Algebra - I
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ALGEBRA I

1. Lecture 72: Generators and relations for Finitely Generated Modules

Let R be any commutative ring and M be a finitely generated R module. If M is generated by $v_1, \ldots, v_m$ consider the map $\phi : R^m \to M$ given by $\phi((a_1, \ldots, a_m)) = \sum_{i=1}^{m} a_i v_i$. Let N be the kernel of phi. By the first isomorphism theorem we have $M \cong R^m/N$.

Now suppose R is Noetherian then we have seen that every sub module of a finitely generated R module is finitely generated- so N is finitely generated by say $x_1, \ldots, x_n$.

Denote elements of R^m as column vectors. Let $x_i = (x_{i1}, \ldots, x_{im})$ and let $X = (x_{ij})$ then X is an $m \times n$ matrix. So when we look at a finitely generated modules of a Noetherian ring R then you can write it as $M \cong R^m/\mathcal{C}(X)$ for the column-space $\mathcal{C}(X)$ of a matrix X.

Example 1.1. *Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/6\mathbb{Z}$. $M \cong \mathbb{Z}/\mathcal{C}(6)$.*

M is also generated by 2 and 3. And so we could define $\phi : R^2 \to M$ $(a, b) \to 2a + 3b$ in $\mathbb{Z}/6\mathbb{Z}$. The kernel of ϕ consists of vectors of the form $3x2y$. And so what we get is $M \cong \mathbb{Z}/\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Example 1.2. *Let $R = F[t]$ and let us take $M = F^2$ and let us define the action of t as the matrix $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.*

We define $\phi : F[t]^2 \to F^2$ taking $(p(t), q(t)) \to (p(A)(1, 0)^t, q(A)(0, 1)^t)$.

We have that the kernel of ϕ is the column space of the matrix $\begin{pmatrix} t & -1 \\ 0 & t \end{pmatrix}$.

So $M \cong F[t]^2/\mathcal{C}(\begin{pmatrix} t & -1 \\ 0 & t \end{pmatrix})$.

More generally let us fix a finite dimensional vector space V over a field F and a linear operator T. Now, if you pick a basis of V $e_1, \ldots, e_n$ and T has matrix A with respect to this basis. We can think of $e_1, \ldots, e_n$ as generators of M as $F[t]$-module. Let $\phi : F[t]^n \to V$

Theorem 1.3. *The kernel of ϕ is the column space of the matrix $tI - A$.*

Proof. Consult lecture 72. ☐