Graph Theory
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Lecture – 12
Part – 1
Graph Connectivity

Welcome to the first part of a 12th lecture. Today we will talk about graph connectivity. Graph connectivity is a basic concept in Graph Theory. It asked the minimum number of vertices or edges that need to be removed from the graph to make the graph disconnected. So, first I will give the formal definition of vertex connectivity. And then we give the definition of h connectivity and then we will study the relationship between the vertex connectivity, and edge connectivity.

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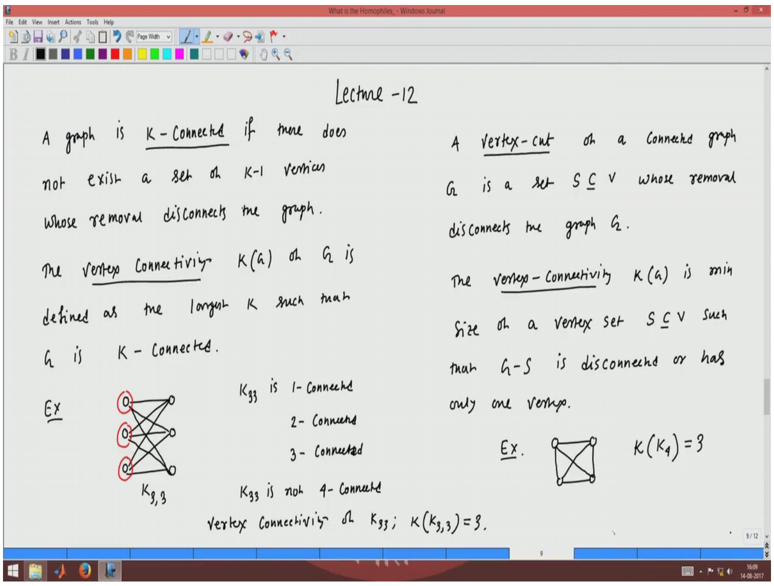

Let me first formally introduced what is vertex connectivity. A graph G is k connected if there does not exist a set of k minus 1 vertices whose removal disconnect the graph. And so, this is the k connected graph. The vertex connectivity say K G of G is defined as the largest k such that G is k connected ok.

Now, I will give example to illustrate this definitions. Let me take this example. I will take a bipartite graph. And let me take complete bipartite graph. So, this is k 3 3. Now you can see that to if I just remove this vertex for example, this graph will not be disconnected. But if I remove this 3 vertices, then the graph we can disconnect graph. So, I can say that this graph k 3 3 is one connected. Because they does not exist set of 0 vertices whose removal disconnect the graph here. I can say the a graph k 3 3 is 2 connected. Because they does not exist a set of one vertex whose removal disconnect the graph. For example, if I just remove this vertex then the graph does not become disconnected. So, it is 2 connected graph well that does exist a set of one vertex whose removal disconnect the graph this k 3 3 is 3 connected, 3 connected because they are does not exist a set of 2 vertices whose removal disconnect the graph k 3 3. But k 3 3 is not 4 connected, because there exist a set of 3 vertices whose removal disconnect this graph s.

So, the graph is not 4 connected. So, it is 3 connected and 3 is the maximum value of k such that the graph is k connected. So, so the vertex connectivity of this graph is 3. So, the vertex connectivity of k 3 3 which is k k 3 3 which is equal to 3. So, this is one way of understanding what is vertex connectivity. There is another parallel definition first we give the definition of vertex cut. A vertex cut of a connected graph G is a set of vertices s whose removal disconnects the graph, graph G. And the other way of understanding this vertex connectivity in terms of vertex cut is a following. The vertex connectivity K G is minimum size of a vertex set is belongs to v, such that g minus s is disconnected or has only one vertex. So, this is also important mostly for the complete graph.

Let me take another example say k 4 is a complete graph with 4 vertices, now here you can see that. So, what is the vertex connectivity of this graph? It is not difficult to observe that the vertex connectivity of this graph k complete graph with 4 vertices is equal to 3. So, if you remove 3 vertices you will be left with only one vertex, that is all. And that is the connectivity of this graph. And in terms of vertex set if you want to understand the vertex connectivity here, then your minimum size vertex cut consists of this 3 vertices if you remove these vertices the graph will become disconnected. So, either it will become disconnected or it will have only one vertex left ok.

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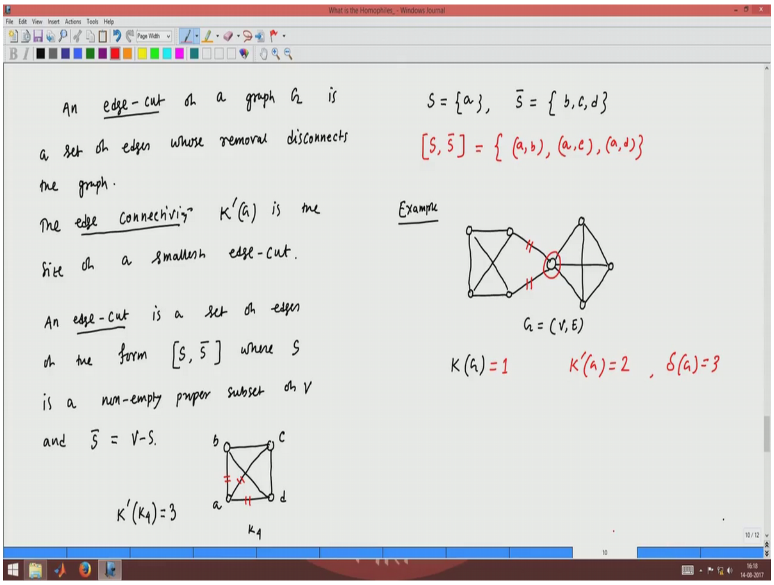

So, next we talk about the edge connectivity, and edge cut similarly like vertex cut and edge cut of a graph G is a set of edges whose removal disconnect, is removal disconnect the graph. And the edge connectivity k primes g is the size of a smallest edge cut. So, edge cut is a set of edges whose removal disconnect the graph and edge connectivity is the size of the smallest edge cut right. And you gen usually use the standard notation to denote edge cut an edge cut is a set of edges of the form s s complement. So, I will explain this one where s is a non empty proper subset of v, and of course s complement is v minus s.

So, let me again consider the complete graph with 4 vertices. So, this is k 4 and the vertices are a b c and d. Now if take my is to be say, a and then it is not at non empty subset of the vertex set and is compliment then; obviously, is b c d. So, the edge cut is consist of this 3 edges the edge cut here the s s complement. So, s s compliment here is consist of this 3 edges. This edge, this edge and this edge. So, s s compliment is the edges between s and s complement their a b a c and a d. So, these are the edges in the edge cut.

Now, you can see that if you remove this 3 edges from the graph if I move these 3 this 3 edges from this graph, the graph will become disconnected. So, that is why this is an edge cut.

So, I hope that you understood the definition of edge cut, and here the edge connectivity of k 4 you can not you need to remove minimum 3 edges to make this graph disconnected. So, k prime k 4 is also equal to 3. And the vertex connectivity was also 3 for this graph.

Now let me take another example to sort of get idea about how this vertex connectivity and the edge connectivity are related. So, I will take this graph. There is no specific name for this graph probably look at this graph. Now the vertex connectivity is the minimum number of vertices that must be removed to make the graph disconnected. So, if I call the graph G V E, then clearly the vertex connectivity K G is what? K G is minimum you need to remove one vertex, if you remove this vertex for sample from the from this graph, the graph will become disconnected.

So, the vertex connectivity is equal to 1 and the edge connectivity k prime G for this graph is 2, you can see that if you remove this edge and this edge the graph become disconnected. So, the edge activity of this graph is 2. And what is the minimum degree of this graph delta G is equal to 3. So, what we can see the that at least for this specific graph, the vertex connectivity is less than or equal less than the edge continuity and the edge connectivity is less than the minimum degree of this graph. Next we prove a theorem which a relates the vertex connectivity edge connectivity and the minimum degree of a graph G. And here relation is that the vertex connectivity is always less than or equal to the edge connectivity. And the edge connectivity is always less than or equal to minimum degree of the graph.

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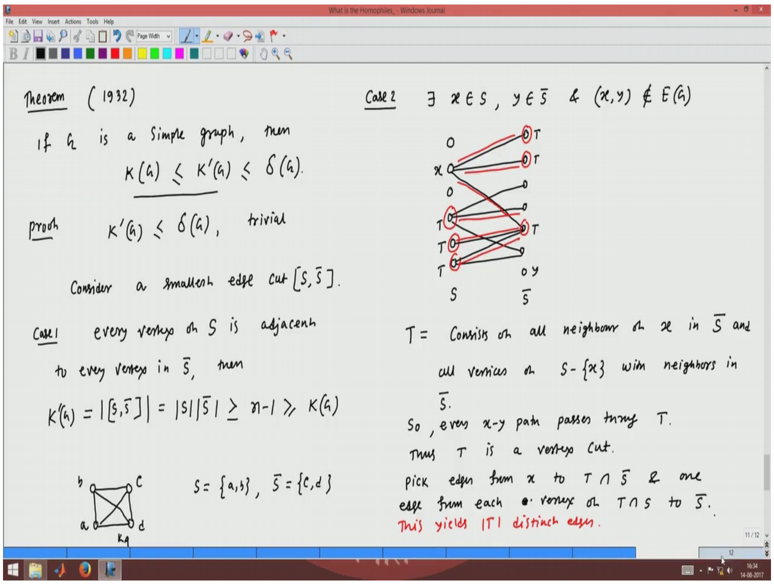

So, let us prove this theorem. This is a 1932. If G is a simple graph, then vertex connectivity of the graph G is less than equal to the edge connectivity the graph G less than equal to delta G this is the minimum degree of the graph G. So, we prove this theorem. So, important theorem k prime G is less than equal to delta G. This is a trivial part, because suppose the minimum degree of let me just refer the previous example. Here the minimum degree is 3 for this graph. So, and this vertex has degree 3 also. So, if I remove this 3 edges from the graph then the graph also become disconnected right, but there is another way of making the graph edge disconnected if I just remove this 2 edges then also the graph become disconnected ok.

So, the edge connectivity is always less than equal to the minimum degree. Because this is easy just you pick the vertex which has minimum degree and remove all the adjacent edges to make the graph disconnected. So, that is a trivial edge cut. So, we have proved this part, now consider now we are going to prove this part basically that vertex connectivity is always less than equal to the edge connectivity. And considered a smallest edge cut, you understand the meaning of this edge cut.

Now the first case one is that every vertex of s is adjacent to every vertex in is complement. So, this happened that when the graph is say complete graph when suppose I take complete graph k 4 again for illustration a b c and d. And if I take this is a complete graph it k 4 vertices k 4 if I take my s to be say a b and s complement to be c d then I can see that every vertex in s is adjacent to every vertex in s complement here right. So, this will happen if the graph is complete graph. And in this case the cardinality of s come s s complement that is the number of edges in this cut is equal to the cardinality of s into the cardinality of s complement. Because every vertex in s is adjacent to every vertex in s compliment. And this number is always going to be greater than or equal to n minus 1, n is the number of vertices. Because see the verticen could be one n minus 1 s has one vertex s compliment as n minus 1 vertex vertices s can have 2 vertices and s complement will have n minus 2 vertices. And then you see that this product is twice n minus 1. So, the minimum possible is n minus 1 when the verticen is one n minus 1 ok.

So, this number is always greater than equal to n minus 1 and vertex connectivity is always smaller than this one, because in the graph with n vertices if you remove n minus 1 vertex that is the maximum you need to remove. So, the vertex connectivity is always less than n minus 1. And this the minimum edge cut. So, this is k prime G. So, in case what I have proved is that the k prime G the edge connectivity is greater than equal to the vertex connectivity. I hope that this is clear.

Now, let me talk about the other case is no not the case that you know not every vertex of s is adjacent to every vertex in s compliment. So, this is this case says that there exists x a vertex x in s and the vertex y in s complement. And this 2 are not adjacent there not an edge in the graph G. Try to think in this way. So, this is these are the vertices in s, suppose their vertices in s and these are vertices in s complement. It is not a bipartite graph there could be a edges between a vertex in vertices between vertices in s.

Suppose this x and this is y and this is no edge between x and y. Now what we want is that we want to remove some of the vertices in the graph, such that there is no path between x and y. So, for that purpose what I have to do is that in order to block all paths between x and y. First I will pick the neighbor of x in s compliment. So, I will construct a set T. So, T consists of some vertices such that if I removed those vertices if I remove the vertices in T n there will be no path between x and y.

So, first T consists of all neighbors of x in s compliment. So, it could be possible that k there could be a path from first from x, you come to this vertex and then there is a path from to y here. So, that possibility we have removed now, because we are including all these vertices in the set T and finally, you want to remove the vertices in T from the graph the other possibility is that there could be first from x you come to some other vertex and from this vertex you go to y or some other vertex in s are ok.

So, what will do is that will in this part in the part is we remove all vertices which has neighbor in s complement. So, these are vertices in T first of all and then and all vertices of s minus x with neighbors with neighbors in s complement. Suppose this is a vertex which has neighbor in s complement this vertex has suppose these are the neighbors of this vertex this is another vertex which has neighbor in s complement. So, this is another vertex which has neighbor in s complement.

So now if you see that the way the set T has been defined, T consists all neighbors of x in s complement and all vertices of s minus x with neighbor in s complement. So, every x y path passes through T, thus T is a vertex cut right. Now what we do is that we show that if you remove this vertices from the from the graph there will be no path between x and y. So, you to remove basically if 6 vertices. And now what you do is that correspond to every vertex you choose an edge. So, pick the edges from x to T intersection s compliment; that means, you pick this edges you pick this edges and one edge from each vertex of from each vertex of T intersection s to s compliment; that means, you pick just one edge from each vertex in T intersection s that is these are the vertices in T intersection s. So, you pick one edge corresponds to every vertex, say this edge. And this yields T distinct edges, but this set of T distinct edges can not disconnect s and x and y.

So, but the set of T vertices can disconnect x and y thus k prime G which is s.

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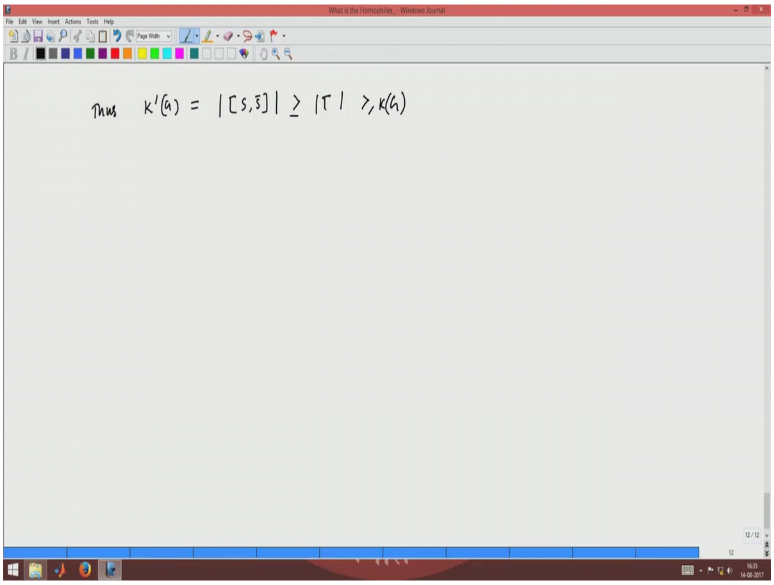

Thus k prime G which is the cardinality of this cut is greater than or equal to T which is the vertex connectivity. So, we have constructed vertex cut and clearly you can see in the previous example. So, this set of vertices can disconnect x and y, but the number of edges between s and s complement, that is larger than the cardinality of T. So, and this size of the cut is the edge connectivity which is larger than the vertex connectivity. So, what we have proved is that, for any given simple graph the vertex connectivity is larger than is always smaller than the edge connectivity and the edge connectivity is always smaller than the minimum degree.

Thank you very much.