Algebra - I
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# ALGEBRA I

## 1. Lecture 81: Deciding similarity

Given matrices $A, B \in M_n(K)$ over a field $K$, when are they similar?

Recall that $M_A, M_B$ are $K[t]$ modules by the action

$$p(t)_A\overrightarrow{v} = p(A)\overrightarrow{v},$$
$$p(t)_B\overrightarrow{v} = p(B)\overrightarrow{v}$$

Suppose $f : M_A \to M_B$ is a $K[t]$-module isomorphism. Then

$$f(p(t)_A\overrightarrow{v}) = p(t)_B f(\overrightarrow{v}),$$

for all $p(t) \in K[t]$.

Let $X \in GL_n(K)$ be such that $f(\overrightarrow{v}) = X\overrightarrow{v}$. Then

$$Xp(A)\overrightarrow{v} = p(B)X\overrightarrow{v},$$

for all $\overrightarrow{v}$. Then

$$Xp(A)X^{-1} = p(B),$$

for all $p(t) \in K[t]$. In particular we have $XAX^{-1} = B$, i.e, $A \sim B$. Conversely, if $A \sim B$ then $\exists X \in GL_n(K)$ such that $XAX^{-1} = B$.

$$\implies Xp(A)X^{-1} = p(B),$$

for all $p(t) \in K[t]$.

So defining $f(\overrightarrow{v}) = X\overrightarrow{v}$ gives a $K[t]$-module isomorphism.

**Theorem 1.1.** *Let $A, B \in M_n(K)$. Then $A \sim B$ iff $M_A \cong M_B$.*

Recall that

$$M_A \cong K[t]^n/\mathcal{C}(tI - A),$$
$$M_B \cong K[t]^n/\mathcal{C}(tI - B)$$

**Theorem 1.2.** *$A \sim B \iff tI - A, tI - B$ have the same Smith Normal form.*

**Example 1.3.** *Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ Then*

$$tI - A = \begin{pmatrix} t-1 & 0 \\ 0 & t+1 \end{pmatrix}$$
$$\approx \begin{pmatrix} 1 & 0 \\ 0 & t^2-1 \end{pmatrix},$$

*since the first diagonal entry is the gcd of the* $1 \times 1$ *minors, the second diagonal entry is the gcd of the* $2 \times 2$ *minors, etc. Similarly*

$$\begin{aligned} tI - B &= \begin{pmatrix} t & -1 \\ -1 & t \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 \\ 0 & t^2 - 1 \end{pmatrix}, \end{aligned}$$

*Thus* $A \sim B$.

**Example 1.4.** *Let* $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ *and* $C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

*Then*

$$\begin{aligned} tI - A &= \begin{pmatrix} t & -1 & 0 \\ -1 & t & 0 \\ 0 & 0 & t-1 \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & t-1 & 0 \\ 0 & 0 & t^2 - 1 \end{pmatrix}, \end{aligned}$$

*since the first diagonal entry is the gcd of the* $1 \times 1$ *minors, the second diagonal entry is the gcd of the* $2 \times 2$ *minors, and the third diagonal entry is the gcd of the* $3 \times 3$ *minors. Similarly*

$$\begin{aligned} tI - B &= \begin{pmatrix} t & 0 & -1 \\ 0 & t-1 & 0 \\ -1 & 0 & t \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & t-1 & 0 \\ 0 & 0 & t^2 - 1 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} tI - C &= \begin{pmatrix} t & 0 & -1 \\ -1 & t & 0 \\ 0 & -1 & t \end{pmatrix} \\ &\approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^3 - 1 \end{pmatrix}, \end{aligned}$$

*Thus* $A \sim B \not\sim C$.