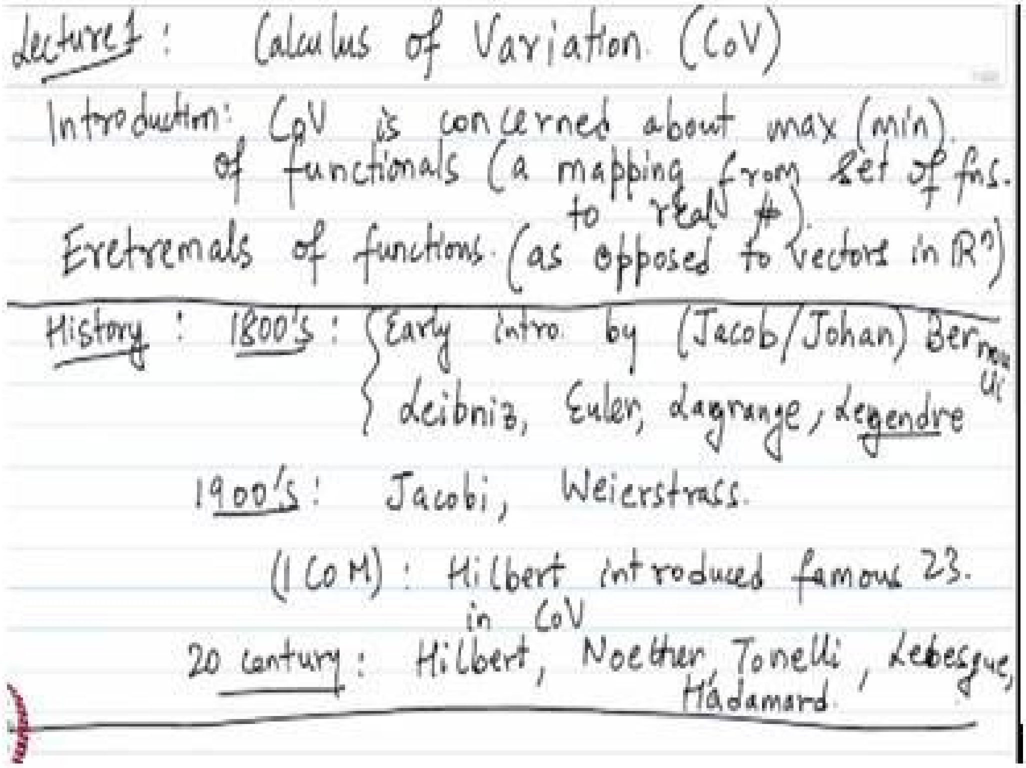
Variational Calculus and its Applications in Control Theory and Nano mechanics Professor Sarthok Sircar

Department of Mathematics
Indraprastha Institute of Information Technology, Delhi

Lecture – 01

Introduction –Euler Lagrange Equations Part-1

(Refer Slide Time: 00:16)



Good morning everyone. So today I am going to start the first lecture of this lecture course series that is on Calculus of Variations. As I have told in my introductory video that this course is primarily on the introduction to calculus of variations with specific applications in control theory and nanomechanics which will be introduced towards the latter half of this course. So just to introduce all the students regarding what this branch is all about.

let me denote calculus of variations by this short hand abbreviations CoV. So CoV is concerned about the maximization or the minimization of functionals. So as we move along the course, I am going to talk about what are functional. So functionals just in Layman's term these are mappings from the set of functions to real numbers. So when we talk about calculus of variations it is all about finding the maxima or minima, also known as the extrema of functions.

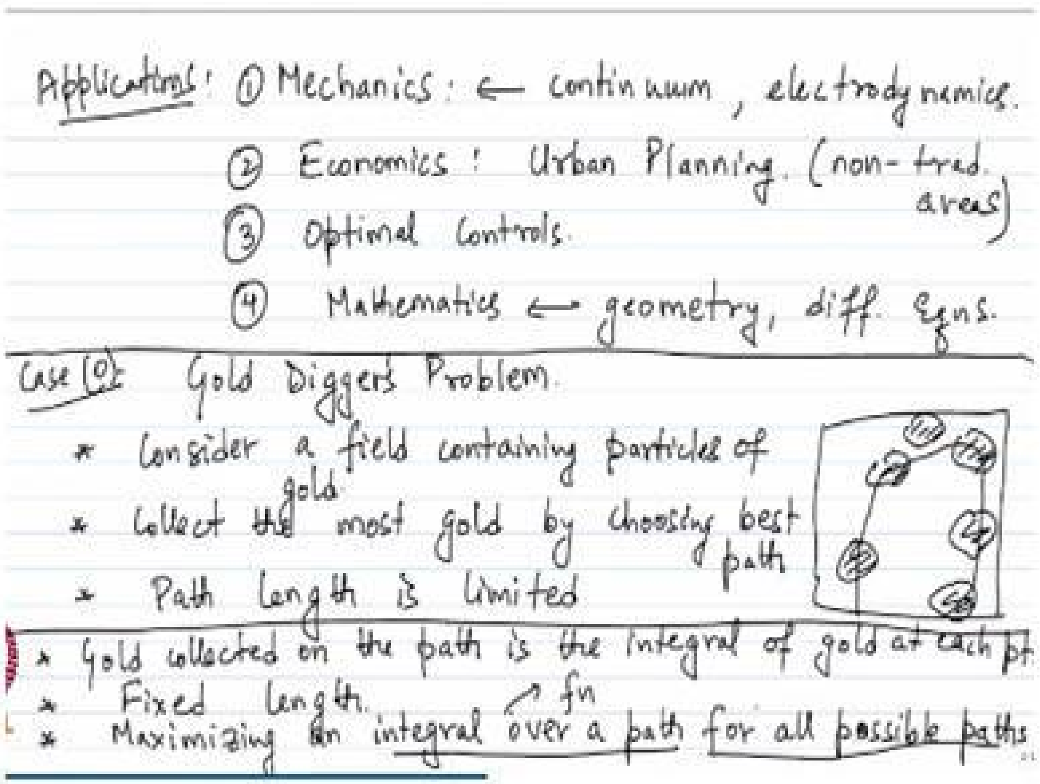
So, the extremals in this case these are not real numbers, but these are functions itself. As oppose to what is regularly taught in multivariate calculus as opposed to vectors in R^n or infinite dimensional calculus. So, with this very brief introduction, let me also talk about the history of this course. So calculus of variations has all along been with us since the 1800's the early mention of this course this topic was done by the Bernoulli brothers.

And I am going to over this course, I am going to reveal some of the problems that were introduced by these mathematicians and then this course was also further introduced by several other mathematicians including Leibnitz, Euler, Lagrange and Legendre. So, as we go along this course I am going to introduce several important results related to this course which were introduced primarily by Euler and Lagrange and towards the end.

When we talk about the sufficient conditions for finding the extremals there is a very vital contribution by Legendre. So in the 1900's more contribution came from Jacobi and from Weierstrass we are going to introduce some set of other important results from Weierstrass as well as Jacobi over the latter half of this course. In fact, in the International Congress of Mathematician it was Hilbert who introduced the famous 23 problems in calculus of variations.

And finally in that 20th Century the contribution came from well the likes of Hilbert, from the likes of Noether's we are going to introduce a very vital result by Noether as well as Tonelli, Lebesgue and finally Hadamard. Well, so since then this topic has primarily found several applications two of which we are going to introduce in this course. Now then, let me also introduce some applications of this course.

(Refer Slide Time: 05:35)



This course this topic is primarily applicable in the areas of mechanics namely classical mechanics starting from continuum mechanics as well as electrodynamics and we will also see that some of the applications that arise in economics namely in urban planning and some of the other non-traditional areas. So, then a very vital application that we are going to introduce over the latter half of this course is the application in control theory.

And finally we time over and over again during the different parts of this lecture series. I am going to introduce several applications in mathematics which includes case studies in geometry and differential equations. So, these problems are widespread throughout the discourse of this lecture series. So let me start with giving some classic examples starting with the very simple case study so let me call this as case 0.

So, let me introduce the topic of extremization of functional with the classic gold diggers problem. So as the name suggests the person who has in this problem there is a field, let us say there is a field that has lots of gold in patches let us say these patches they represent gold and the idea the objective of the person who is digging the gold out of the field is to traverse find a path such that he or she can collect the maximum amount of gold.

So the problem is as follows, so we consider a field containing particles of gold and the objective is to collect the most amount of gold by choosing the best path. So, we are given so out of the family with different paths we have to choose the best paths so that we can collect the maximum amount of gold and however the problem that we have is that the total path length in this case is limited.

So whatever path we choose we have to make sure that the total length of that path cannot exceed a certain value. So, in this problem we can see that we are maximizing a function of a function. So what are we doing? We are collecting gold so gold is collected on the path and that path is the integral of the gold at each point and further I have mentioned that the length of the path is fixed.

So we can see that we are maximizing an integral which is our path length and integral over a path for all possible paths, that is we are maximizing a function of a function. So we can see that the integral over the path in this case is the function.

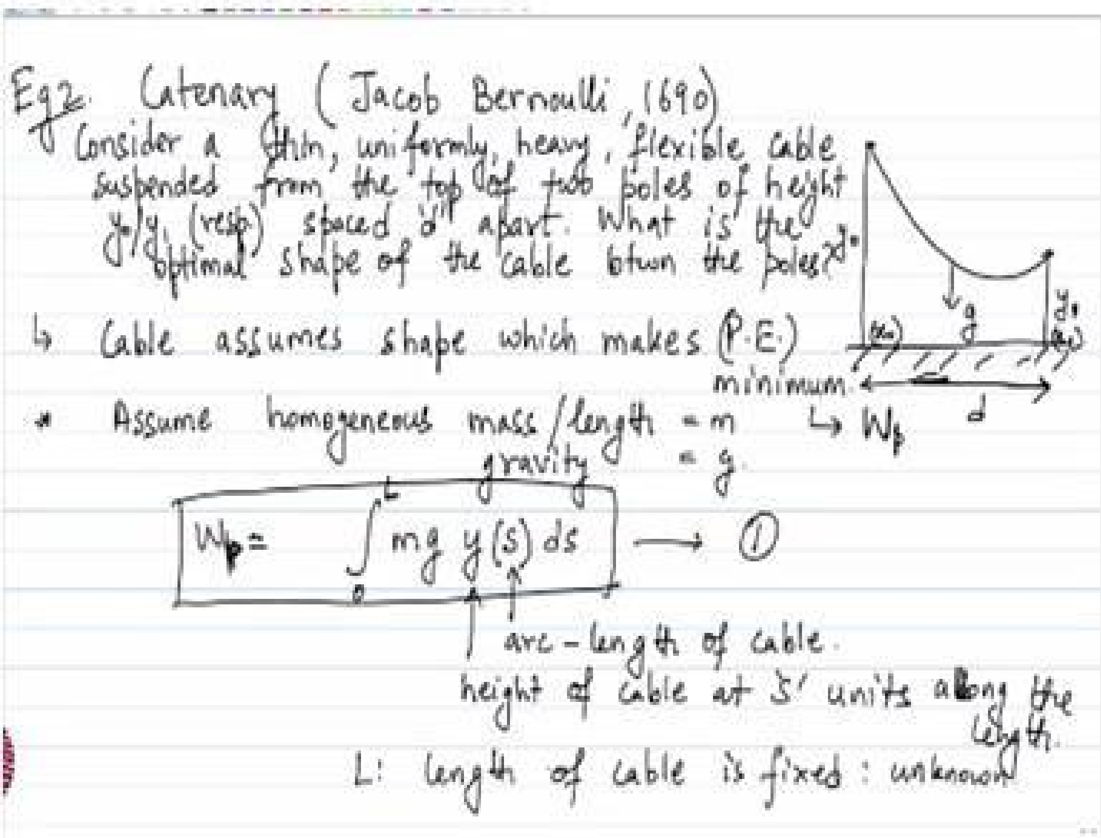
(Refer Slide Time: 11:07)

↳ Gold collected : Another fn.

Maximizing : $\underline{fn}\,(\underline{fn})$.

And the gold collected is another function. So, we are maximizing a function of a function where this function is the amount of gold collected and this function is the integral over all possible paths that are available with fixed path length.

(Refer Slide Time: 11:43)



So, the second problem that I want to discuss is the problem of catenary that was originally introduced by the younger of the Bernoulli brothers Jacob Bernoulli in 1690. So I am going to state the simplest version of this problem and over the course of our lecture series I am going to show you the solution as well as the different variations of this problem. So the simplest version is as follows.

So let us assume that we have two poles of different length. Let us say the first pole is length y_o and the second pole is of length y_1 and the two poles are separated by a distance d apart pole 1 has an x coordinate x_o and pole 2 has a x coordinate x_1 and between the two poles we have a rope which is hanging so that the end points of the rope are fixed on the top of these two poles.

And the shape of the rope is completely influenced by the action of gravity. So it is completely weight which is determining the shape of the pole. So the problem is as follows, let us consider a thin uniformly heavy flexible cable suspended from the top of the two poles as I have shown in the figure and the poles are of height y_o and y_1 respectively as shown in the figure again.

And these are spaced a distance of d apart. So the problem says we have to figure out the optimal shape of the cable between these two poles. So now the problem says that we have to find the optimal shape. So this question is what is it that we are trying to optimize. It turns out this we are trying to optimize the total energy of the system or we are trying to minimize the total energy of the system.

So as the cable is not moving the total energy is just the potential energy. So we see that the cable assumes a specific shape which makes the potential energy so I denote the potential energy by PE which makes this potential energy the minimum. So that is the governing factor of the shape of this cable. Now, further there are some simplifying assumptions we assume as a problem says we assume homogeneous mass per unit length.

And let us say that mass per unit length is given by m. So the cable is a homogeneous cable the mass is distributed homogeneous throughout the length of the cable and then further the gravitational constant is denoted by g. So in this case let us say our potential energy let us denote it by w_p. So in this case my potential energy w_p is denoted by the total summation or the total integral of mass times the gravitational constant time the height of the cable above the ground.

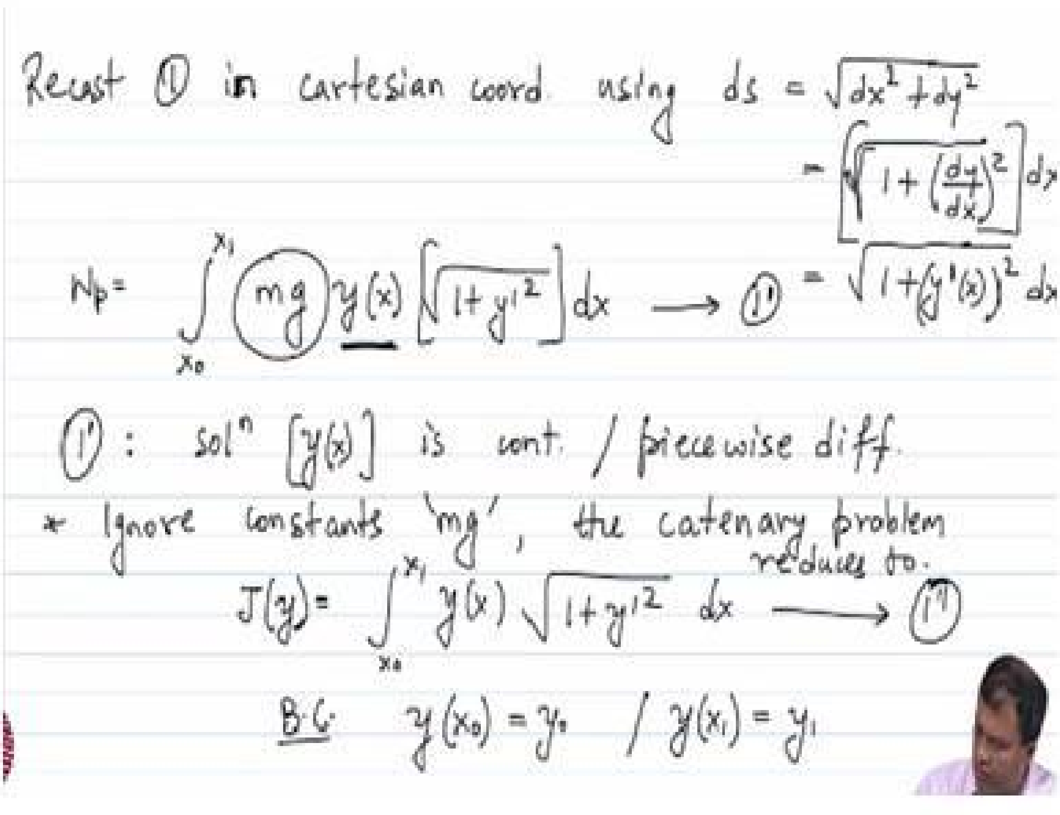
And we see that this particular integrating variable s is the arc length of the cable and this particular variable y is the height of the cable at s units along the length of the cable. Now then further we assume we assume that the length of the cable is fixed so in fact this potential energy is integrated over the entire length of the cable integral 0 to L i.e

$$w_p = \int_o^L mgy(s)ds \qquad \mathbf{1}$$

the simplest version of this problem we assume that this length although it is fixed, but it is an unknown in the problem.

Now, so this is the original we see that this particular let me denote it by **1**, so 1 denotes an integral of a function, y is the height function and w_p is the function or the integral of a function and hence w_p the potential energy is a functional. So we can then the next step of this problem is to recast this particular problem denoted by this **1** in a simpler form.

(Refer Slide Time: 18:45)



So, we recast **1** in the Cartesian coordinates. Let us say that let us say my x coordinate is along this direction and my coordinate is along this direction. So we recast our problem in the Cartesian coordinates using the fact that the length element ds is

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx = \sqrt{1 + (y'(x))^2}dx$$

’ is the derivative with respect to x. So, finally my potential energy now can be recast into the Cartesian variables. So we have in terms of x, end points in the x coordinates are x_o and x_1 and new functional is

$$N_p = \int_{x_o}^{x_1} mgy(x)[\sqrt{1 + y'^2}]dx \qquad \mathbf{1}'$$

We see that the unknown of the problem or what we are trying to optimize is or find the optimal solution is the solution to this y(x) which is the height of the cable at a distance x. So we assume one assumption

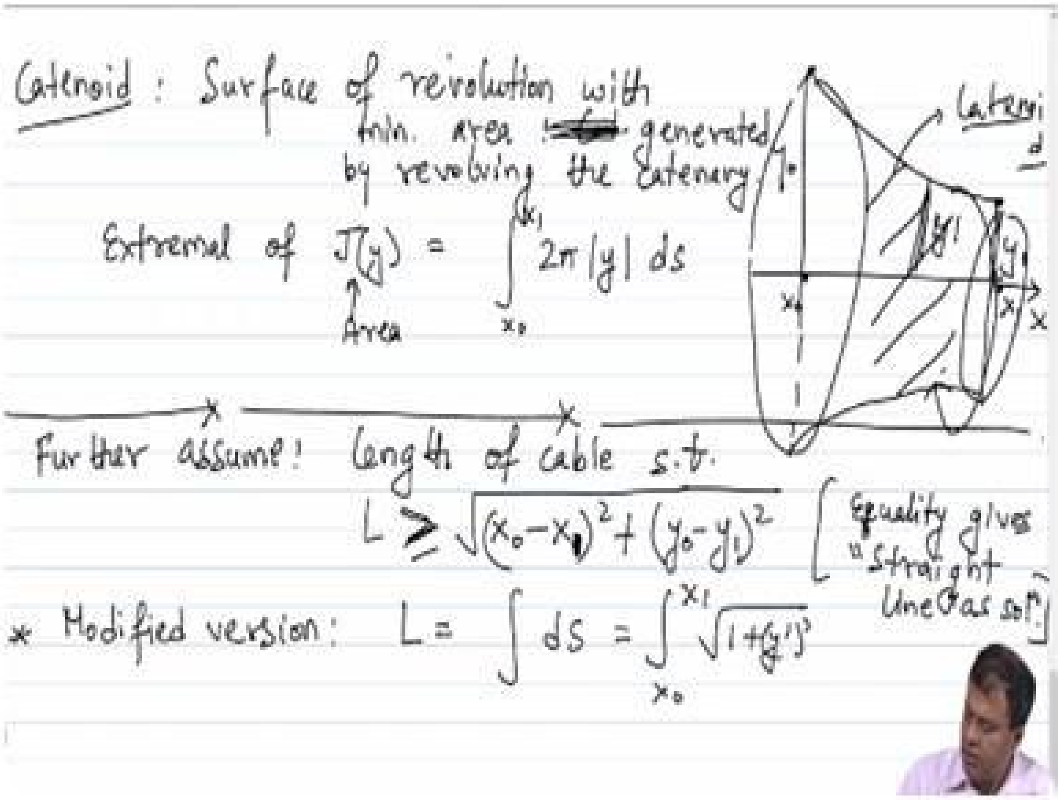
$_0$ in this optimization problem is we assume that the solution to **1** the solution y(x) is continuous and piecewise differentiable at the barest minimum.

And then further we can ignore some of the constants in the problem. We can ignore constants like mg so this is just a constant in the setup and we can take it equal to 1 or ignore it and in that case, the new case functional in this catenary problem reduces to the following functional. Let us denote this new functional by J(y)

$$J(y) = \int_{x_o}^{x_1} y(x)\sqrt{1 + y'^2}dx \qquad \mathbf{1}''$$

further this optimization is for fixed boundary conditions. So from now on I am going to denote my boundary conditions with this short notation B.C. Boundary conditions are $y(x_o) = y_o$ and $y(x_1) = y_1$. So this is the case of a fixed boundary condition problem. A similar and a more general version of this catenary problem is the problem of catenoid.

(Refer Slide Time: 23:08)



So what is the problem of catenoid? So let us again go back to our diagram of catenary. We have this rope which is hanging between the two cables y_o to y_1 and further suppose we are able to rotate this cable along this x axis. So we can see this it is going to generate a surface of revolution about the axis of rotation x and this particular surface that I am talking about this particular surface is the catenoid.

So catenoid is the surface of revolution with minimum area that is generated by revolving the catenary. So in this case the extremial we want to find the extremal of the following functional J of y which is my this is my area functional. So we are trying to minimize the area of the surface of revolution. So we are trying to minimize the area

$$J(y) = \int_{x_o}^{x_1} 2\pi|y|ds$$

where absolute value of y is the height of the catenary, ds is the lateral length of this cable or the arc length. So moving along one major assumption in the catenary problem is that we have to assume certain constraint on the length of the cable the length of the cable L has to be greater than or equal to the distance between the two points.

The length of the cable is has to be greater than or equal to the distance between the two points because if L is equal to this particular square root distance. We see that the solution to the catenary problem is just gives straight line as solution. So a modified version of this problem is we can assume that

Length of cable s.t $\quad L \geqslant \sqrt{(x_o - x_1)^2 + (y_o - y_1)^2}$ (equality gives straight line) and

$$L = \int ds = \int_{x_o}^{x_1} \sqrt{1 + (y'(x))^2}dx$$