will become free, and 3 will be match to with 2; that means, 3 2 will become engaged. I hope that you understood this is the step that 2 is at this moment engaged. So, 2 is engaged and she is engaged with one that is m prime is m prime is one and now she has a new proposal from m which is 3.

So, see look at her preference list and she can see that this is true, because woman 3 prefers, sorry woman 2 prefers 3 to 1 prime, that is why 3 2 will become engaged. And it will make one free break this engagement at the third iteration. Now move to the 4th iteration, now we can see that 4 is free at this moment 1 is also free. So, let us move to 4 4 the highest rank woman enforce preference list is 4 and 4 is free. So, 4 4 will become engaged at the 4th iteration.

And then we broke this engagement during the third iteration here I should write that. Now you still you can see that the man 1 is free at this moment. So, you go to man 1 again and look at the woman in her in his preference list highest rank woman whom he has not proposed is 8 is 4. So, w will be at this woman w will be equal to 4. So, 1 will propose to 4. So, 1 will propose to 4 , but 4 is not free. So, 4 is already engaged with 4. And 4 has a new proposal from 1. So, this is woman 4 this is also woman 2.

Now, see will look at her preference list whom she prefer more 1 or 4. So, 1 is higher ranked man than 4. So, she will break this engagement at the iteration 4, and she will be engaged with 1. So, 1 4 will be engaged. Now we can see at this moment 4 is free now again. So, you go inside this while loop and in the 6th iteration. Now 4 is the only man who is free. So, 4 will propose to the highest rank woman whom he has not proposed yet. So, 4 will propose to 1. And 1 is free at this woman you can see that the woman 1 is free.

So, 4 will be engaged with 1. And you can see that at this moment after the 6th iteration all the main man are engaged. Man 1 is engaged with a 4 man 2 is engaged with 3, man 3 is engaged with 2 and man 4 is engaged with woman 1. So, you are outside this while loop at this moment you are outside this while loop. And you return this sort of all engaged prayer and that is the stable matching. So, I hope that you understood the algorithm. And now we will talk about the correctness of this proof. This is a theorem.

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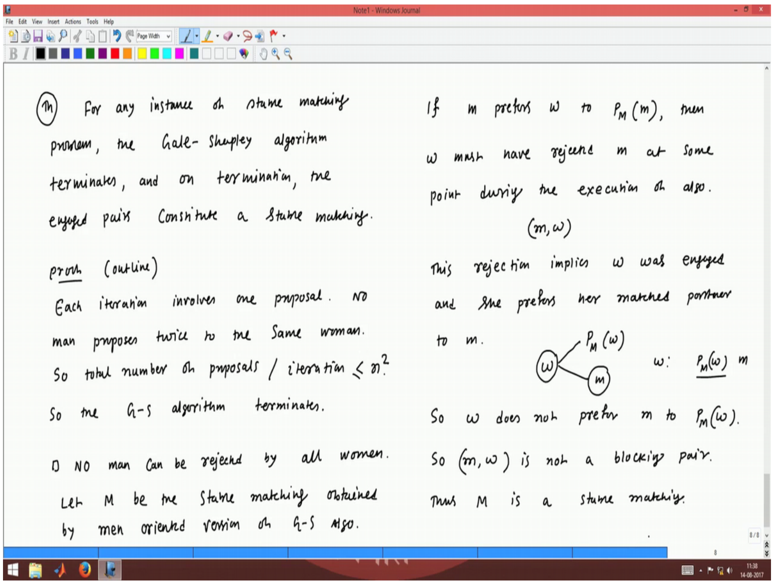

And it says that for any instance of stable matching problem the gale Shapley algorithm, algorithm terminates and on termination the engaged pair constitute a stable matching. So, this theorem is like proving 2 things. First thing is that this algorithm will terminate. Because you can see that when a free man is proposing to free woman proposing to a woman who is free, if then they become engaged. And then in the latter part of the algorithm some time we break that engagement. So, how do you ensure that the algorithm will terminate at some point of time.

So, this is the first part of the theorem that it says that the gale Shapley algorithm will terminate. And the second part is that the set of engaged pairs that you returned at the end of the algorithm, that will form a stable matching; that means, there is no blocking pair with respect to the matching that you get from the gale shapley algorithm. So, will prove this theorem and this theorem is sort of the correctness proof of the gale Shapley algorithm that it returns stable matching indeed ok.

So, there is some observation to make here. So, will just give the outline of the proof is not that um a giving the complete proof. So, first thing is that each iteration. So, if you have seen that in the previous example we had like 6 iterations. And each iteration involves one proposal. And no man proposes twice to the same woman. This is true because if you look at the definition of w. So, w is the highest rank woman in m's preference list whom he has not yet proposed.

So that means, no man will propose twice to the same woman. So, so total number of proposals or iterations is always n square. N is the number of men and women. So, there are total n square pairs. And at the worst case we might have to explore all the possible pairs. So, the total number of proposals or iterations is always less than or equal to n square. So, the G S algorithm; that means, gale shapley algorithm terminates ok.

So, this is the end of the first part of the proof that the gale shapley algorithm terminates. And the second part is that on termination the engaged pairs constitute a stable matching. First of all this is an observation just that no man can be rejected by all women. This is a simple observation that you have to make; that means, this ensure that no man can be rejected by all women this will ensure that the there exists a matching. And then the next part of this proof is that we will proof that the matching that the algorithm returns is a stable matching there is no blocking pair ok.

So, let m be the stable matching obtained by men oriented version of G S algorithm. This is the one we talked about is the men oriented, because you know we start with the men proposing the woman. So, this is called the men oriented version of the G S algorithm.

If now we are talking about the possibility of a blocking pair, if m prefers w to it is match partner P M m, then w must have rejected m at some point during the execution of algorithm. So, we are exploring the possibility that m w is a blocking pair. So, we are assuming that m prefers w more compared to it is match partner in m, let us see what happened, what happened?

Now this rejection this rejection implies w was engaged, and she prefers her matched partner to m. So, when w will reject someone when she is engaged and she is she prefer her matched partner more compared to the new proposal from m; that means, So w does not prefer m to P M w. This is I hope this is clear because at some point of time w has rejected m; that means, in if you look at the W's preference list P M w whoever be her matched partner at that point of time of the algorithm. This is in higher preference list then m.

So, she prefers P M w to m. So that means, w does not prefer m to P M w. So, this is this proves that m w is not a blocking pair. So, so m w because of this fact the w does not prefer m to it is match partner. So, m w is not a blocking pair. Thus m the m is the set of engaged pair that the gale Shapley algorithm returns. So, m is a is a stable matching ok.

So, we have proved that the gale Shapley algorithm terminates though there are some engagement and then you break the engagement again in all these things, but finally, it will terminates and on termination it returns a stable matching. So, the one the gale Shapley version that we have given. Here it is a man oriented stable matching. You can just change the role of man and woman you will get a woman oriented stable matching. And the woman oriented stable matching will be different from the man oriented stable matching that you can check.

Thank you very much.