Graph Theory
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Lecture – 15
Part 1
Flow Network

Welcome to the first part of lecture 15 on Graph Theory. So, in this lecture we learn flow network. So, first we learn what is that network a network is a directed graph, such that each edge has some capacity. So, directed graph also I never talked about what is directed graph. So, a graph is directed if every edge has a direction. Say in case of directed graph if I say there is an edge from a to b so, that will be indicated that you can only move from a to b not from b to a.

So, let us learn first what is network formally and then we learn what is a flow in the network. And finally, will see how to find maximum flow in a network.

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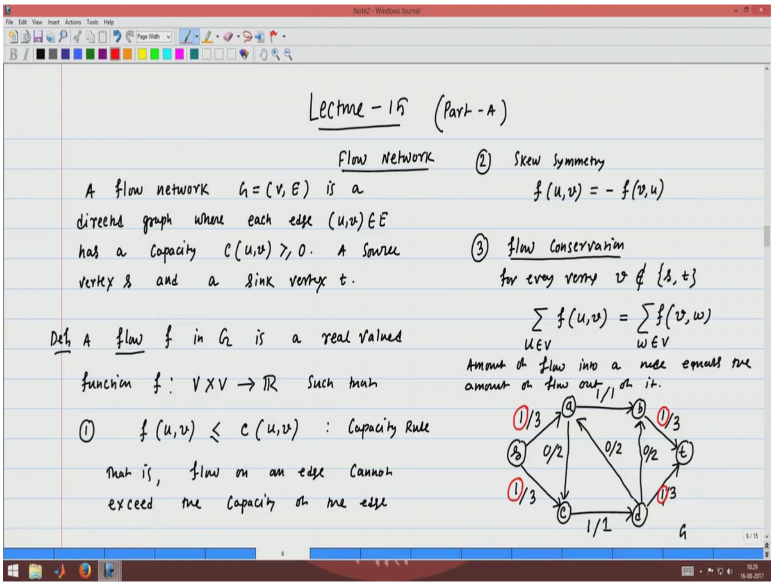

So, here is the definition of network. So, a flow network denoted by G V E is a directed graph where each edge say u v in E has a capacity denoted by c u v and this is always greater than equal to 0. And it has a source vertex s and a sink vertex t. So, maybe I can just draw a network here. So, this is what the network is this is the source vertex, and this is the sink vertex t and it has other nodes of course.

So, this is a b c and d. And as I said a network is a directed graph. So, this is the meaning of s to a, you can only move from s to a not from a to s using this edge. So, and this has capacity 3; that means, 3 unit of flow you can send from s to a, a to b this has capacity 1. T to b sorry, b to t it has capacity 3. S to c it has capacity 3. C to d capacity 1, d to t capacity is 3, a to c capacity 2 and d to b capacity 2 and d to a this edge has capacity 2 again. So, this is what a network is. So, g is a network here. So, it is a directed graph as I said it is a directed graph and each edge has capacity c u v. So that means, here c s a is equal to 3, that is what the meaning of c s c u v. Now I hope that there is no doubt about what is network. Now let me introduce what is the flow. So, a flow denoted by f in the network G is a real valued function f, which is from v cross v to R, such that some conditions are satisfied.

The first conditions is that the flow amount from u to v is always less than equal to the capacity of the edge u v; that means, the meaning of this one is that if the flow value from s to a cannot be more than 3. You cannot send more than 3 unit of flow from s to a because the capacity maximum capacity is 3. So, you can along s to a, you can send 0 unit, you can send one unit of flow, you can send 2 unit of flow you can send maximum 3 unit of flow, not more than that. This is what the capacity rule this is this is called this is called capacity rule. So, the meaning of this one is that that is flow on an edge cannot exceed their capacity of the edge. I am sure that you understand this thing.

Now the second rule is the second conditions of the flow we are talking about the flow definition right definition of flow. So, this is all under the definition of flow the second rule is skew symmetry, skew symmetry skew symmetry. So, this is the flow from u to v is equal to the negative of the flow from v to u. So, how do I explain this one? So, the flow suppose I sent one unit of flow from s to a; that means, my flow from s to a is 1, then the flow this will be also can be considered as the flow from a to s will be considered as minus 1, that is all. That much is for the time being. Now the third condition which is very important, that is called flow conservation. So, the meaning of this one is that for every vertex v other then the source and the sink vertex, the prow going into a vertex u or vertex v if u v the amount of flow that is going to the vertex b I will explain this one from all other vertices. So, u belongs to v. So, this is the amount of flow that is going into v is equal to the amount of flow that is going out of v; that means, v to w and w belongs to v.

So, the meaning of this one is that the amount of flow into a node equals the amount of flow out of it. So, I am received that may be this is this part is not clear. Let me give an example of a flow. Say I said that flow is a real valued function; that means, for every edge you put some real number. Such that that real number whatever number you are putting is less than the capacity this will be some how true automatically. And the flow conservation rule is true let me just given examples and then I will explain what is a flow valid flow. So, I assigned for this edge I assigned one unit of flow. So, the meaning of this one is that this is the notation that I will we following in this lectures that flow capacity this is the flow this is the capacity. So, this is the flow, this is the capacity. So, along this edge I will send one unit of flow, along this edge I will send 1 unit of flow, along this edge from a to c I will send 0 unit of flow.

Let me write this 2 here. Along this edge I will send 0 amount of flow, along this edge the capacity was 2. So, it is from d to a I will send 0 amount of flow. From d to b I will send 0 0 unit of flow, from s to c I will send 1 unit of flow. C to d I will send one unit of flow and from d to t I will send one unit of flow. Now the question is whether this is a valid flow, whether this flow. So, flow I said it is a function which assigned some real number to every edge.

So, I have assigned some real numbers now 1 1 1 0 0 0 1 1 1 here. Now I need to check whether this allocation of flow satisfy all this conditions. First the capacity rule is satisfied because every where you can see that the flow amount is less than or equal to the capacity right. And here also the second is fine it will be automatically true the third one now flow conservation. So, one vertex con conserve some amount of flow the amount of flow coming in 2 node, must be equal to the amount of flow going out of that node. So, the amount of flow that is coming into a is 0 plus 0 plus 1. So, the amount of flow that is coming into a is 1. And the amount of flow that is going out of a is also 1.

So, you can see that this flow conservation rule is true for a you can also check for b for b the amount of flow going into b is 1 plus 0. So, this is 1 and the amount of flow going out of b is 1. And similarly for c the amount of flow coming into c is 0 plus 1 1. And the amount of flow going out of c is 1. Finally, check for d also. The amount of flow coming into d is 1 and the amount of flow going out of d is also 1. So, the flow conservation rules are is true for all vertices except the source vertex, and the sink vertex. So, from the source the flow only go out right. So, the 2 unit of flow is going out of s and 2 unit of flow is coming into t the sink vertex.

So, I hope that you understood what is a network and what is a flow in a network. Flow has some conditions this capacity rule must be true for a flow to be valid, skew symmetry and the conservation rule this 3 rules must be true for a flow to be valid. Now then what is the problem here what is the optimization problem that we are trying to solve in the flow network is that a given of network you have to find what is the maximum flow that you can send from the source to the sink vertex. This is the optimization problem, maximizing the amount of flow. We talk about that problem now.

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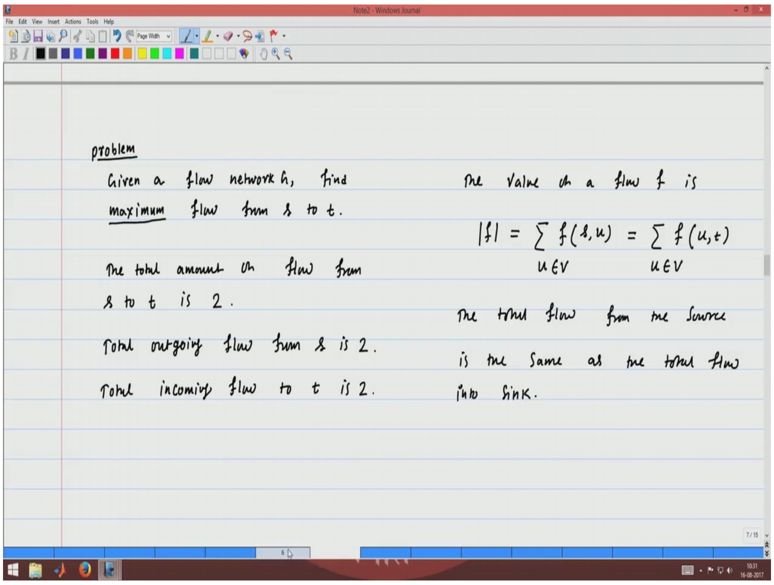

So, the problem we consider here is given a flow network a flow network find maximum flow from s to t. So, this is the problem that here going to solve. And this is not a trivial problem how to find a maximum flow from s to t. Now in my previous network you might have observed that the total amount of the total amount of flow from s to t is 2. And the total out going flow from s is 2. And total incoming flow to t is 2. So, what I am trying to say is that the total outgoing flow from s look at the previous network. The total outgoing flow from s is 1. This one unit of flow plus 1 unit of flow that is the total 2 unit of flow is going out of s. And that will be same as the total incoming flow to t total incoming flow to t is also 2 because this should be true. That the amount of flow going out of s is equal to the amount of flow coming into t, this is this should be true because of the flow conservation rule.

No note can conserved some flow that is why the amount of flow going out of s is equal to the amount of flow coming into t. Now formally the value of a flow of a flow f is denoted by cardinality of f I mean mod f basically that is the flow that is going out of s s to u u is a vertex b, and which is also same as and the amount of flow going into t.

So, were u is a vertex in v. The total flow from the source is the same as the total flow into sink. So, we have learned all the technical terms. So, we know what is the network what is the flow what is the value of a flow, and what are the conditions of flow. So, to satisfy and now the question is what is the maximum flow that one can send from s to t is this 2 is the a maximum flow possible in this network, or you can send some more amount of flow from s to t. So, that is the problem here. And in order to solve this problem how to find maximum flow from source to sink in a flow network. We will talk about 2 important concepts one is called residual network. So, the residual network is a with respect to a given flow.

Also we talk about augmenting path. Let me talk; let me start with the definition of residual network. And probably we will give an example to explain what is residual network in the second part of this lecture.

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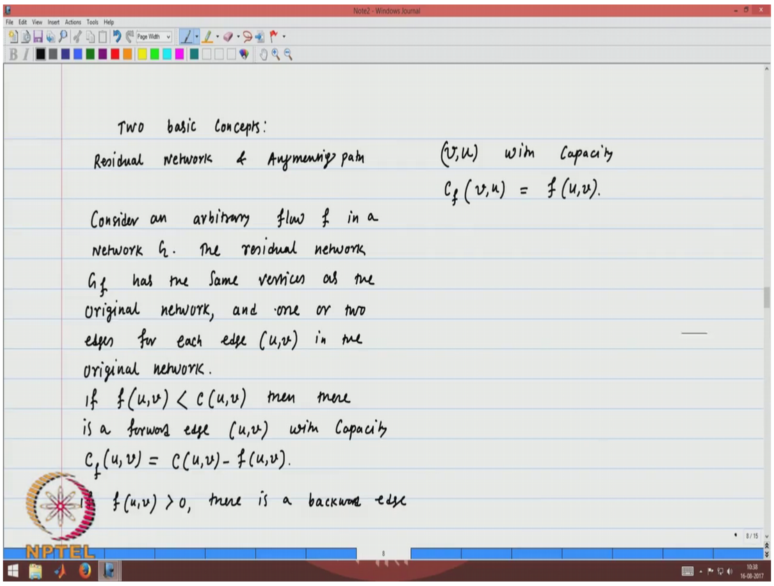

So, I will start with the definition of residual network. So, 2 basic concept one is the residual network, and the other one is augmenting path. So, I will just mention the definition at this moment considered an arbitrary flow f in a network G. So, arbitrary flow means it just satisfy all the condition is not necessary the maximum flow. Even you can assign 0 flow for every edge, that is also fine then.

The residual network residual network G f as I said that residual network is with respect to a flow has the same vertices as the original network. And one or 2 edges, I will explain all this things using example, but let me just state the definition at the beginning. So, it has a same set of vertices as the original network and one or 2 edges for each edge u v in the original network. So, how we construct the edges? If the flow in the edge u v is strictly less than the capacity c u v, then there is a forward edge forward edge u v with capacity c u v let me put this f also to denote that this capacity is for the residual network for the edge u v, this is equal to c u v minus f u v, this is 1.

And the other case is that if f u v is strictly greater than 0, there is a backward edge backward edge v u with capacity c f v u is equal to f u v. So, this is what the definition of residual network I did not explained it yet. So, in the next part of this lecture the second part of this lecture. I will take example to illustrate what is this residual network. And after that I will talk about augmenting path what is a augmenting path in a residual network.

Thank you very much.