Introduction to Probability and Statistics
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Lecture – 13
Rules of Probability

In this lecture we continue our discussion on the basic concepts of probability. In the previous lecture we looked at sample space and the events. And, then we saw the first principle that the sum of the probabilities of all the events adds up to 1.

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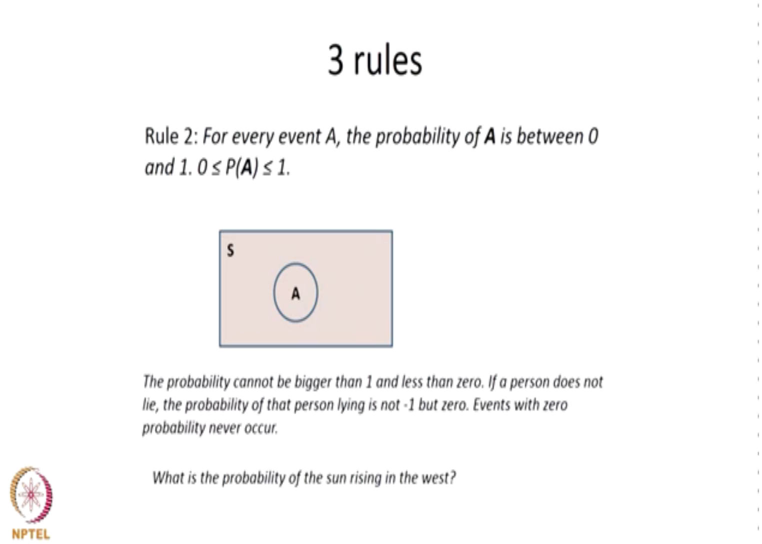

Now, we look at the second rule which says for every even A the probability of A is between 0 and 1. So, P of A is less than or equal to 1. The probability cannot be bigger than 1, and it cannot be less than 0. Sometimes we say this, if a person does not lie, then we say the probability of that person lying is minus 1. In a conversation we say that to stress the point that this person will not lie at all.

Or if someone says what is the probability of the sun rising in the east, then sometimes we say 1.1, saying that it is simply that the sun will not risen any other direction. So, in all these instances we have to understand that the actual answers are 0 and 1 respectively, and not any number less than 0 or greater than 1. So, what is the probability of sun rising in the west? Is 0 and not anything less than that.

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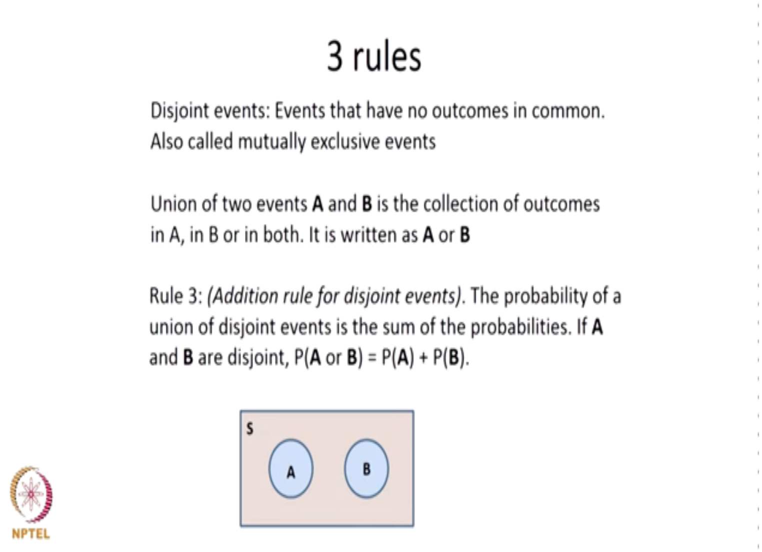

Let us continue; now we have something called disjoint events. So, events that have no outcomes in common are called disjoint events, sometimes also called mutually exclusive events. Now we define union of 2 events A and B is the collection of outcomes in A and B or in both is written as A or B. Third rule which is addition rule for disjoint events. Probability of a union of disjoint events is the sum of the probabilities, if A and B are disjoint P of A or B is P of A plus P of B. So, if A and B are mutually exclusive P of A or B is P of A plus P of B.

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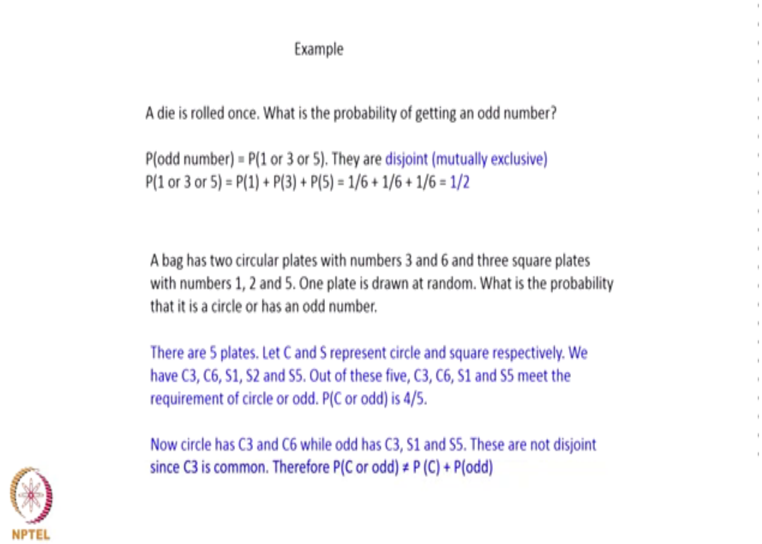

Now, let us look at examples. A die is rolled once, what is the probability of getting an odd number? So, probability of getting an odd number is probability of getting either 1 or 3 or 5. They are disjoint, mutually exclusive therefore, probability of getting a 1 or getting a 3 or getting a 5 is probability of getting a 1 plus probability of getting a 3 plus probability of getting a 5. In each of these cases the probabilities 1 by 6 because, it is a die and therefore, the answer is 1 by 6 plus 1 by 6 plus 1 by 6 which is half. Of course, there is another way of looking at it, you either get an odd number or 0 plus even number.

So, you have 3 plus 3 therefore, the answer is half. Now look at another example. A bag has 2 circular plates with numbers 3 and 6 and 3 square plates with numbers 1, 2 and 5. One plate is drawn at random. What is the probability that it is a circle or it has an odd number? Now there are 5 plates. So, let us C and S represent the circle, and the square plate respectively. So, we have one plate; which is called C3; which means a circular plate with number 3 we have another plate which is C6 which is a circular plate with number 6 written and the 3 square plates are S1, S2 and S5, one out of these 5; so, C3, C6, S1. Out of these 5 C3, C6, S1 and S5 meet the requirement of a circle or an odd. So, the question is what is the probability there it is a circle or it has an odd number. So, out of these 5, 4 of them meet the requirement C3, C6, S1 and S5 meet the requirement.

Therefore, probability is C or odd which is 4 by 5. Now we realize carefully that the circle has C3 and C6 while odd number is C3, S1 and S5. These are not disjoint, because we have C3 common in both. Therefore, probability of C and odd is not equal to probability of C plus probability of odd. Once again in this particular problem or example, we found out that there are actually 5 plates. Out of which 4 out of 5 meet our requirement and therefore, the probability was 4 by 5. But if we looked at both of these separately, there are 2 circular plates, and there are 3 plates with odd number. But there is one which is common which is C3. Therefore, probability of circle or odd is not equal to probability of circle plus probability of odd.

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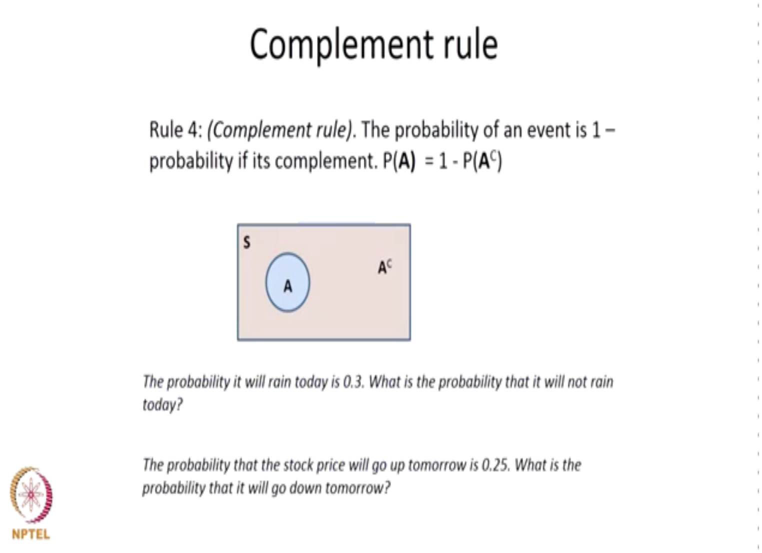

Now let us look at the third rule which is called the compliment rule. Probability of an event is 1 minus probability of it is complement. So, P of A is equal to 1 minus P of A compliment. Probability that it will rain today is 0.3, what is the probability that it will not rain today? 0.7; probability that the stock price will go up tomorrow is 0.25, what is the probability that it will go down tomorrow? Could say 0.75, but it could remain the same one could say it is slightly less than 0.75.

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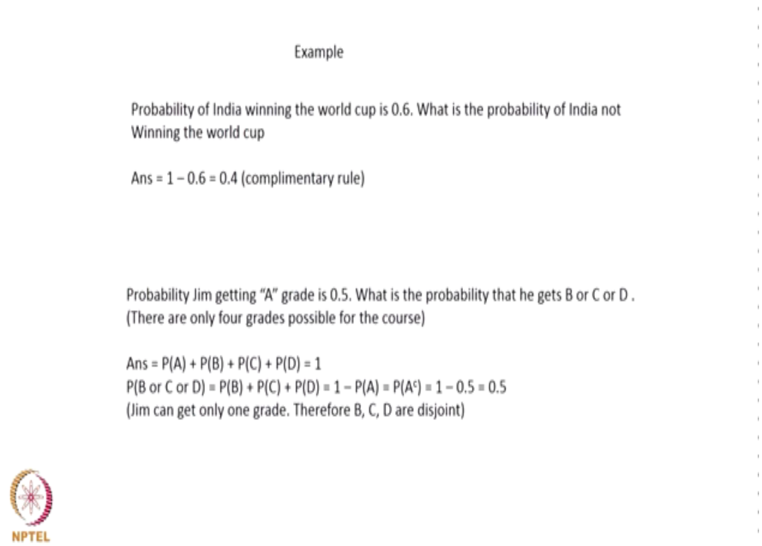

Example probability of India winning the world cup is 0.6, what is the probability of India not winning the world cup? Is 1 minus 0.6, which is 0.4. Probability of Jim getting an a grade is 0.5, what is the probability that he gets B or C or D? There are only 4 grades possible for the course. So, probability of getting a grade is 0.5, probability of getting another grade that is not a grade is also 0.5. Since the person can get only one grade B C and D are disjoint.

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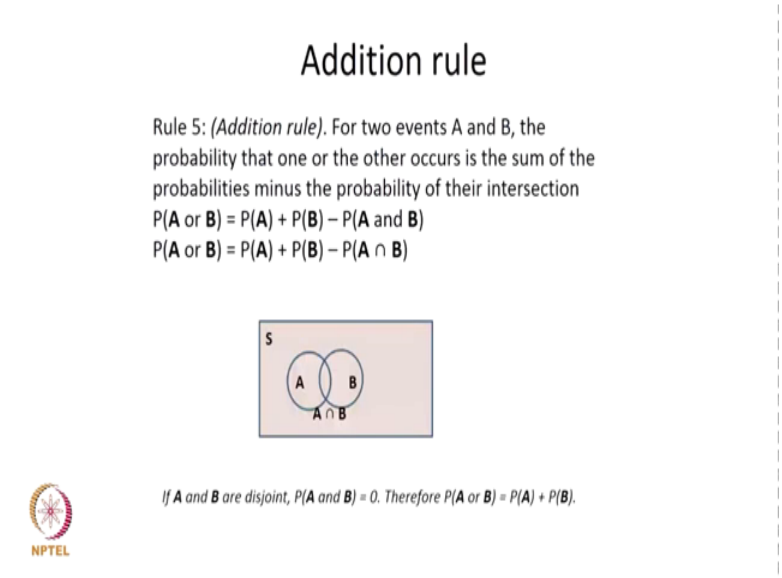

Rule 5 addition rule, for 2 events A and B the probability that one or the other occurs is the sum of the individual probabilities minus the probability of intersection. So, P of A or B is equal to P of A plus P of B minus P of A and B. So, P of A or B is equal to P of A plus P of B minus P of A intersection B.

The Venn diagram shows what we are discussing. If A and B are disjoint and P of A and B is 0 therefore, P of A or B is equal to P of A plus P of B. Now we realize that the rule that we saw for disjoint comes under the general addition rule.

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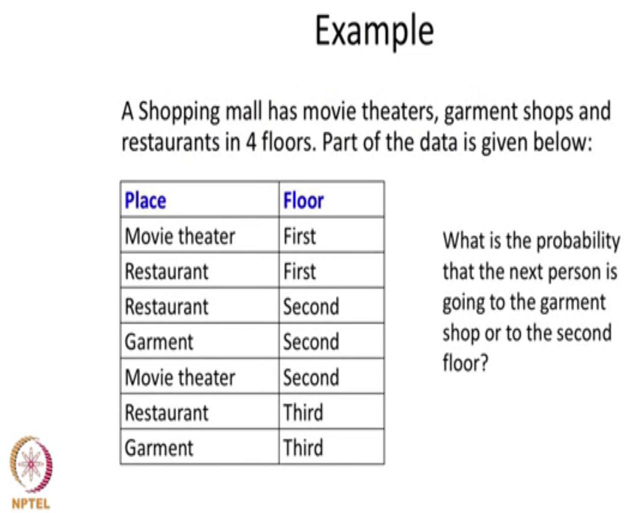

Place	Floor
Movie theater	First
Restaurant	First
Restaurant	Second
Garment	Second
Movie theater	Second
Restaurant	Third
Garment	Third

Example. A supermarket has movie theaters, garment shops and restaurants in 4 floors. Part of the data is given below. So, we have a movie theatre in the first floor, we have a restaurant in the first floor, we have a restaurant in the second floor, we have a garment shop in the second floor, we have another movie theater in the second floor, we have another restaurant in the third floor and a garment shop in third. What is the probability that the next person is going to the garment shop or to the second floor?

So, if the person is going to the garment shop, the person can go to a garment shop in the second floor as well as third floor. And if the person is going to the second floor, the person could be going to either the restaurant or a garment shop or a movie theater.

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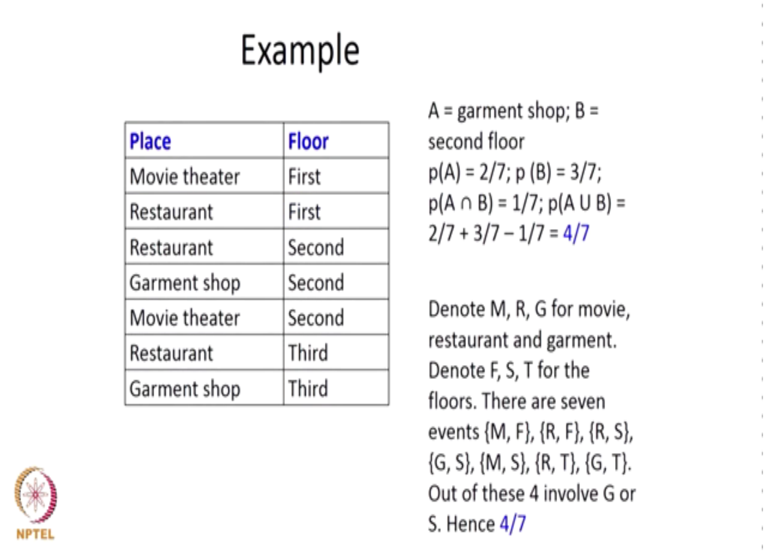

So, let a represent the garment shop and B represents the second floor. So now, probability P A is 2 by 7 because there are 7 items and 2 of them are garment shops. Let us assume that equally likely that people go to each one of these. So, P of A is 2 by 7, P of B is 3 y 7, because B is second floor, there are 3 things in the second floor. So, 3 by 7 P of A, intersection B is 1 by 7, because there is also a garment shop in the second floor so, it is 1 by 7. Therefore, P of A union B is equal to P of A plus P of B less P of A intersection B. So, 2 by 7 plus 3 by 7 minus 1 by 7 is equal to 4 by 7.

We can also do this slightly differently. Let M R and G represent movie restaurant and garment shop, and let F S and T represent the floors. So, there are 7 events M F, R F, R S, G S, M S, R T, G T. Out of these for involve G or S which is garment shop or second floor and therefore, the probability is 4 by 7. So, the same example or problem they can do it in multiple ways. Sometimes we follow this event sample space way, sometimes we use these formulae and slowly as we move along we will have to understand both the ways of doing it. And more importantly understand how both of them are related to each other.

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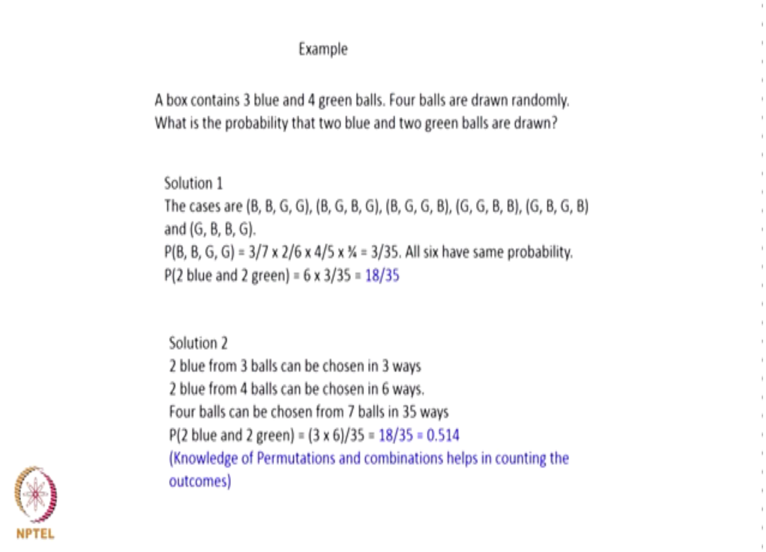

Example: A box contains 3 blue balls and 4 green balls. 4 balls are drawn randomly, what is the probability that 2 blue and 2 green balls are drawn? So, since 4 balls are drawn randomly. The first way to do it is it can be blue, blue, green, green, blue, green, blue, green, blue, green, green, blue, green, green, blue, blue and so one. So, we have looked at all of these. So, probability of doing a blue and blue and green and a green 3 assuming that these are not replaced.

So, 3 by 7 into 2 by 6 into 4 by 5 into 3 by 4, which is 3 by 35; there are 6 ways of doing it, all 6 have the same probability of 3 by 35. So, the answer is 6 times 3 by 35, which is 18 by 35 which is 0.514. Now we do this in a in another way. So, 2 blue balls from 3 balls can be chosen in 3 C 2 which is 3 ways. 2 blue balls from 4 balls can be chosen in 4 C 2 which is 6 ways. 4 balls out of 7 can be chosen in 7 C 4 which is equal to 7 C3, which is equal to 7 into 6 into 5 by 1 into 2 into 3; which is 35 therefore, the probability is 3 into 6 by 35 which is 18 by 35 which is 0.514.

So, knowledge of permutations and combinations helps in counting the outcomes. So, this is another way of solving these kind of problems. Sometimes we do it in the first way; where we compute individual probabilities and multiply. The same thing is done slightly differently with the number of ways of doing something and then we compute. So, this is another thing that we have to understand as we progress in our study of probability; that there are multiple ways of looking at time solving the problem. The concepts are all the same, except that we have learn to understand how each one works and relate each one of them.

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Example

In a card game a pack of 52 cards is dealt to 4 players. What is the probability that each player gets 1 ace?

Take person 1. The ace should come in one out of the 13 picks. Take the case where the ace is in pick 1 and the remaining 12 do not have an ace.
P(ace in position 1 and no ace in 12) is 4/52 x 48/51 x 47/50 x .... x 37/40 = 0.03376
The ace can come in any one of the 13 positions. Total probability = 13 x 0.03376 = 0.4388
For player 2 it is 3/39 x 36/38 x 35/37 x 34/36 x .... x 25/27 = 0.03556
The ace can come in 13 positions. P = 0.4623
For player 3 it is 2/26 x 24/25 x 23/24 x .... x 13/14 = 0.04. The ace can come in 13 positions P = 0.52
For player 4 it is 1/13 and since ace can come in 13 positions P = 1
Total probability = 0.4388 x 0.4623 x 0.52 = 0.1054

Now, let us look at another example. So, in a card game a pack of 52 cards is dealt to 4 players. So, each player gets 13 cards, and what is the probability that every player gets 1 ace. So, there are 4 aces in a pack of 52 cards. So, what is the probability that each player gets 1 ace? So, this is slightly more involved example and where the some of the things that we have learnt till now will be put to use.

So, let us try to see how we solve this. So, let us take the first person. The ace should come in one out of the 13 picks, because we assumed that somebody is dealing the cards. So, this person one is going to get one card per pick 13 times. So, look at the case where the ace is in pick one and the remaining 12 do not have an ace. So, ace in position one, and no ace in the remaining 12 positions is 4 by 52, because the first position there are 52 cards are there the person gets one ace so, 4 by 52. Now if you leave out the ace 4 aces there are 48 non aces.

So, 48 by 51 into 47 by 50 and so on; this 48 by 51 comes because this second pick out of the 51 cards that are remaining. So, he could get any out of these 48, because these 48 do not have an ace. Remember, again this is equivalent of the ball not being replaced so, the card is does not go back to the deck. And it is completely given to this 4 peep. So, the

12 remaining picks we will start with 48 by 51, 47 by 50 and go on till 30 7 by 40. And therefore, this probability is 0.03376.

So, for the first person, getting an ace in pick one, and not getting an ace in the remaining 12 picks is 0.03376. Now if we take the same person, we are trying to go back to the problem to see that this person gets only one ace. So, this person can get that ace in his or her first pick second pick up to 13th pick. So, it can come in any one of the 13 positions, and we can quickly realize that the probability is actually the same. Therefore, the total probability is 13 into 0.03376 which is 0.4388. So, this is the probability that the first person gets an ace and one ace out of the 13 picks that this person has.

Now, for prayer 2 we have 3 by 39 into 36 by 38 and so on. So, how we get these numbers? So, we assume what normally happens in a when 4 people play cards is we start putting one for each person and do it 13 times. Right now we are not assuming that, we are going to assume in some way that the packet is well shuffled. And the first person gets the first 13 cards; the second person gets the next 13 cards and so on. And therefore, for the second person, there are only 39 cards that are remaining. And since the first person has got only one ace, 3 aces are remaining in these 39 cards and therefore, this person gets 3 by 39 is the probability of getting an ace in the first pick.

Now, probability of not getting an ace in the remaining picks are 36 by 38 into 35 by 30 7 and so on, because with every pick you realize that the denominator is reducing by 1. Because one card is given the 36 by 38 comes because the second player is already got an ace in that 3 by 39. So, there are only 2 more aces remaining, and 36 non aces remaining out of 38 remaining cards. And therefore, 36 by 38 and then it moves on till 25 by 27 which is 0.03556. Now again this ace can come in any one of the 13 positions. Therefore, it is 13 times 0.03556; which is 0.4623. For player 3 now both players 1 and 2 have got their 13 cards.

So, only 26 cards remain, and players 1 and 2 who have got one ace so, aces remain. So, 2 by 26 and then out of these 26, there are 2 aces that remain 24 non aces remain therefore; we get 24 by 25 into 23 by 24 and so on to get 0.04. And this ace can come in 13 positions therefore, 0.04 into 13 which is 0.52. Now for player 4 the answer is actually 1, because the prayer 4 does not have any choice. The player 4 has to get back take back the remaining 13 cards that are available. And therefore, probability is 1 by 13, and we will realize that it happens 13 times. So, 13 into 1 by 13 which is 1 and therefore, the probability that each one gets one ace is 0.4388 into 0.4623 into 0.52 into 1, which is 0.1054.

So, this is a more involved example. And it is very common to study and to look at these kind of examples either from tossing a coin or rolling a die or picking something from a pack of cards, whenever we study probability. But that we can understand is the if the card problems become a little more complicated than the die problem or the ball picking problem. So, we will look at some of these interesting problems as we move along. What is also required is to understand what happens and in what sequence or what order it happens. And once we understand that the whole thing comes nicely for us to get this. So, we will quickly realize that 0.1054 is the probability that a person all 4 gets an ace. Please note that we have not qualified it by saying that the first person gets the ace of spared and so on, it becomes even more involved do that, but each person getting one ace is happens 10 percent of the times 0.1054.

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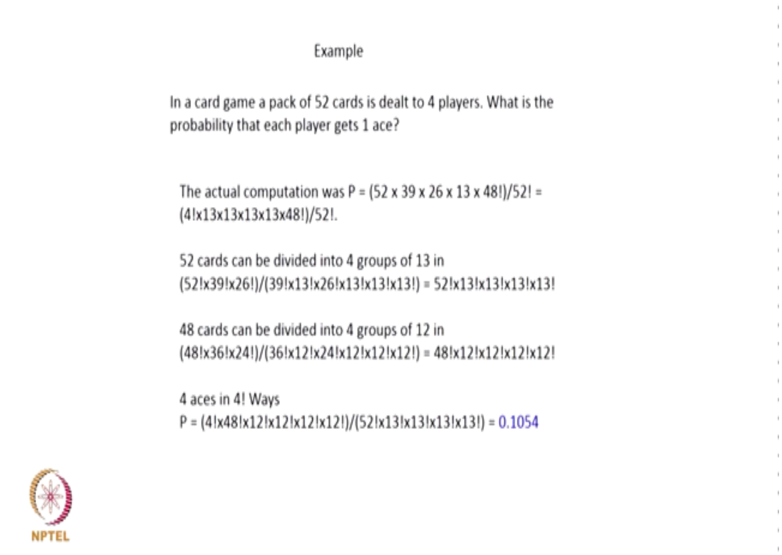

Now, let us look at the same problem done in a completely different manner using permutations and combinations. So, in a card game a pack of 52 cards is dealt to 4 players, what is the probability that each player gets one ace? So, the actual computation was the is equal to 52 into 39 into 26 into 13 into 48 factorial by 52 factorial which is given here. Now, 52 cards can be divided into 4 groups of 13 in this many ways. 52

factorial into 39 factorial into 26 factorial divided by 39 factorial into 13 factorial into 26 factorial into 13 factorial and so on.

Now, 48 cards can be divided into 4 groups of 12 in these ways, 4 aces in 4 factorial ways. And therefore, the probability will be 4 factorial into 48 factorial into 12 factorial into 12 factorial into 12 factorial into 12 factorial, divided by 52 factorial into 13 factorial into 13 factorial into 13 factorial which is 0.1054. So, this looks a little more complicated, but one has to kind of understand how this actually happens. So, it is a lot easier to look at this problem in the previous method, where we took the case where we take one person, and then say this person gets an ace in the first pick, second pick third pick 13 picks and then find the probability, then we move to the second person and so on. This involves a lot more of factorials which means we do it the permutation combination way where the first one was done using the probability way.

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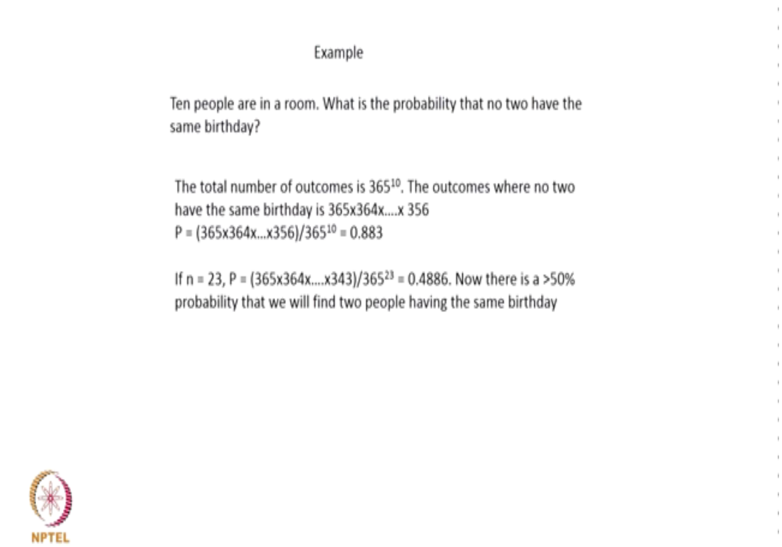

Now let us look at another example. 10 people are sitting in a room. Now what is the probability that no 2 have the same birthday? Now assume that the year is made up of 365 days and so on. So, total number of outcomes is 365 into the power 10. The outcome where no 2 have the same birthday is 365 into 364. So, on into 356 and therefore, the probability is 365 into 364 etcetera up to 356. 10 people divided by 365 into the power 10 is 0.883. So, there is an 80 percent probability that 2 people no 2 people will have the same birthday.

But as we increase the number of people when we do n equal to 23, we realize that this probability becomes 0.4886, it is quite understandable that as more people are in the room the probability that 2 people have a common same birthday is higher. So, as n increases to 23, we realize the probability of no 2 people having the same birthday reduced to 0.4886. And therefore, we say when there are about 23 people there is a 50 percent chance that we will have at least 2 people having the same birthday; one instance of 2 people having the same birthday.

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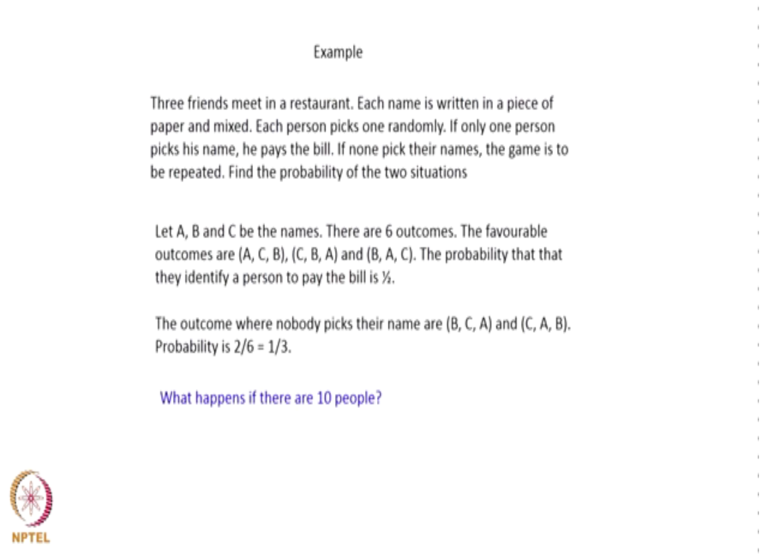

Now, let us look at another example to understand this. 3 friends meet in a restaurant. Each the name of each person is written in a piece of paper and it is mixed. So, each person picks one paper randomly. If one if somebody manages to pick his name, assume that all 3 are men and picks his name, he pays the bill for all of them if nobody picked their names the game is to be repeated now find the probability for both the situations.

So, let a B and C be the names there are 6 outcomes. The favourable outcomes are A, C, B, C, B, A and B, A, C. So, these are the favourable outcomes. Therefore, the probability that they identify a person to pay the bill is actually half. So, let us go back to the problem. Each person picks one randomly if a person picks his name he pays the bill. So, when we do A C B, we assume that a picks his name whereas, B and C do not pick the name.

So, when we do C B A, B picks his name C and a do not pick the name and so one. So, we are not looking at the case where all of them pick their names. You know, we do not look at the case of A B C. So, we have a situation where if only one person picks the name and the other 2 do not pick the name, then the person who picks the name pays the bill. So, we look at that situation and say that out of the 6 possible outcomes 3 outcomes meet the requirement. So, probability that they actually identify a person to pay the bill is half.

If nobody pick their names, then the game is to be repeated. So, when nobody picks their game what happens? It is either B C A or C B A, because a picks the name B, B picks the name C, C picks the name A and the other way. So, 2 out of these 6 outcomes nobody picks the their name so, it is 2 by 6 1 by 3. The only outcome that is left out is all of them pick their names. So, one may assume that if that happens, they share the bill and paid. So, this is another example where we have not looked at it using the permutation way we could do that as well. For a smaller example such as this, it is nice to do this in the probability way of solving this problem.

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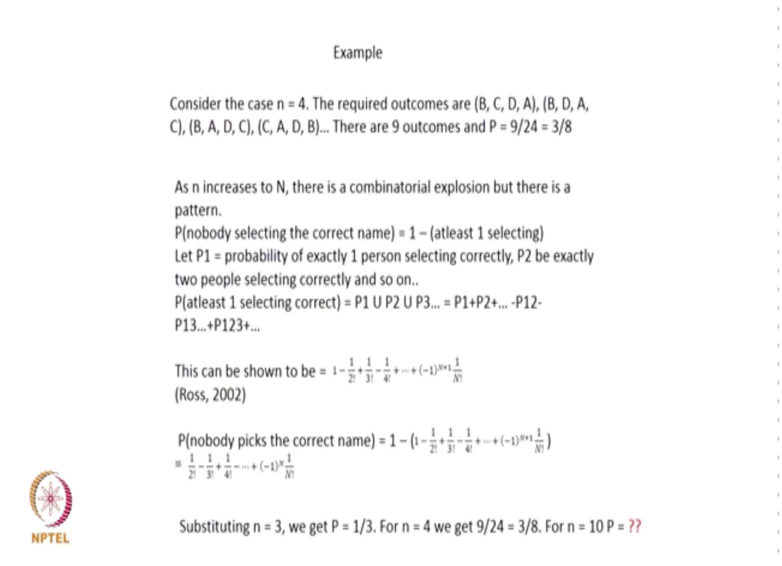

Now, we look at another example. So, look at case n equal to 4, the outcomes are B C D A, which means a picks name B, A, B, C, D are the 4 people. So, a picks B, B picks C, C picks D and D picks A. So, nobody picks the correct name. And then we look at B, D, A, C again nobody picks the correct name B A D C and C, A, D, B nobody picks the correct name. Therefore, the game is to be repeated or the game is to be repeated to find out who will actually pay the bill. So now, in this case there are 9 our comes out of 24 possible outcomes. So, there is the probability of 3 by 8 that the game has to be repeated and we are not able to find the person to actually pay the bill in the first iteration of this game.

Now, what happens is, as n increases to capital N which means as there are more and more people there is a combinatorial explosion because we have to find the number of outcomes increases, but there is a pattern. So, probability of nobody getting selecting the correct name is 1 minus probability of at least one selecting the correct name. Let P 1 be the probability of exactly one person selecting the correct name. P 2 be the probability of exactly 2 people selecting the correct names and so on. So, probability of at least one person doing it correct is P 1 union P 2 union P 3 etcetera; which is the P 1 plus P 2 etcetera and less intersection P 1 2 P 1 3 and so on.

So, this gets into a nice expression and therefore, nobody picks the correct name can also be written through an expression. And then we can substitute for n equal to 3, we get 1 by 3 and for n equal to 4 we get 9 by 24 etcetera. So, this is how we solve these kind of problems, and look at probability and try to understand probability in further.

So, remaining concepts of probability we will look at in the next lecture.