Graph Theory
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Lecture – 15
Part 2
Residual Network and Augmenting Path

Welcome to second part of lecture 15 on Graph Theory. Today we will learn 2 basic concepts residual network and augmenting path in a network with respect to augment flow. These 2 concepts are important to find maximum flow in a network.

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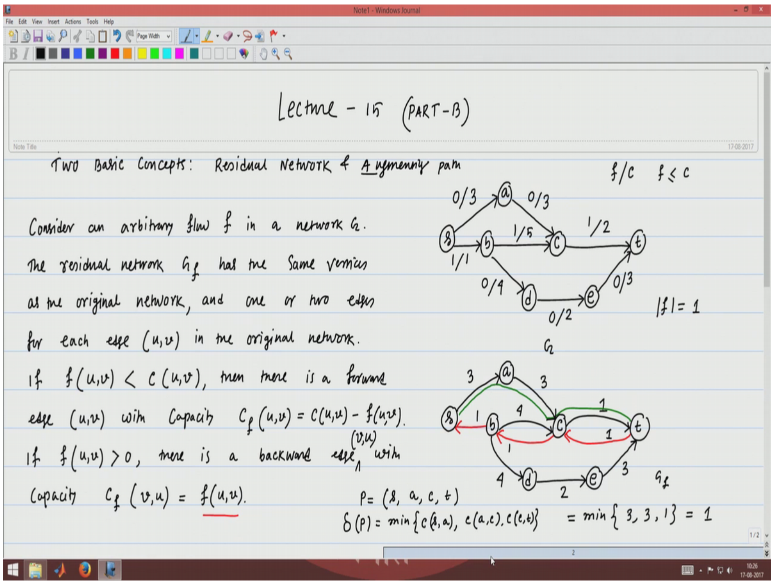

So, I will start with residual network first. So, we talk about this 2 basic concepts - one is residual network and augmenting path. So, this residual network with respect to a given flow, so consider an arbitrary flow f in a network G, then the residual network denoted by G f has the same vertices as the original network and 1 or 2 edges for each edge u, v in the original network.

So, I wrote this definition in the previous part also let me just repeat once more if the flow amount in the edge u v is strictly less than the capacity of the edge u v then there is a forward edge u v with capacity C f u v this the capacity of the edge forward edge u v in the residual network and that capacity is C u v minus f u v. And the other condition is that if the flow u v is flow around the edge u v is strictly greater than 0 there is a backward edge with capacity C f the backward is v u with capacity C f v u equal to f of u v.

Well, as I said that of course, this is a very important concept I will slowly explain the significance of this one also let me give example to illustrate this one. So, I will consider this network this is the source node and these are the other nodes this is the sink node t. So, there is an edge from a to a which has capacity 3, there is an edge from s to b which has capacity 1, there is an edge from b to c which has capacity 5 and edge from a to c with capacity 3, c to t with capacity 2, b to d with capacity 4 and d to e with capacity 2, e to t with capacity 3. So, this is the network G and I said that this residual network concept it is with respect to a flow.

Now let me just assign a flow here, so I will send 0 amount of flow along from s to a 0 amount of flow from a to c and then from b is to b 1 the flow amount is 1, and b to c 1, c to t 1 and here b to d 0, d to e 0, e to t 0. Now first we need to understand that this is valid flow because you can see that everywhere. So, this is the notation right f flow capacity. So, everywhere you can see that the flow amount is less than or equal to the capacity right. So, that the capacity rule is satisfied now look at the conservation rule.

Now, you can see the of flow that is going through a is 0 and the amount of flow going out of a is 0 similarly if you look at the node c the amount of flow that is going into c is 1 and the amount of flow that is going out of c is 1 look at b the amount of flow going into b is 1 the amount of flow that is going out of b is also 1 it is 1 plus 0. And similarly the flow conservation rule is true for d and e because the amount of flow going into d is 0 and the amount of flow going out of the t is also 0.

Now, just for if I remove this one by say if I put 2 then the capacity rule is true that 2 is less than equal to 5, but the conservation rule is not true because here you can see that the amount of flow that is going into b is 1 and amount of flow going out of b is 2 plus 0. So, the flow conservation rule is not valid. So, that is why you cannot put 2 here, I just wanted to clear such concepts. So, this is a valid flow. So, this residual network is with respect to arbitrary flow; that means, valid flow of course. So, now, I will draw the residual of the network G with respect to this flow.

So, it says that the residual network G f will have the same set of vertices. So, the vertices are again s a b c t d and e. Now it says that 1 or 2 edges for each edge in the original network. Now in the original network there is an edge from s to a. So, corresponds to this edge there could be 1 edge in the residual network or there could be 2 edges now see in this case there will be only 1 edge that is the forward edge since here if you can see that the flow is strictly less than the capacity. So, there is a forward edge u v with capacity 3 minus 0. So, there will be a forward edge from s to a with capacity 3 that is all.

And there will not be any backward edge because the flow is no flow is 0, flow is not greater than 0 if the flow is greater than 0 strictly greater than 0 then only there will be a backward edge. Similarly from a to c this is very important I will explain why this is required why this residual network is required at some point of time. So, there will be a forward edge because the capacity is strictly less than sorry the flow is strictly less than the capacity. So, there will be a forward edge with capacity c minus f.

Now, look at this edge s b. So, here you see the first condition this condition is not true the flow is not strictly less than the capacity it is less than or equal to. So, there will not be any forward edge here and here you can see that the flow is strictly greater than 0. So, there will be only 1, so there will be a backward edge. So, in the residual network let me draw the backward edge using a different color. So, there will be backward edge from s to b with capacity, with capacity equal to the flow amount that is 1, so the capacity is 1.

Now look at this edge b c. So, here you can see that the capacity is strict sorry the flow is strictly less than the capacity. So, there will be a forward edge with capacity 5 minus 1 that is 4 and also there will be a backward edge because the flow amount here is strictly greater than 0, so there will be a backward edge. So, I am drawing all the backward edge with different color. So, there will be backward edge of capacity 1. What about c t? So, there will be forward edge of capacity 1 right because the flow is strictly less than the capacity.

So, there will be a forward edge of capacity 2 minus 1 and there will be a backward edge of capacity 1 because the flow amount is 1. Now for b d it is clear there will be only 1 edge that is the forward edge with capacity 4, for d e there will be only 1 forward edge right and that is of capacity 2 and for e t there will be only 1 edge that is forward edge with capacity 3 and this is called the residual network of this network G with respect to the flow f. So, I hope that you understood what is residual network.

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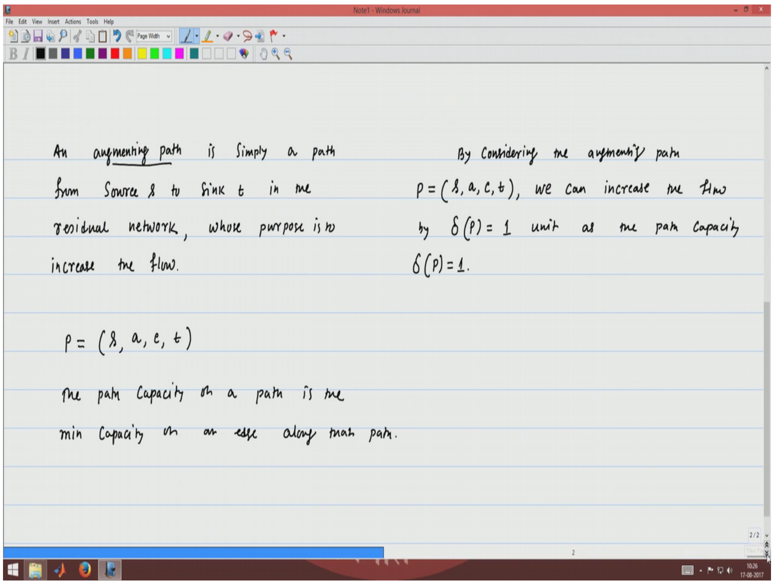

Next, we talk about augmenting path an augmenting path is simply a path from source s to sink t in the residual network and this is a purpose whose purpose is to increase the flow. So, I will explain how an augmenting path increases the flow amount first let me find out augmenting path. So, here we can see that there is an augmenting path; that means, we have to find the path from s to t. So, that path could be there are many maybe, but I can see that one path is s to a and then a to c and then c to t. So, the augmenting path here is P. Let denote this augmenting path by P. So, P is s a c t.

And now we talk about the capacity of this path. So, the capacity the path capacity of a path is the minimum capacity of an edge along that path. So, what you want to mean by this one is that, so here we took the path we took the augmenting path let me write here we took the augmenting path P to be s a c t. So, the path capacity which is denoted by delta P is that you take the minimum of the capacity of s a, the capacity of a c. So, these are the edges on the path P and the capacity of c t.

So, minimum of these 3 is equal to minimum of. So, the capacity of s a is 3 the capacity of a c is again 3 and the capacity of c t is equal to 1. So, this is the path capacity is 1. So, this is what the meaning of path capacity. So, what we do using this one is that by considering the augmenting path P in that example of course, which is s a c t we can increase the flow we can increase the flow by delta P amount that is the capacity of the flow which is equal to 1 here unit as the path capacity is.

So, once you have an augmenting path which is a path from the source to sink in the residual network you compute the capacity of that path. If the capacity of the path is 1 you can increase the flow amount by 1 unit. So, if you recall in the first flow that we took there the flow value was 1 I did not mention that I will go back to that. So, now, using this new this augmenting path we can augment 1 unit of flow in that network so that means, we can increase the flow by 1. So, the final flow in the network after augmenting this one unit of flow it will become the flow value will 2. So, we will understand this part now.

So, let me just mention here. So, this was our residual network here you can see the flow what is the flow value here what is the flow value that we, so we started with the flow and the flow value the value of the flow is 1. Now using this augmenting path we can make the flow of this we can increase the flow by 1 unit; that means, the flow will become 2. So, see we considered this path so that means, we sent 1 unit of flow along this path. So, the flow here will become we will replace this 0 by 1 we replace this 0 by 1 and we replace this 1 by 2.

So, we can we are adding 1 unit of flow or we are augmenting 1 unit of flow with respect to this augmenting path. So, let me just draw the new flow now.

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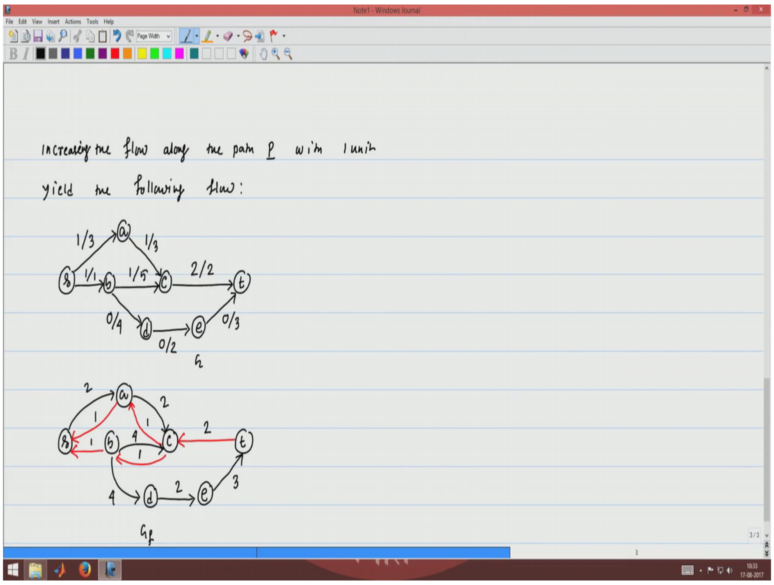

So, I can move towards the next page. So, increase the flow along the path P the P means the one we considered in the example increasing the flow along the path P with 1 unit yield the following flow. So, what flow we get now is that this one. So, this is the source and the other vertices are a, b, c, sink vertex t, d and e. So, now, here the flow value along s a it was 0 before now it will become 1. So, 1 and capacity is 3. So, along a c the flow was 0 before now it is 1 and the capacity is 3 along c t before it was 1 now it will become 2 and the capacity is 2. So, s b it was 1 before now it is 1 and b c it was 1 before now also it is 1 and b d it is 0 4, d e it is 0 2 and e t it is 0 3.

So, we have increased the flow amount from 1 to 2 and the question is can we increase it further. So, we will talk about Ford Fulkerson algorithm it says that when you have a flow in a network you compute the residual network and if you can find an augmenting path in the residual network we can increase the flow in that case so; that means, so we have 1 new flow now again we will compute the residual network with respect to this new flow and see whether there exit a augmenting path in the new residual network. So, this is how the algorithm I will talk about the algorithm of course, but this is sort of we are trying to explain the algorithm or sort of before studying the algorithm I am illustrating the algorithm.

So, let me just compute the residual network for this new flow with respect to this new flow and then we will stop. So, here the residual network with respect to this new flow is this one is the same set of vertices. So, the residual network this is the network G and the flow is given and this is my residual network. So, here as you remember there will be a forward edge with capacity 2 and there will be a backward edge with capacity 1. So, a c there will be forward edge of capacity 2 and there will be a backward edge of capacity 1 here there will be no forward edge because the flow is equal to the capacity.

So, there will be a backward edge of capacity 2 and this is 1. So, in this case for s b there will be backward edge only of capacity 1 and for b c there will be a forward edge of capacity 4 and a backward edge of capacity 1, of capacity 1 and for b d there will be a forward edge of capacity 4 no backward edge for d e there will be a forward edge of capacity 2 and for e t there will be a forward edge with capacity 3.

So, this is the residual network with respect to the new flow and in the next lecture we will learn how to find maximum flow in a given network using the idea of residual network and augmenting path.

Thank you very much.