Algebra - I
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Last time, we saw the axioms for groups. Now, today I am going to tell you about how to construct new groups from old. One of the simplest ways of constructing a new group from an old group is by taking a subgroup. A *subgroup* is a subset of G such that $id \in H$ and it is closed under the binary operation. A 3rd axiom is, if $if\ x \in H$ then $x^{-1} \in H.$

Observe that you can leave out the first axiom here, because it follows from 2 and 3. Because if $x, x^{-1} \in H,$ and H is closed under the binary operation, then the identity belongs to H.

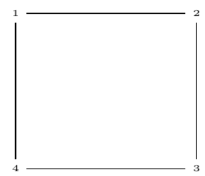
Example 1.
Let $s_1 = (12),$ in the cycle notation. This means $s_1(1)=2, s_1(2)=1, s_1(3)=3.$ Observe that $s_1^2 = id.$ I claim that $\{s_1, id\}$ is a subgroup of $S_{3.}$
Remember we only need to check two things, that if we have two elements x, y in H then their product is in H. This is clearly true.
If an element is in H then its inverse in H. We have only two elements in H are id and $s_1.$ Their inverses are respectively id and $s_1,$ which are both in H.

Example 2.
Let $C = (123).$ So $C^2 = (132), C^3 = id.$ It is easy to verify that $H = \{id, C, C^2\}$ is a subgroup of $S_{3.}$

Let me point out that, a subgroup is a group in its own right by restricting the binary operation on $G \times G$ to $H \times H:$ that means, just apply it only to elements of H.

Now, let me show you a nice way of constructing many interesting groups. So, let us start with an example.

Example 3.
So, what I am going to do is, I am going to construct an interesting subgroup of $S_4.$ I am going to draw the square.



If you know some graph theory, this figure is called the cyclic graph on 4 vertices. So, what is a graph? A *graph* is a pair(V,E)- a set of vertices V and a set of edges E. An edge is a set of two vertices, indicating a line drawn between these two vertices. In this case the set $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}.$ I am going to look at the subgroup of S_4 , which I will call $Aut(C_4),$ comprising automorphisms of the above graph $C_4.$ This is defined as the set of permutations $\{w \in S_4 : \{i, j\} \in E\ iff\ \{w(i), w(j)\} \in E\}.$
Firstly, let us consider if 1 goes to 1 under an automorphism. Then 2 must go to 2 or 4 since those are the only vertices adjacent to 1, and {1,2} being an edge, {1,w(2)} must also be an edge. If 2 goes to 2, 3 must go to 3 and 4 must go to 4. This gives us the identity element **1234**.
Now if 1 goes to 1 and 2 goes to 4, since {2,3} is an edge, the image {w(2),w(3)} must also be an edge, so 3 must go to 3 (or 1 but that is already taken by 1). This gives us the element **1432**. These are the only automorphisms that fix the element 1.
Next suppose 1 goes to 2. What is the image of 2? It must be adjacent to the image of 1- so 2 must either go to 1 or to 3. If 1 goes to 2, 2 goes to 1- this fixes the position of 3 and 4. We get the element **2143**.
If 1 goes to 2 and 2 goes to 3, it forces 3 to go to 4 and 4 to go to 1, giving us the element **2341**.

Similarly, given any vertex there are 2 automorphisms that take 1 to that vertex. Thus the total number of automorphism of the square is $2\times 4=8$. Thus $Aut(C_4)$ is a subgroup with 8 elements.

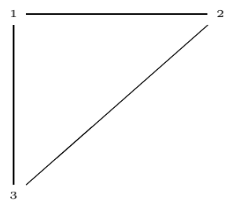
Let us look at the elements 2143 and 2341 in the above example. 2143 is a reflection of the square about an axis bisecting the square vertically. What about 2341? It is a rotation of the square 90 degrees clockwise around its centre. Now consider 3412. You can see that this is a rotation by 180 degrees. The element 4123 is a rotation, counter clockwise by 90 degrees; 4321-its a reflection about a horizontal axis passing through the centre.

So the subgroup $Aut(C_4)$ consists of all rotations of the square around its centre by multiples of 90 degrees and reflections about either vertical or horizontal or diagonal axis passing through the centre.

This group is called the *dihedral group* D_4 - some people call it D_8 - I will call it D_4.

The construction of graphs with [n] as the set of vertices allows you to construct a large family of subgroups of permutation groups as the automorphism groups of those graphs.

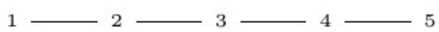
Example 4.
Let us take the complete graph on 3 vertices, denoted K_3 : there is an edge between every two vertices.



Notice that since every two vertices have an edge between them, every permutation in S_3 preserves the set of edges of the graph. Thus $Aut(K_3)=S_3$.

More generally $Aut(K_n)=S_n$. So S_n itself can be seen as the automorphism group of a graph. The group of automorphisms of discrete graph with n vertices- where the set of edges is empty- is also S_n.

Example 5.
Consider the graph L:



The number of edges incident on a vertex (edges that contain it) is called the degree of the vertex. L has 2 vertices of degree 1- 1 and 5, and 3 vertices of degree 2- 2,3 and 4. If an automorphism of the graph took any of the vertices 2,3 or 4 to the vertex 1, the two vertices adjacent to them would have to be bijectively assigned to the single vertex adjacent to 1, which is not possible. Thus, $Aut(L)$ has 2 elements- the identity element which retains the original labelling and the element that acts on L by reversing the labelling of its vertices. In general one observes that an automorphism of graphs preserves the degrees of all vertices- that is, $degree(x)=degree(w(x))$ for all $x\in V(G), w\in Aut(G)$.

To recapitulate:

- A subgroup of a group G is a subset of G that is closed under the binary operation of G and that contains the inverses of each of its elements.
- A graph G is a set of vertices V(G) and edges E(G). An edge is a set of 2 vertices. The number of edges incident on a vertex is its degree.
- An automorphism of a graph is a permutation of its vertices such that there exists an edge between two vertices if and only if there exists an edge between their images under the automorphism.
- The set of automorphisms of a graph is a group. The automorphism groups of graphs on n vertices create a large collection of subgroups of S_n.
- The automorphism group of a square (also called the cyclic graph C_4) is the *dihedral group* D_4. Its elements act on the square by rotating it a multiple of 90 degrees around its centre and by reflections along the horizontal, vertical and diagonal axis passing through the centre.