Algebra - I
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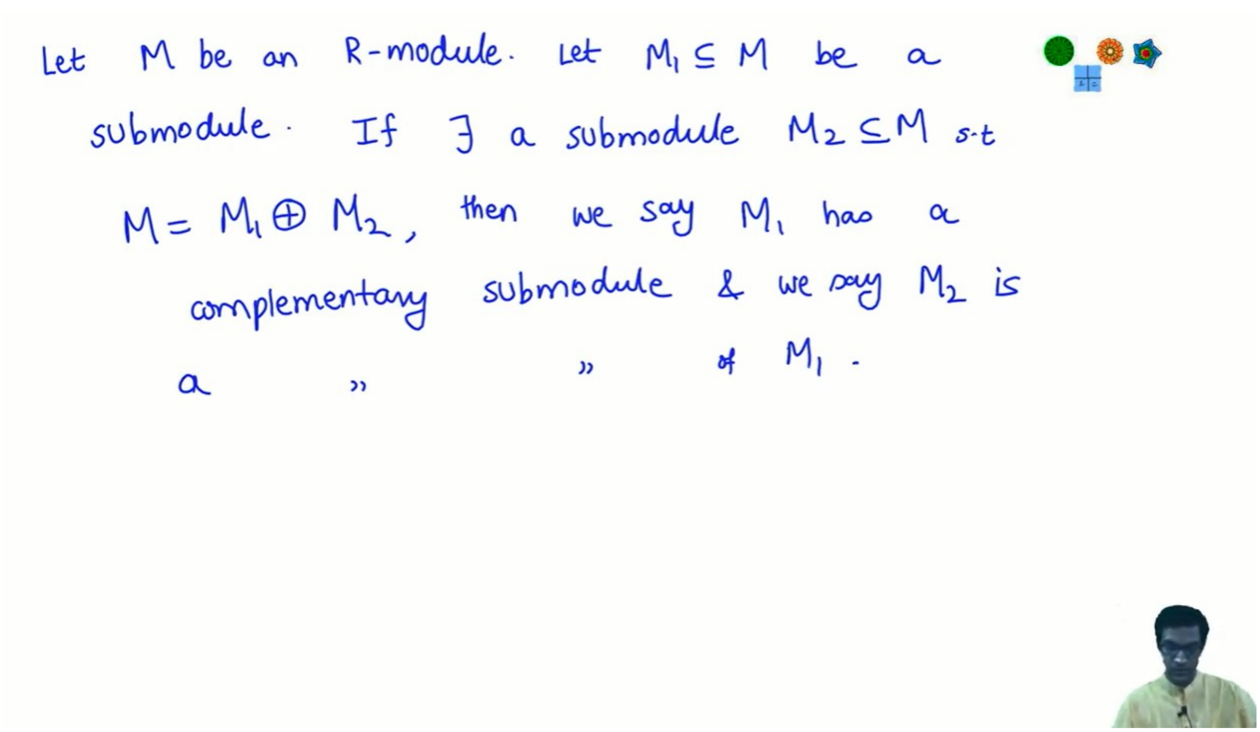
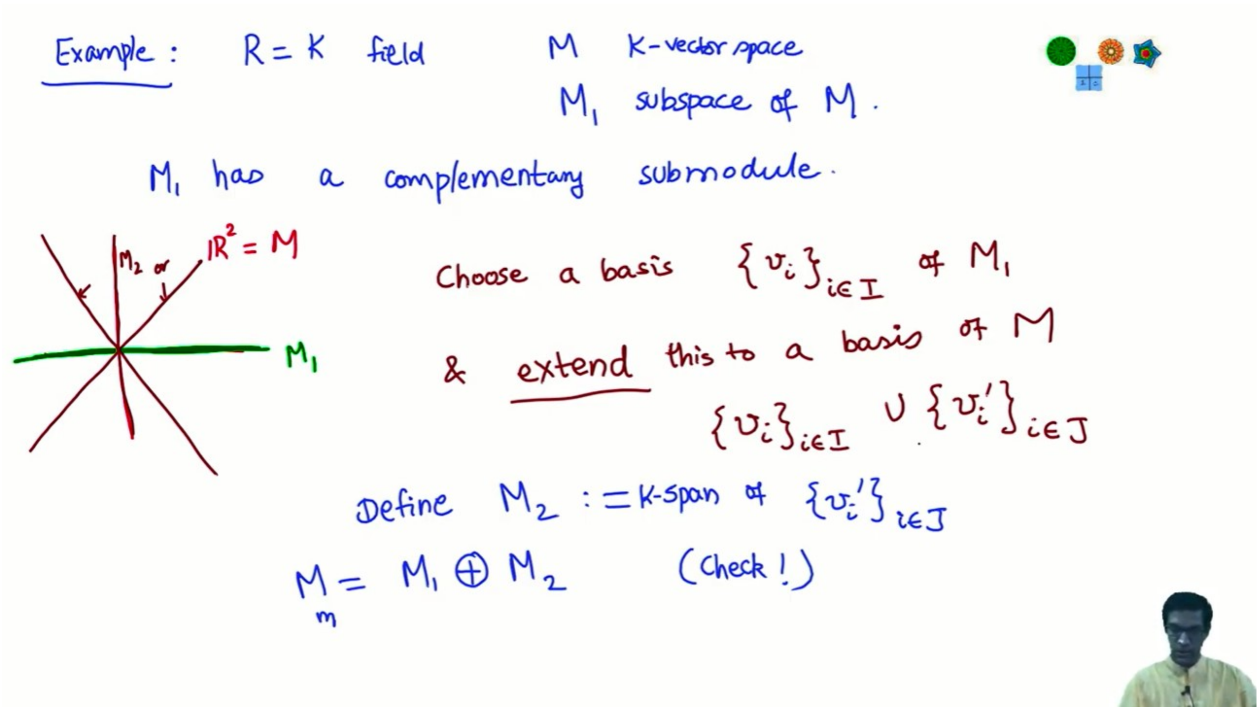
Lecture 63 [Complementary submodules]

Let us talk about Complementary submodules, complementary submodules . So, this related to the notion of direct sums that we talked about before. So, let M be an R module . Let M_1 be a submodule. Now, if M_2, so, if there exists a submodule M_2 of M, such that M is the internal direct sum or the direct sum of $M_1 \oplus M_2$ then we say that M, M_1 has a complementary subspace.

So, we say M_1 has a complementary subspace or a submodule as a complementary submodule ok and in this case the the complementary submodulus is M_2. So, we say M_2 is a compliment of M_1 and we say further that M_2 is a complementary submodule of M_1 ok. Now, the the point here is that you may not in general have compliments given an M_1 there may not exist an M_2 that is something that happens and the other important thing is even if it does have a complement the complement need not be unique.

There are often many different submodules. Each of which serves as a complement to the to the given module ok. So, let us just look at some some examples here ah. The most





familiar one is that of subspaces or vector spaces. So, this is the case when R is a field K and so, M_1 and M; so, let us say M is a K vector space. Otherwise, it is a module over the ring $R = K$ and M_1 is a subspace ok. So, I am given all this.

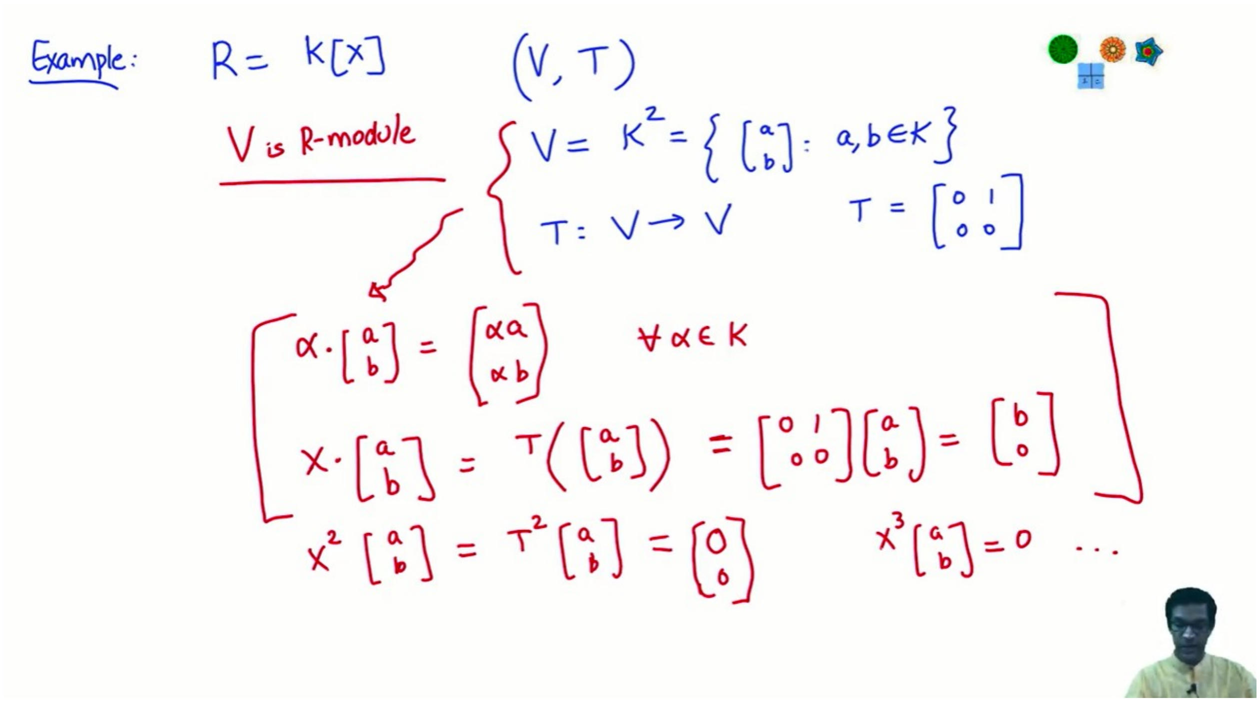
So, the question now is does M_1 have a complement ok, does it is there a complementary submodule of M for M_1, does M_1 have a compliment? Ok. And the answer in this case of vector spaces or in the case when R is a field the answer is always yes ok. So, M_1 always has a compliment. So, claim is that I can manufacture a complement for M_1, M_1 has a compliment ok. So, this is you know complementary submodule really.

So, as a complimentary submodule, why? Well, yes how we usually do this in vector spaces is probably familiar to you. So, let me just draw sort of a schematic diagram. So, let us say my vector space is so, as an example suppose my ring was my field was the field of real numbers and suppose my vector space was $M = \mathbb{R}^2$ ok.

And suppose I gave you M_1 to be let us say the X axis. So, let us say this space here is M_1 ok and what I need to do is to is to find a compliment to M_1 ok. Now, as you can see there are in fact, tons of different complements. Of course, I could take the y axis as a here is one possible compliment, but in fact, there are many others. I could take any line other than the X axis. Each line other than the X axis will serve as a complement. So, these are all different choices of M_2.

I can take this to be M_2 or I can take this to be M_2 or I can take this to be M_2 and so on ok. So, I have many choices here and what is it that we are really doing? Well, what we are doing in this picture can be done in general. So, how do we find the complement? Well, here is the construction. Let us just do this via basis for this vector space ok. So, how do we manufacture a complement? Choose a basis. Let us call it v_i 's .

So, at this at this point I am just allowing everything to be maybe even infinite dimensional I do not care. So, choose a basis v_i of M_1 ok and the key point here is that this basis can



always be extended. So, and extend this to a basis ok, let us call this. So, I already have some basis vectors to that I I add a few more. Let us call it $v'_{ii\in J}$. So, I can given a basis for a subspace or given in general a linearly independent set in a vector space you know that you can extend it to a basis.

You can add on some additional vectors. So, that the whole set is a basis of the ambient vector space ok. Now, this is a very very non obvious and important property that vector spaces have which general modules do not, ok. So, let us do this. You can extend this to a basis and now what does that get you, now, what we can do is we can use this extended basis to define the complementary subspace. So, I define M_2 to just be the span ok. So, this is span over K k span of the remaining basis vectors $v'_{ii\in J}$ ok.

Now, once you do this, so, like we did in this in this figure observe that the ambient vector space is nothing but M_1 direction M_2 ok. Why? Well, again I leave this for you to check, but you can already see why this must be true because if I given arbitrary element M in M, I first write it as a linear combination of these basis elements.

Some terms involve v_i 's, the other terms could involve the v_i primes. So, the portion which is a linear combination of the v_i 's I call it as M_1, the portion which is a linear combination of the v_i primes I call it M_2 ok. I now use this to show that this decomposition is unique and so on ok. So, if you are in the context of vector spaces then any submodule of my module M or rather any subspace.

Therefore, can you can always manufacture a complementary submodule for it ok. Now, this fails in general if we are talking about arbitrary rings. So, this is not true in general and let us see an example where this fails ok. So, for this we will just take R to be the ring $K[X]$, the ring of polynomials and one variable X and if you remember what this what modules over this ring are given by they are just pairs (V, T), where V is a vector space and T is a linear operator on V, ok.

This is really how any module over this ring can be obtained. So, in our case let me you know take V to be something very simple. Let me take V to be the vector space K^2, ok. So, its I think of it as all column vectors all 2×2 column vectors $\begin{bmatrix} a \\ b \end{bmatrix}$; a and b coming from K . So, this is my vector space and I need to also specify a linear operator on this vector space and I will just give you the matrix of this linear operator.

So, this is $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. So, recall we have looked at this example once before when we talked about quotient modules and so on. So, T is just this this operator. Now, what is this mean? Again to recall how pairs V comma T correspond to modules over $K[X]$. This the translation is the following. When I take a scalar from my field, so, if I take alpha scalar from my field and I make it act upon an element of V, it just acts as usual.

It just multiplies now this is the; this is the definition of scalar multiplication in in K^2, right. I will just multiply alpha on both components. So, this for all alpha coming from K ok. So, I think of alpha as as the constant polynomials which is you know they only have a constant term no X or X^2 and so on.

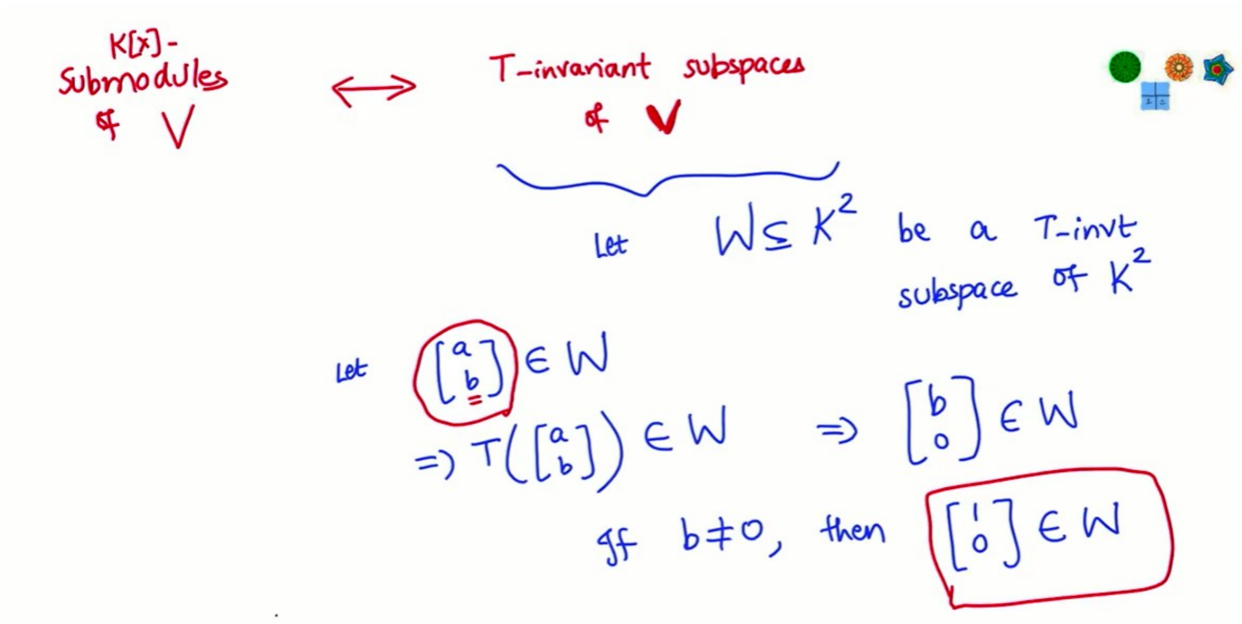
Next, I tell you what X does, how does X act. Well, X acting on an element of this vector space is given by the action of T on that element ok. So, T really performs the role of X and in this special case what is $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$, In this particular case it is easy to check if I take X square for example. So, X square is supposed to be T square right. I I perform T twice, but if you take this particular matrix and square, it just gives you the 0 matrix. So, in fact, this is just going to be the 0 vector. So, X square well in fact, all higher powers of X; X cubed, X power 4 and so on, all of them act as 0 ok.

So, in some sense all the information is only contained in in these first two actions that of alphas that is constant polynomials and the polynomial X ok. So, that is my module. So, what have I done for you? I have defined for you a module. So, V in this case is a module over R ok, V is an R module via these definitions ok. Now, I claim that in this context I can find an example of a submodule which does not admit a complement ok.

So, next task is to try and understand what submodules of V look like and remember this again is something that we have addressed before. If I need to find submodules, so, this is all over the ring $K[X]$. So, the $K[X]$ submodules of V are the same as T invariant subspaces of V ok. So, look back again on on the lecture on submodules that, we talked about what submodules of modules over $K[X]$ look like. These are just T invariant subspaces ok. Now, what is a T invariant subspace in this particular example that is all we are we are down to. So, let us try and understand what are the T invariants subspaces. So, suppose, so, let me work on the the right hand side. So, let me try and look at this.

So, let M subset of K^2 be a T invariant subspace. Maybe you should not call it M, let us use some vector space notation. Let me call it W may be that W subset of K^2 be a T invariant subspace of K^2 ok. So, what does that mean? So, let me pick an element in W.

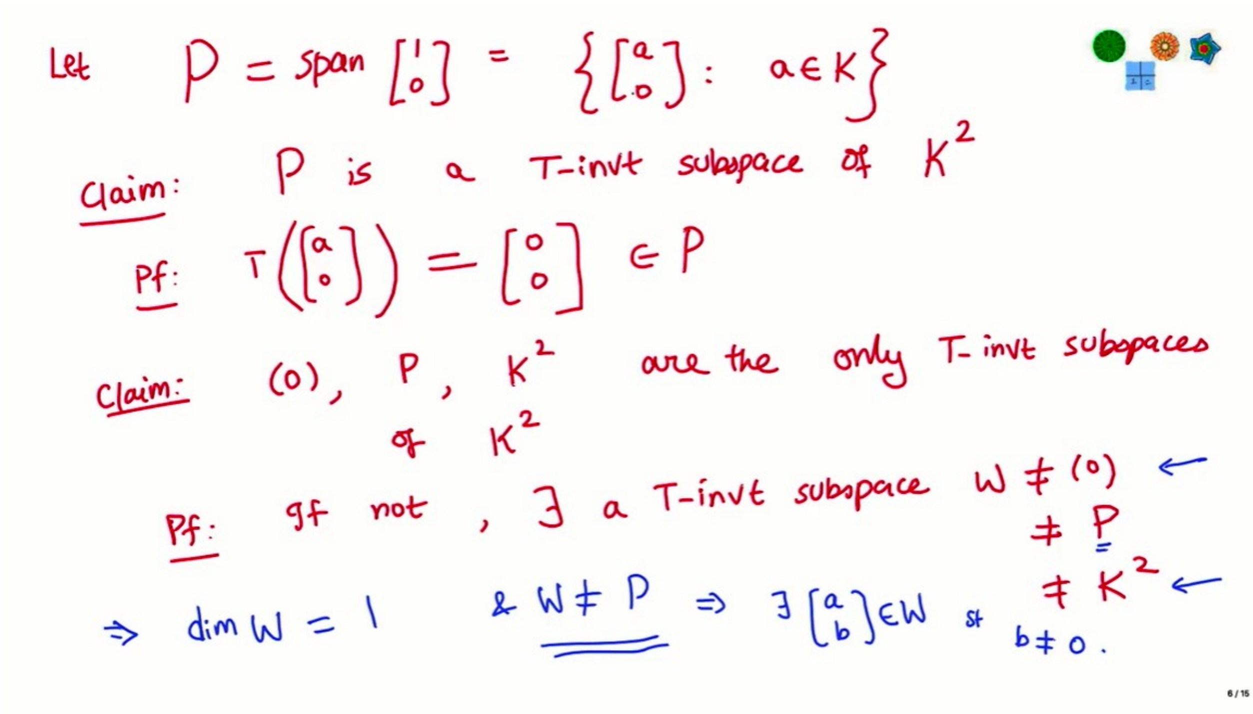
So, suppose I take the vector are $\begin{bmatrix} a \\ b \end{bmatrix} \in W$. Now, saying that W is T invariant means that when I act $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) \in W$ the answer is again in W ok. Now, what does this mean? In this case recall T does this special thing, it just sends $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix} \in W$.



Now, if b is not 0 if b is not 0 then this in fact, tells me that you know I can divide by b, I can conclude that the vector 1 , 0 is in W ok. So, that seems to be some conclusion we can draw that if suppose my invariant subspace has a vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in which this element b is not 0, then I can conclude that the the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$ ok. So, what does this mean? Let us just take well this this sort of already tells us gives us a candidate for a subspace. So, let us look at the following candidate. So, let us define a subspace. So, let me call it maybe P, it is just a span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ok. In other words it is all vectors of the form $\{\begin{bmatrix} a \\ 0 \end{bmatrix} | a \in K\}$ ok . So, I claim that P is in fact, P is an T-invariant subspace.

So, I claim that P is a T invariant subspace of is a T invariant subspace of K^2 ok. Why is that? Well, because proof if I just take; you know take any element $\begin{bmatrix} a \\ 0 \end{bmatrix} \in P$. In fact, you see that what you get is just $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ ok because it it b is 0 in this case. So, T of a 0 is just $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So, of course, that belongs to the subspace P ok. So, this is T invariant for very obvious reasons that it maps every element there to 0 ok. So, this is this is sort of trivial, but I claim more. In fact, that this is the only T invariant subspace of K^2 other than the two trivial possibilities, which is 0 and the whole ok.

So, claim the stronger claim. In fact, the 0 subspace the sort of 1-dimensional subspace P and the 2-dimensional subspace K^2 are the only T invariant subspaces of K^2 ok. So, I claim that there are only these three T invariant subspaces. So, let us prove that. So, we have just shown that P is T invariant. These other two guys are of course, T invariant. I claim there cannot be any others ok. Why not? Well, that is what we just try to show here. Suppose W is a T invariant subspace of K^2 ok. So, let us pick such a guy. What do we conclude? If



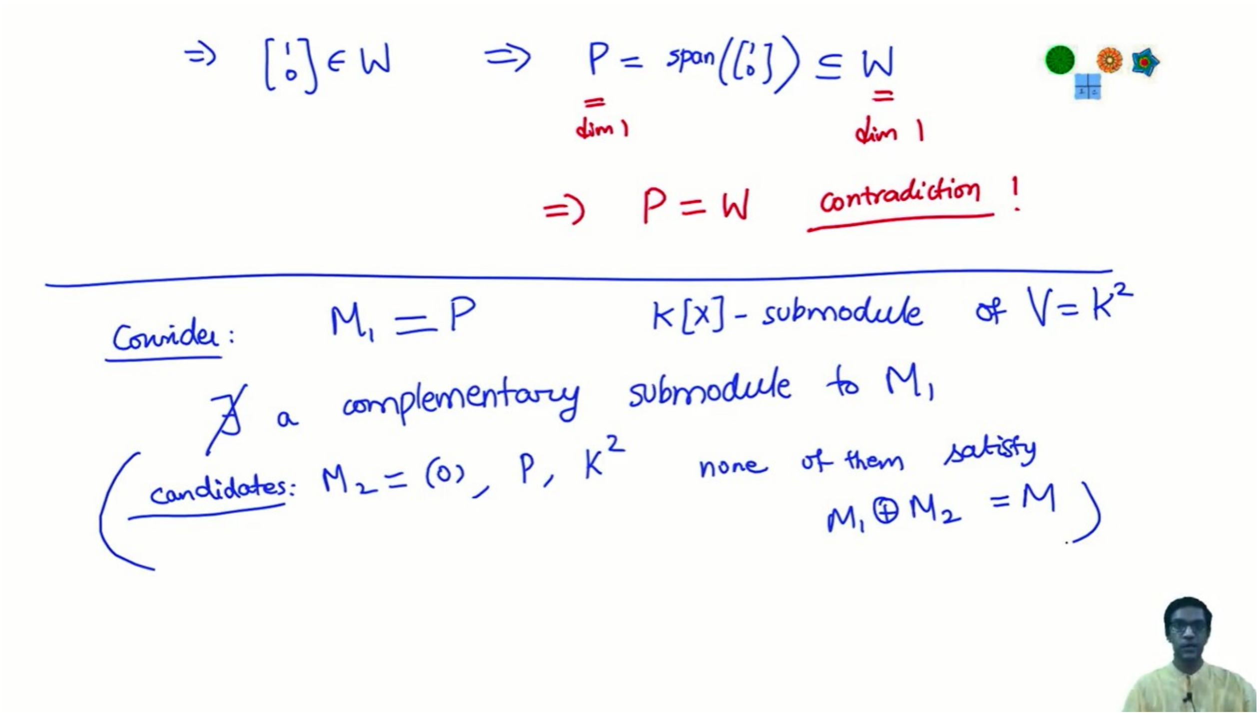
suppose I have a vector $\begin{bmatrix} a \\ b \end{bmatrix} \in W$ in which b is not 0 ok then I conclude that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$ ok.

So, recall that is that is what this argument showed. So, let us now use that in this proof. So, I I claim that these are the only T invariant subspaces. So, maybe let us argue by contradiction. If not there exists a T invariant subspace W which is not one of these three , W which is not equal to 0, which is not equal to this space P, which is not equal to K^2 right.

Now, what does that mean, ok? So, what does that mean in particular? It says that W is not 0-dimensional, W is not 2-dimensional. So, this tells you that W is not a 0-dimensional space, W is not the whole thing. So, W can only be some 1-dimensional space right. So, that is what we have conclude so far.

It is a 1-dimensional subspace, but further it is not equal to this particular special 1-dimensional subspace P, it is some other 1-dimensional space ok. So, W is 1-dimensional. So, all this is now thought of as vector spaces over K. So, we know for sure it is 1-dimensional and it is not this particular 1-dimensional guy P. Now, what is P? If you remember P is just everything in the form a , 0 ok and W is not this 1-dimensional space, which means that I can surely find a vector $\begin{bmatrix} a \\ b \end{bmatrix} \in W$ such that b is not 0 right. Why can I do that? Because W is not equal to P right. If it if it is not P then at least one vector in W must have a component the second component here, which is not 0 ok.

So, there is a vector in W for which its second component b is not 0, but now, we just use the argument that we used right here the previous page which said that if b is not 0 then I conclude that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$ ok. So, let us conclude that this means that 1 , 0s and W belongs



to W ok, but observe what does that mean? 1 , 0 is exactly the span of 1 , 0 is exactly P right. P is the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ok. So, if $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$ then the span of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is *a* subset of W right. So, I conclude that P must in fact, be *a* subspace of W, but that is not quite possible because this guy is also 1-dimensional and this guy is also 1-dimensional ok. So, both are 1-dimensional and I conclude one of them is *a* subspace of the other then that the only possibility is that they are equal ok.

But recall that is *a* contradiction because we started out under the assumption that P and W are not equal to each other ok. So, look back again on this proof. The one basic observation is this this thing that we just made that if you can find *a* vector in W whose second component is not 0, you can conclude that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in W$ ok. Now, what does that get us? Let us go back to this this claim here. So, this claim tells us something something very very interesting. It says that there are only three subspaces which are T invariant ok and recall T invariant subspaces are the same as $K[X]$ submodules.

Now, if these are the only three T invariant subspaces then let me do the following. Let me take my my submodule M_1 to be P itself here. So, now, let us do the following. Let us take so, consider M_1 to be P . So, this is definitely *a* $K[X]$ submodule of K^2 ok because its T invariant. But observe this does not have *a* complementary subspace ok. Why not? Well, because there are not any subspaces possible. What are the only possible invariant guys, you can either take P 0 or K^2 right and well let us see what do you need for *a* complementary subspace. At the very least you need that the intersection of that with P should be 0 right. So, I cannot take P. P cannot be the complement of itself because the intersection is not 0, K^2 cannot be the complement of P. The only possibility is 0 maybe because the intersection is 0, but P intersection 0 is 0, but P plus 0 is not the whole thing ok. So, P plus 0 would just give me P itself. So, observe that none of these three candidates can serve as a complementary subspace to M_1. So, observe that there exists there does not exist a complementary subspace or a complementary submodule .

Why? Because the only candidates, so, there are only three candidates, P and the whole none of them satisfy the requirement to be M_1 direction M_2 is M ok . So, complementary submodules do not always exist. You you know vector spaces are somewhat special in this regard ok. So, there are many other instances where they where they do exist as well, but this is just to reinforce that what we are familiar with in the vector space context, one should be a little careful with with trying to use those same sorts of reasoning in the case of general rings ok .