Algebra - I
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ALGEBRA I

1. Lecture 83: Jordan Canonical form

Recall that the rational canonical form is a block diagonal matrix of the form:

$$\begin{pmatrix} C_{f_1} & 0 & \dots & 0 \\ 0 & C_{f_2} & \dots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \dots & C_{f_r} \end{pmatrix}$$

with $f_1|f_2|\dots|f_r$. Given $f(t) = t^n + a_1t^{n-1} + \dots + a_n$, we have

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -a_n \\ 1 & 0 & \dots & 0 & -a_{n-1} \\ 0 & 1 & \dots & 0 & -a_{n-2} \\ 0 & \dots & \ddots & 0 & \dots \\ 0 & \dots & 0 & 1 & -a_1 \end{pmatrix}$$

Example 1.1. *Let* $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, *the rational canonical form of* A *is*

$$\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$$

For an R-module M and a prime $p \in R$, the p-primary part is defined as

$$M_p = \{m \in M | p^k m = 0 \text{ for some } k > 0\}.$$

M is p-primary iff $M = M_p$.

If $M = M_{\text{tors}}$ then $M = \oplus_p M_p$

For an algebraically closed field K and $R = K[t]$,

$$\text{prime ideals of } R \leftrightarrow (t - \lambda),$$

for $\lambda \in K$.

Given $A \in M_n(K)$, $M_A = K^n$, $t \cdot \overrightarrow{v} = A\overrightarrow{v}$, denote the generalised eigenspace for λ as V_λ. Then $V_\lambda = M_{A,t-\lambda}$ where

$$M_{A,t-\lambda} = \{\overrightarrow{v} \in K^n | (t-\lambda)^k \overrightarrow{v} = 0 \text{ for some } k > 0\}$$

For the primary decomposition $K^n = \oplus_{\lambda \in \mathbb{C}} V_\lambda$, with $A(V_\lambda) \subseteq V_\lambda$. Let $A_\lambda : V_\lambda \to V_\lambda$ be the linear map obtained by restricting A to V_λ. Then

$$A \sim \begin{pmatrix} A_{\lambda_1} & 0 & \dots & 0 \\ 0 & A_{\lambda_2} & 0 & \dots \\ 0 & \dots & \ddots & 0 \\ 0 & 0 & \dots & A_{\lambda_m} \end{pmatrix} \tag{1}$$

for some $\lambda_1, \dots, \lambda_m \in \mathbb{C}$.

M_{A_λ} is $(t-\lambda)$-primary iff

$$M_{A_\lambda} \cong K[t]/(t-\lambda)^{k_1} \oplus \dots \oplus K[t]/(t-\lambda)^{k_r}.$$

Suppose $M = K[t]/(t-\lambda)^k$, consider a basis of M given by

$$\begin{aligned} e_0 &= 1, \\ e_1 &= (t-\lambda), \\ &\dots \\ e_{k-1} &= (t-\lambda)^{k-1}. \end{aligned}$$

Then $te_i = e_{i+1} + \lambda e_i$. Thus t acts by the matrix

$$J_{k,\lambda} = \begin{pmatrix} \lambda & 0 & \dots & 0 \\ 1 & \lambda & 0 & \dots \\ 0 & 1 & \lambda & \dots \\ 0 & \dots & 1 & \lambda \end{pmatrix}_{k \times k}$$

Then with A as defined in (**??**), we have

$$M_{A_{\lambda_i}} \cong K[t]/(t-\lambda_i)^{k_{i1}} \oplus \dots \oplus K[t]/(t-\lambda_i)^{k_{ir_i}}.$$

Then we have

$$A_i = \begin{pmatrix} J_{k_{i1},\lambda_i} & 0 & \dots & 0 \\ 0 & J_{k_{i1},\lambda_i} & 0 & \dots \\ 0 & 0 & \ddots & \dots \\ 0 & \dots & 0 & J_{k_{ir_i},\lambda_i} \end{pmatrix}$$

Example 1.2. *Let A be a matrix such that $M_A \cong K[t]/(t-2) \oplus K[t]/(t-2)(t^2-5t+6)$. Then*

$$A \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & 5 \end{pmatrix}$$

in its rational canonical form. Since $t^2 - 5t + 6 = (t-2)(t-3)$,

$$K[t]/(t-2)^2(t-3) \cong K[t]/(t-2)^2 \oplus K[t]/(t-3)$$

Then $M_A \cong K[t]/(t-2) \oplus K[t]/(t-2)^2 \oplus K[t]/(t-3)$. *Thus*

$$A \sim \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Example 1.3. *Let*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then calculating the Smith normal form we have

$$\begin{aligned} M_A &\cong K[t]/(t^2-1) \oplus K[t]/(t^2-1) \\ &\cong K[t]/(t-1) \oplus K[t]/(t+1) \oplus K[t]/(t-1) \oplus K[t]/(t+1) \\ &\cong K[t]/(t-1) \oplus K[t]/(t-1) \oplus K[t]/(t+1) \oplus K[t]/(t+1) \end{aligned}$$

Thus the JCF of A *is*

$$A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$