Algebra - I
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# ALGEBRA I

## 1. Lecture 82: Rational Canonical form

Let $M = K[t]/p(t)$ with $p(t) = t^d + a_1 t^{d-1} + \ldots + a_d$. $M$ has basis $\{1, t, \ldots, t^{d-1}\}$ on which $t$ acts by

$$t \cdot t^i = \begin{cases} t^{i+1} & \text{if } i < d-1 \\ -a_1 t^{d-1} - \ldots - a_d & i = d-1 \end{cases}$$

Thus the matrix for the action of $t$ is given by

$$C_{p(t)} := \begin{pmatrix} 0 & \ldots & 0 & -a_d \\ 1 & 0 & \ldots & -a_{d-1} \\ 0 & 1 & \ddots & \vdots \\ 0 & \ldots & 1 & -a_1 \end{pmatrix}$$

**Theorem 1.1.** *Let $A \in M_m(K), B \in M_n(K)$ and $M \cong M_A, N \cong M_B$. Then*

$$M \oplus N \cong M_{\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}}.$$

*Proof.* Let $x_1, \ldots, x_m$ be a basis of $M$ such that $t$ acts as multiplication by $A$ on this basis; let $y_1, \ldots, y_n$ be a basis of $N$ such that $t$ acts as multiplication by $B$. Then $M \oplus N$ has a basis $\{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ on which $t$ acts as multiplication by $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$. $\square$

Given $A \in M_n(K)$,

$$M_A \cong K[t]/p_1(t) \oplus \ldots \oplus K[t]/p_r(t),$$

for monic polynomials $p_1, \ldots, p_r$ with $p_1 | p_2 | \ldots | p_r$. Then

$$M_A \cong M_{C_{p_1}} \oplus \ldots \oplus M_{C_{p_r}}$$

Thus

$$M_A \cong M_{\operatorname{diag}(C_{p_1}, \ldots, C_{p_r})}.$$

**Theorem 1.2.** *Every matrix $A \in M_n(K)$ is similar to a unique matrix of the form*

$$\begin{pmatrix} C_{p_1(t)} & 0 & 0 & \dots \\ 0 & C_{p_2(t)} & 0 & \dots \\ 0 & 0 & \ddots & \dots \\ 0 & \dots & 0 & C_{p_r(t)} \end{pmatrix}$$

*where $p_1|p_2|\dots|p_r$.*

**Example 1.3.** *Let*

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

*Then*

$$tI - A = \begin{pmatrix} t+1 & -1 & -1 \\ 0 & t & -1 \\ 0 & -1 & t \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & t+1 & 0 \\ 0 & 0 & t^2-1 \end{pmatrix}$$

*Then $M_A \cong K[t]/(t+1) \oplus K[t]/(t^2-1)$. Thus*

$$\begin{aligned} A &\sim \begin{pmatrix} C_{t+1} & 0 \\ 0 & C_{t^2-1} \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$