Algebra - I

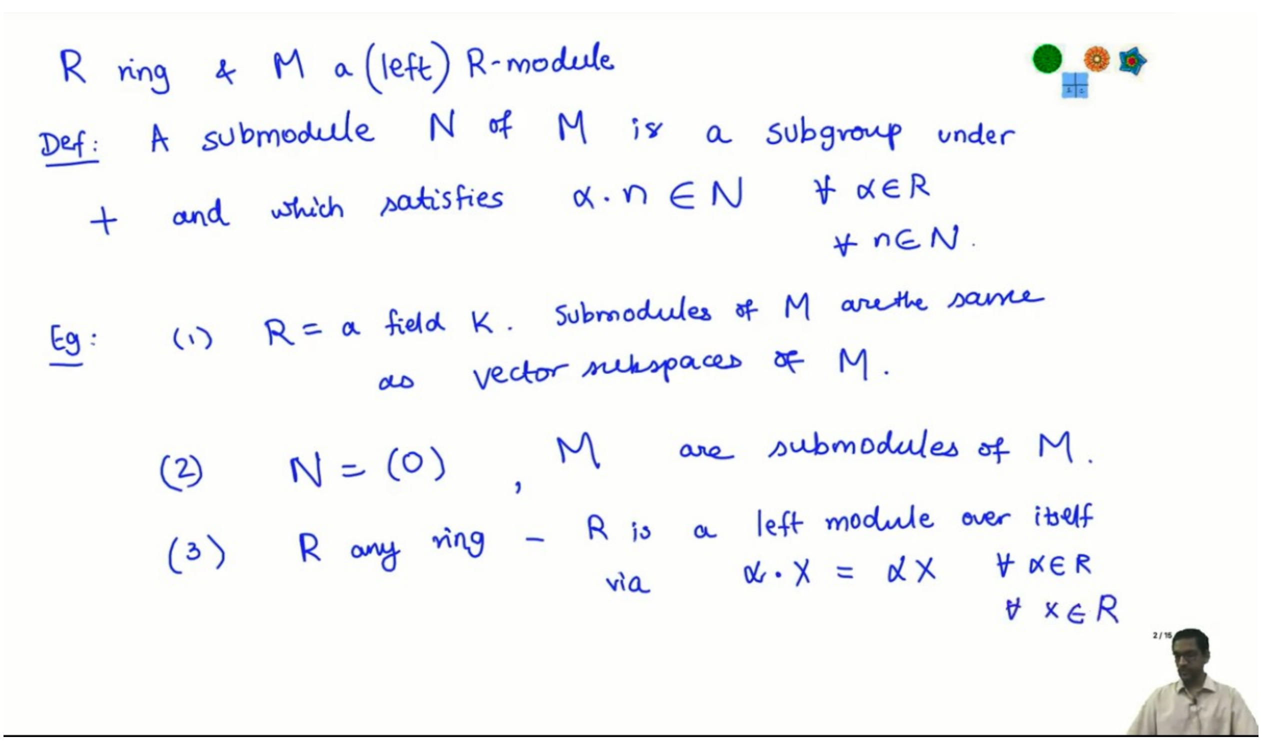
Prof. S. Viswanath & Prof. Amritanshu Prasad
Department of Mathematics
Indian Institute of Technology, Madras

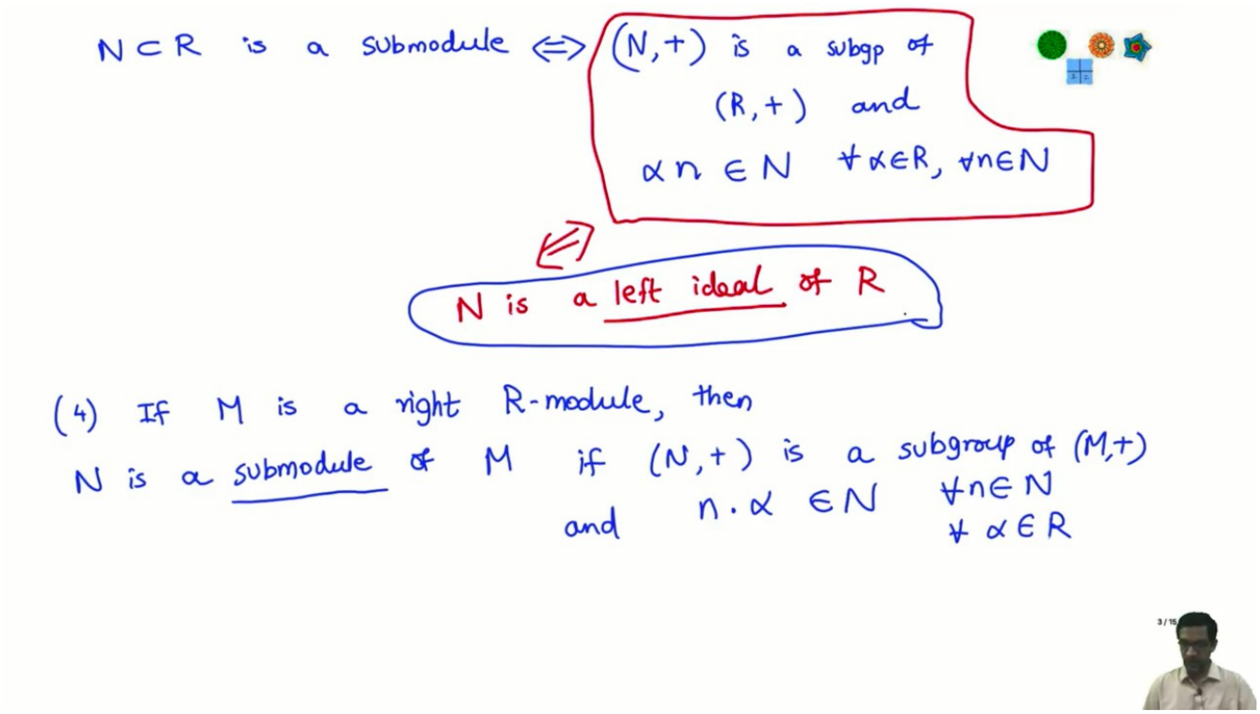
Lecture 55 [Submodules]

We will start talking about Submodules next. So, what is the Submodule? So, here is the definition. So, if R is a ring and M is a module and M left R module . So, again I as I said a few times already, we will usually just suppress this this word left and when we say just R module, we usually mean left R module .

So, suppose I have an R module, then definition a submodule of M a submodule N of M well it satisfies two conditions. Firstly, is a subgroup is a subgroup under under the addition operation subgroup under addition ok and which satisfies which satisfies the following property that it is closed under scalar multiplication, ok.

It satisfies the following property $\alpha \cdot n \in N$ for all $\alpha \in R$ and for all elements $n \in N$, ok. In other words, the set N is closed under addition and scalar multiplication. So, that is what a submodule is. Here it is a natural definition there the the obvious examples if R is a field K .





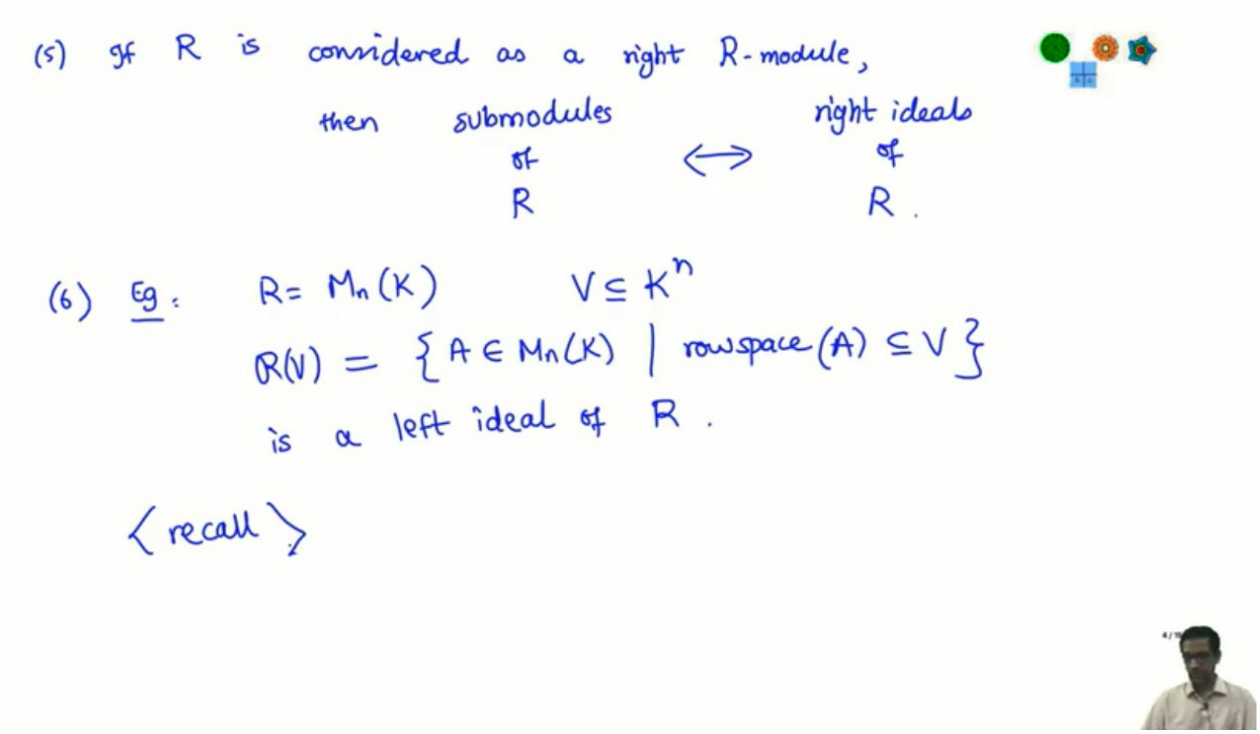
This case recall that R modules are just a same as vector spaces over the field K and of course, it comes as no surprise that submodules are just the same as subspaces. So, in this case recall submodules of M. Recalling the definition of subspaces are the same as subspaces vector subspaces of the vector space N, ok ok.

Now, um there are some obvious submodules always. So given any module N, I can construct two obvious submodules the 0 module and the the entire module M. So, these are sometimes called the the two trivial submodule, the the 0 on one end and M on the other on the other end ok and let us take another example recall that last time we spoke about a ring R. So, if R is any ring , then R is a left module over itself by the left multiplication operation via the following action $\alpha \cdot x$ is just αx for all α in the ring R. So, that is I think of that are scalars and x from the ring R again that is the module . So, this is the the left multiplication um action that makes R into a into a module over itself and in this module, what are the submodules.

So, observe that if I take a subset of R is now a submodule for this particular way of making R into a module. It is a submodule means that well by definition that N is an additive subgroup is a a subgroup of the additive group of R and further condition number 2 says that if I do αn now, this is the usual multiplication in the ring $\alpha n \in N$ for all $\alpha \in R$ and for all $n \in N$, ok.

Now, recall from the lectures on rings that this pair of conditions that we have just written out these two guys is exactly the definition of a left ideal ok. So, in other words submodules of the ring R under this left multiplication action is are the same as left ideals of R, ok and you know I leave this as an exercise for you. You can also look at the the right submodule.

So, let us just quickly define that notion as well um. If R or rather if M is a right module , then we can talk about submodules, then submodules of M are ah. So, now we need to define this notion of a sub modules. So, let me just say n is now a submodule of N. Then



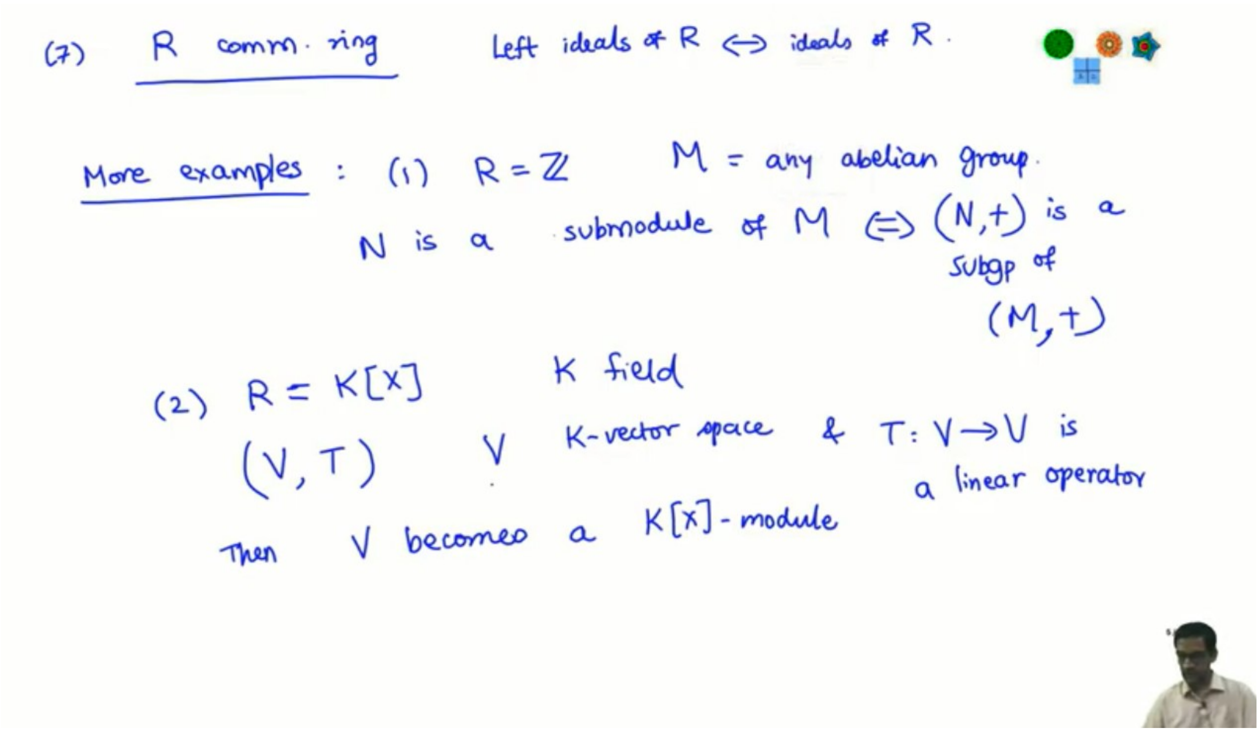
we say n is a submodule of M if its closed under addition if $(N, +)$ is an additive subgroup . Well it is a it is a subgroup of $(M, +)$ a is a subgroup and we have you know the closeness on the other side n acted on the right by $\alpha \in N$ for all $n \in N$ for all $\alpha \in R$ ok.

So, this is the the right action and um observe that just like you know we did the earlier thing. We said if I take R thought of as a left module over itself, then left ideals and left submodules are the same ah. Similarly, if you think of R as a as a right module over itself, then the the submodules turn out to be just right ideal.

So, if R is considered as a right module over itself considered as a right R module , then submodules of R turn out to be submodules of R are the same as right ideal of R ok by the same argument that we just gave, ok. So, these already give us plenty of examples. So, if I you know think of R as a module over itself, so if you look back on the lectures on on rings and left and right ideals and rings and so on . Ah you can write down lots of examples excuse me. So, for example, I can take you know make various choices of rings.

For example, I can take the matrix rings $M_n[K]$ and in fact, the left and the right ideals of $M_n[K]$ were determined in in one of the earlier lectures . So, if you recall the the notation there, the the ideals look like this. I can take any subspace V of Kn and you had $R(V)$ which was the set of all matrices in $M_n(K)$ whose row space which means the span of the rows is contained in this fixed subspace V.

And then there was C of V which is similarly the the set of matrices whose column space is contained in V. So, let me just talk about the row space for the moment . I am just looking at left ideal. So, this is in fact a left ideal . So, in other words its a its a submodule, ok. So, this I am I am sort of recall from the previous lectures ok. So, that gives you an example of a left submodule of a of a submodule of the ring of matrices acting upon itself on the left OK. And of course, similarly I mean this was a non-commutative ring ah. You can



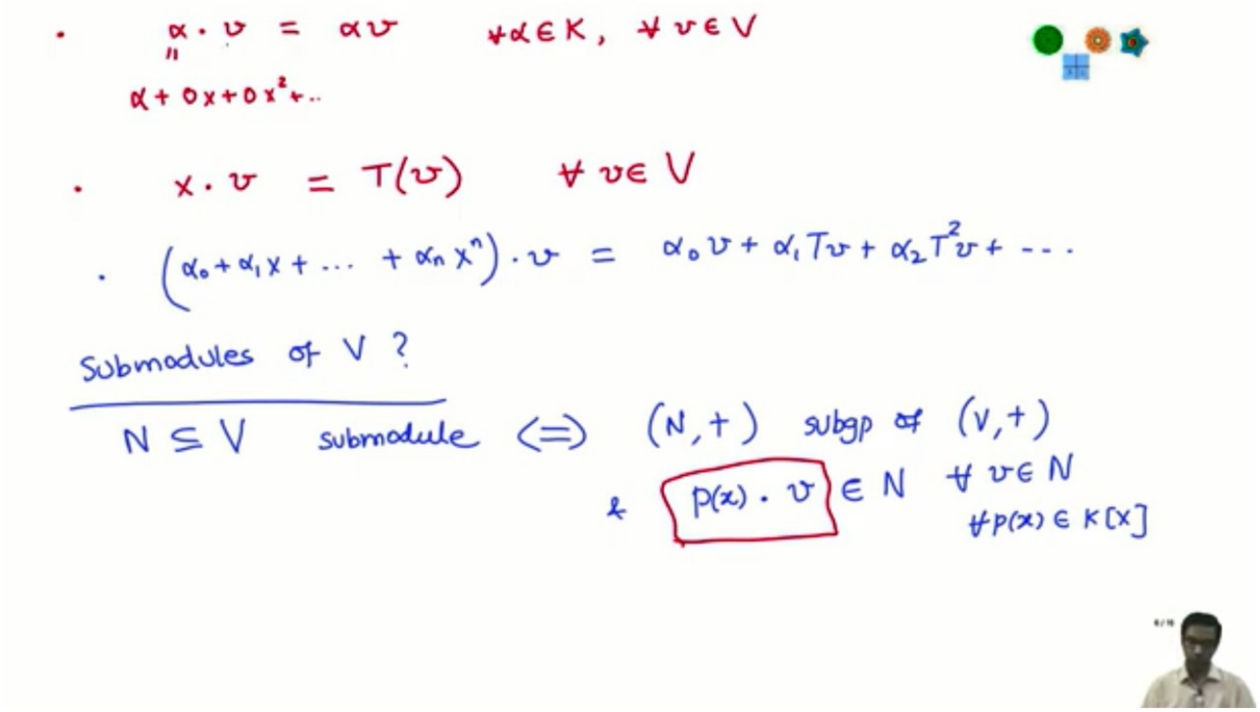
look at commutative examples as well. So, I could for example take any commutative ring . Commutative rings, of course you have seen many examples already. So, look back on the um the examples of ideals and commutative rings.

So, now of course recall that left and right there is no distinction anymore. Left ideals of R are really the same as what we call two sided ideals or ideals ok and of course, you know ideals were determined in many different examples before, ok.

So, this is just to say that you already know lots of examples of submodules because you already know lots of examples of ideals in commutative rings or left ideals in non commutative rings ok. So, that is already many examples but let us do a few more. So, here are some more examples of submodules ah. So, we number it one again. So, if I take the ring to be the ring of integers, then M recall $\mathbb{Z}$ module is the same as an Abelian group. So, N is any Abelian group ok . So, that is a $\mathbb{Z}$ module and in this case, it is easy to show and I will leave that as an exercise that N is a submodule. It is a submodule.

So, let me just put $\mathbb{Z}$ submodule of M or maybe just say submodule. So, we already said the ring is $\mathbb{Z}$, this is a submodule of M is actually the same as saying it is a subgroup. There it there is no additional condition here. $(N+)$ is a subgroup of $(M+)$, ok and and the reason is not too hard to see ah.

Recall that the $\mathbb{Z}$ module structure on an abelian group is more or less forced, ok. You cannot do much. You have to use the the addition in the in you know the underlying addition to define the multiplication by scalars ok and so, in some sense the the same sort of argument proves this statement as well that submodules are just the same as subgroups. You just pick a subgroup that is automatically a $\mathbb{Z}$ submodule ok because multiplication by a number N is just a repeated addition.



So, if it is closed under addition, it is automatically closed under repeated addition ok. So, example 2 which is slightly less trivial which is a ring of polynomials in a single variable. So, let us take K to be a field and R to be the ring of polynomials and now here recall from one of our earlier lectures that how do you construct modules over this ring $K[X]$ the way you do. It is well a module can be obtained by starting with this pair V, T.

So, if I start with V which is a K vector space just in the usual sense of the word and I need to start with a linear operator on V and so, T from V to V is a linear operator ok or a linear transformation. So, if I give you these two things, then it uniquely defines a a module a $K[X]$ module structure on V, then recall V becomes a $K[X]$ module by the following prescription. So, this this these two pieces of information that V is a vector space and T is a linear operator, these two things define the actions of .

So, how does $\alpha \cdot v$ ah? This is just the usual scalar multiplication. So, this is for all α coming from K and for all $V \in V$ ok. So, when I say α I think of it as the constant polynomial $\alpha_0 + \alpha_1 x + ... + \alpha_n x^n$. If you wish you know I can just put zeros on all the other higher powers of x. So, the constants, the constant polynomials α act on v by just scalar multiplication. So, that is the first and the special polynomial x power 1, the degree 1 polynomial homogeneous polynomial x power 1 or x. The way x acts on v is given by T ok.

So, this is you can think of these two as the two defining properties and of course, these two automatically tell you what a general polynomial does if I take $\alpha_0 + \alpha_1 x + ... + \alpha_n x^n$ acting on V . It is just $\alpha_0 v + \alpha_1 T v + ... + \alpha_n T^n v$so on ok. So, that was recall how we made V into $K[X]$ module, ok. Now, the question is what are submodules now ok. So, let us now ask what are submodules of V now ok. So, these are you know KX submodule. I am thinking of V as a $K[X]$ module now . So, what what is the definition? So, let see when do you say something is a submodule.

So, suppose N is a submodule means that firstly, it is closed under addition $(N+)$ is a subgroup of $(V+)$ and secondly, it is closed under scalar multiplication ok. So, it means if I take any polynomial $p(x)$ from $K[X]$ and I act it on V, the answer is in N again for all V coming from N and for all polynomials $p(x)$ in my ring $K[X]$ ok. Now, what we will do is in particular we will apply this this condition here that $p(x)v$ belongs to N. We will apply it to those two special polynomials. Let us take constants and let us take X power 1.

So, in particular this implies the second condition means if I take a constant or if I take X, ok take these two types of polynomials I act it on v, the answer is again in N for all $v \in N$ and this belongs to N for all V in N again and this is also for all scalars α and K ok. So, what do I conclude? The first condition which says that if I taken a vector from N and I multiplied by any scalar from K, the answer is again in K, ok. So, this property together with the fact that so recall also we know something more about N. We also know that N is closed under addition is a subgroup ok. So, these two properties tell you exactly that N is a subspace. This is exactly what a vector subspace is, ok.

So, these two properties what I marked in green say that N is a vector subspace of of V ok and now, this additional property let us see what that implies to you. Remember? It is just I am sorry. xV remember is just $T(v)$. So, this says that if I take an element of N and I get on it, I get back an element of N.

In other words, this says that the subspace N if I on it, I get back I mean it it maps N to N vectors in N to vectors in N . So, what does that mean? Well these two properties are exactly the definition of what is called T invariant subspace. So, N is a T invariant subspace of V ok and of course, you can just reverse this entire chain of arguments and conclude that a a submodule is the same as a a a T invariant subspace. I can just put all my arrows back ok and you have to check one last thing that if $T(v) \in N$ for all v then in fact the more general polynomial. So, remember this is our general polynomial X. Ah the action of a general polynomial on a vector from N will also give me a vector in N ok, but that is because if it is T invariant, then it is also T square invariant and T cubed invariant and so on, ok.

So, I will just leave this last bit of checking to you and here is the final conclusion that N, so let me just record the conclusion that N is a submodule of V. It is the same as saying that N is a T invariant subspace of V, ok and this is the the T here is it comes from the way V has been made into a into a $K[X]$ module. X is acting by T, ok. That is the T we are talking about ok now ah. So, that is the the second important example.

$\Rightarrow$ $\alpha \cdot v \in N \quad \forall v \in N \quad \forall \alpha \in K$ & $(N,+)$ is a subgp of $(V,+)$

$X \cdot v \in N \quad \forall v \in N$

$\Downarrow$

$T(v) \in N \quad \forall v \in N$

$\Downarrow$

$T(N) \subseteq N$ $\quad$ $\Leftarrow$ N is a vector subspace of V

$\Downarrow$

N is a T-invariant subspace of V.