**Variational Calculus and its Applications in Control Theory and Nano mechanics**
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**Lecture 14**
**Generalization/Numerical Solution of Euler Lagrange Equations Part 2**

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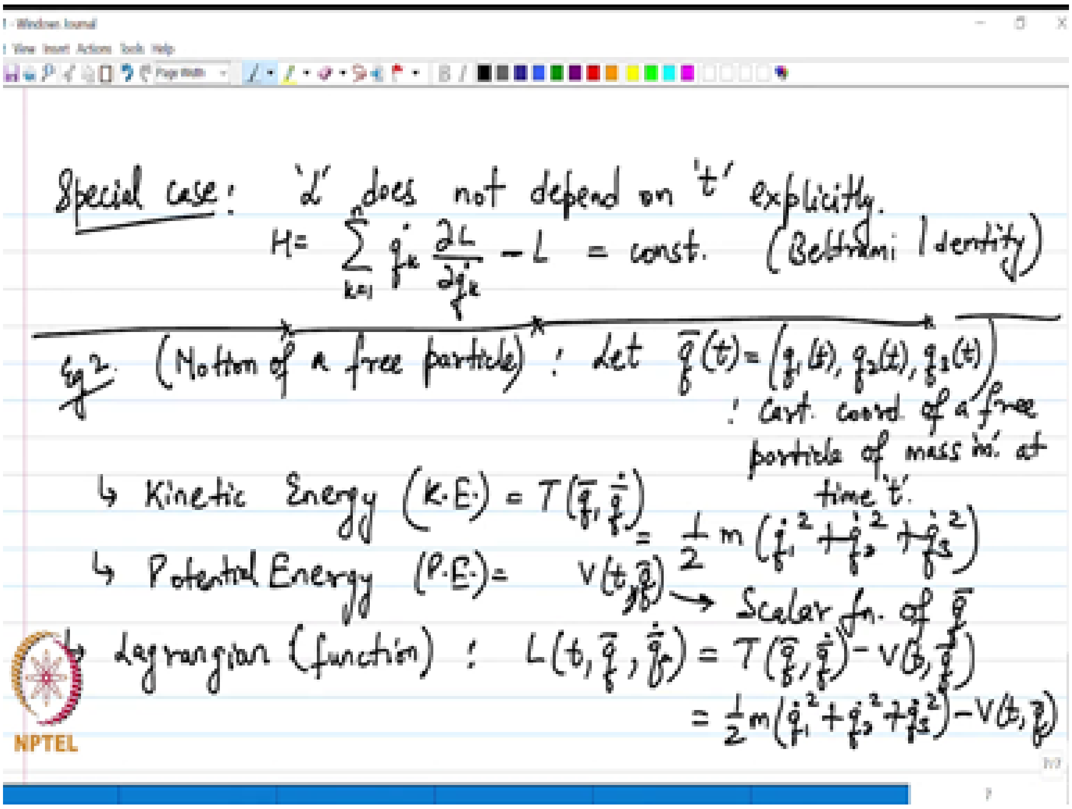


For the n variable case. So, let us look at an example of the special case scenario, example involves motion of free particle, to describe the motion of a free particle let us define the coordinate vector of the particle $\bar{q}(t) = (q_1(t), q_2(t), q_3(t))$ which we say that this is the Cartesian coordinate of a free particle of mass m at time t.

So, then in that case the motion is going to be described using the sum of its kinetic energy and the potential energy, let me define, describe it using this abbreviation K.E is defined by K.E $= T(\bar{q}, \dot{\bar{q}}) = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$

Potential energy denoted by the abbreviation P.E. we define P.E = V(t,$\bar{q}$) this potential energy is a scalar function, because this is an energy, so it must be a scalar function and also it can describe the forces.

The force is described by the gradient of this potential with respect to this coordinate vector, so this is standard Newtonian mechanics, the gradient of the potential with respect to the position gives us the net force on the particle. let us continue, we also now describe the so called Lagrangian, so what is Lagrangian?

Lagrangian is a functional which is defined by $L(t, \bar{q}, \dot{\bar{q}}) - V(t, \bar{q}) = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - V(t, \bar{q})$

in this particular example we will see that the Euler Lagrange equation reduces to the conservation of energy that is the the beautiful result in this example. So, let us look at the Euler Lagrange equation, but before that let me introduce a new concept.

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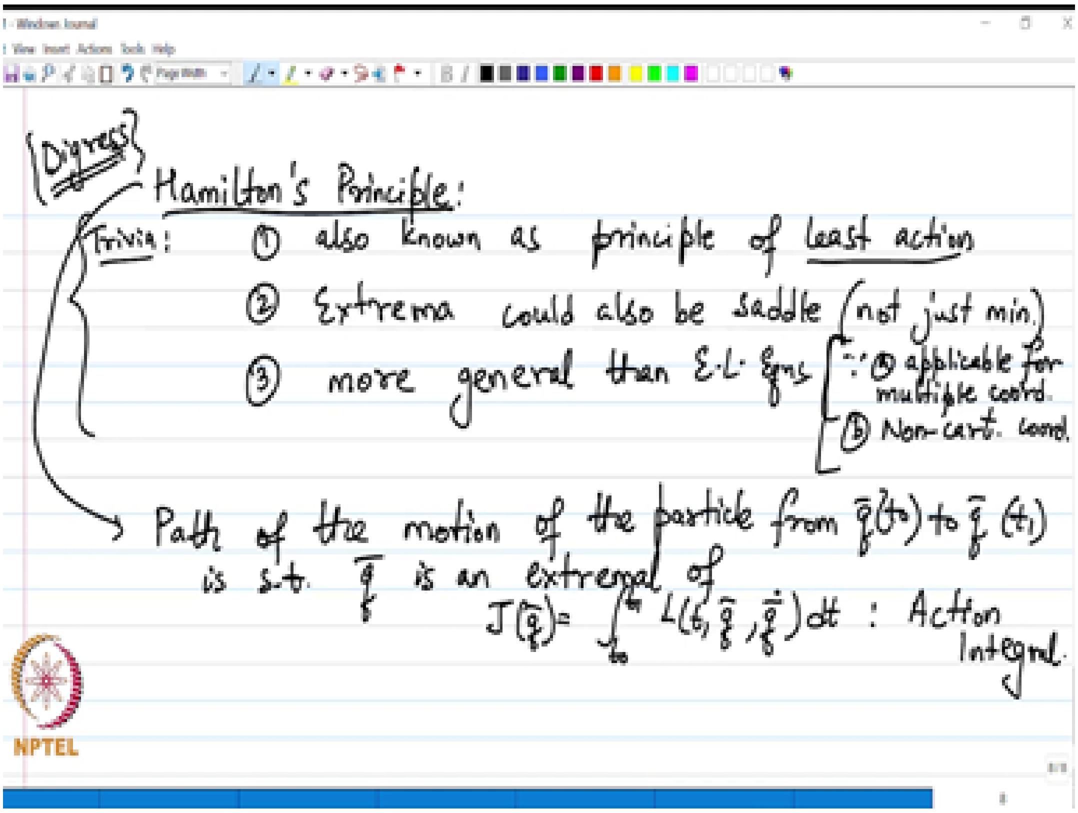

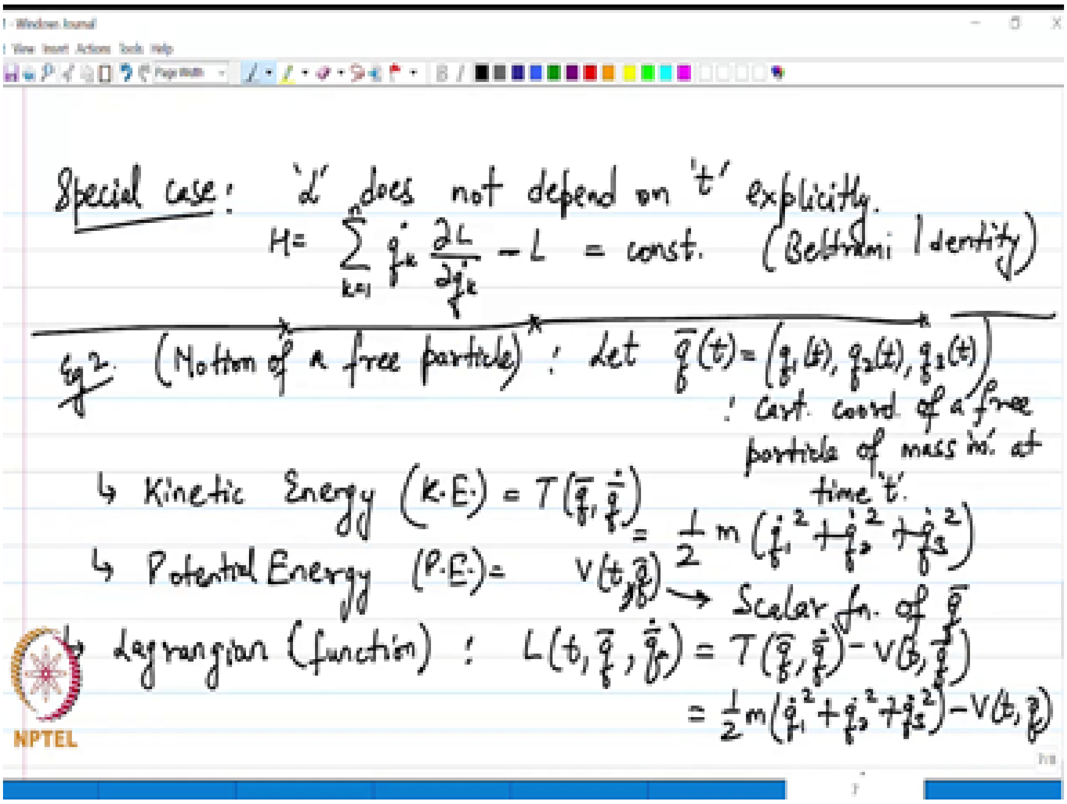


So, let us now digress, so there is a brief digress from our original discussion, let me now introduce the Hamilton's Principle, it is nothing but it is just the physic language of the variation calculus. So, let me write down, little bit of trivia about Hamilton's Principle. It turns out that Hamilton's Principle is also also known as the principle of least action.

All these words will become very clear as we move along our course discussion. So, another fact is that when we find the extremals for these examples and other similar physical examples, not necessarily the extremals will be either maxima or minima; the extremals could very well be saddle points. We are going to describe and distinguish between the different types of extrema later on when we describe the sufficient condition for the extremals.

So, what I just said is that finding extremal does not guarantee us to find max or min, so extrema could also be saddle point, not just minima and Hamilton's Principle is sometimes more advantageous than just looking at plain Euler Lagrange because it is more it is more general than Euler Lagrange equations because it is applicable for multiple coordinate system, so functions of several variables, dependent variables and it is also more often than not applicable for different coordinates, so in particular it is also applicable for non-Cartesian coordinates. So, that is some of the advantage of using Hamilton's Principle.

Now, coming back, what is Hamilton's Principle? Hamilton's Principle says that the path of the motion of the particle from $\bar{q}(t_o)$ to $\bar{q}(t_1)$, is such that $\bar{q}$ is an extremal of the form $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$ So, the path of the motion of the particle from point a to b is such that $\bar{q}$ is an extremal , this particular integral also known as the action integral.

So, the Hamilton's Principle is nothing but the Euler Lagrange equation stated in the words of physics. So, essentially this is the Euler Lagrange equation. So, moving on, let us go back to the previous slide, we are going to find the extremum for the motion of the free particle, we are going to use Hamilton's Principle and find the least action of the extremum of the action integral.

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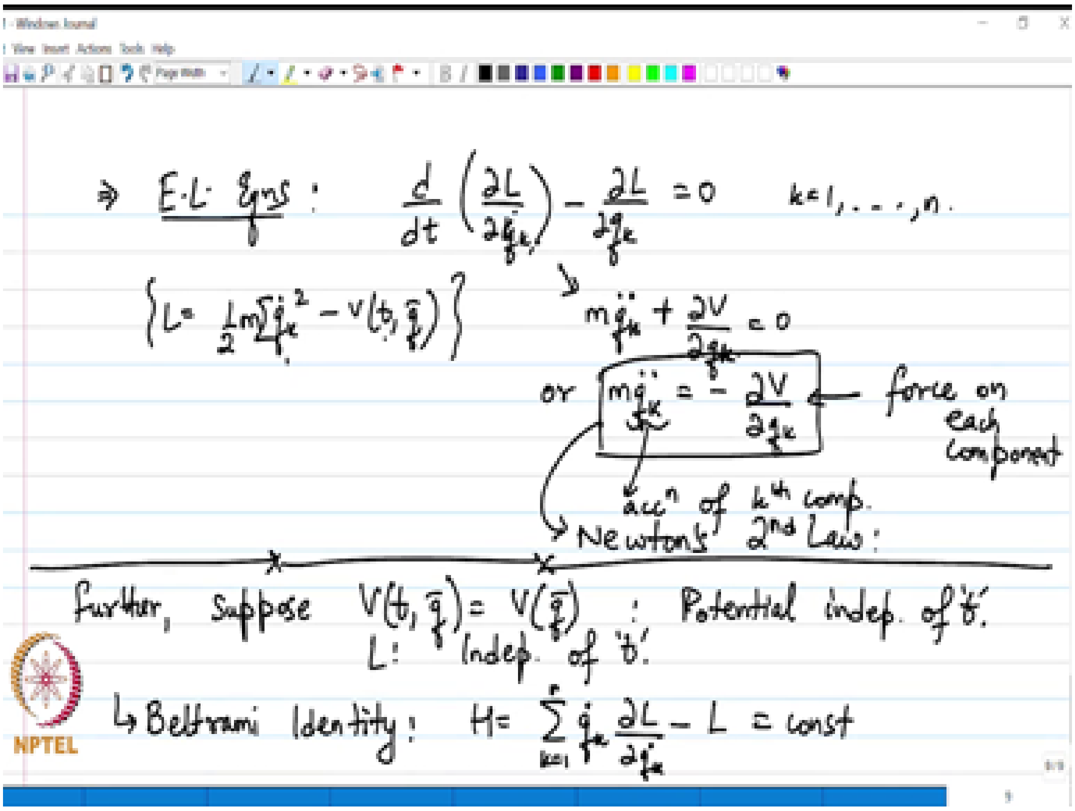

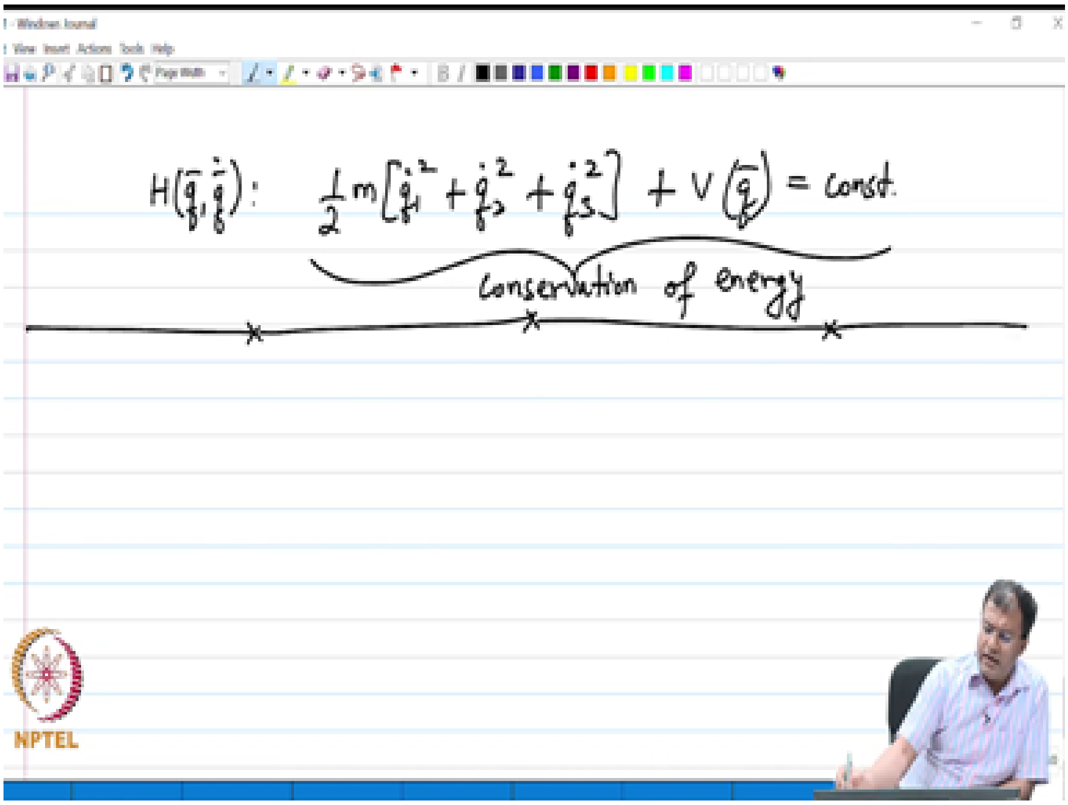


We are solving the Euler Lagrange equations and we are applying Hamilton's Principle, essentially Euler Lagrange equations for the n variable case, so when we do that we have the following set of equations, and this is for k = 1,.....,n Now let us now plug in the value of L, so notice that $L = \frac{1}{2}m\sum \dot{q_k}^2 - V(t, \bar{q})$.

Why minus? Because as we increase the coordinate of the particle, physically it is assumed that the potential energy decreases so it has an inverse relation, so L is the Lagrangian, so this, after plugging in the value of L, we get that this is also equal to $m\ddot{q_k} + \frac{\partial V}{\partial q_k} = 0, k = 1, .....n$

or I have $m\ddot{q_k} = -\frac{\partial V}{\partial q_k}$ and people who have done basic Newtonian mechanics, we see that the derivative of V with respect to q or the gradient of V with respect to q gives us the force on each component.

So essentially I am saying that and this particular quantity on the left hand side is nothing but the acceleration of the $k^{th}$ component. I am sure students can easily recognize that this equation is nothing but Newton's second law that is mass time's acceleration is equal to the net force on each component. So, we have essentially written Newton's second law while solving the Euler Lagrange equation.

So, the physics follows beautifully in this case, so suppose now we can look at a special case where our potential is independent of t, notice that t only appears, the independent variable t only appears in V, so suppose if the potential also independent of t, then my Lagrangian will be completely dependent on $\dot{q}$ and q, it is independent of the independent variable t. And then we could directly use the Beltrami Identity to reduce our Euler Lagrange.

So, what is just said is the following. So, further suppose, $V(t, \bar{q}) = V(\bar{q})$ which is a potential independent of t and the Lagrangian L will also be independent of t, . So in this case my Euler Lagrange reduces to the Beltrami Identity $H = \sum_{K=1}^{n} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L = \text{Constant}$

So, we just plug in the values of this function L and while we see that my Beltrami Identity reduces to

$$H(q, \dot{\bar{q}}) = \frac{1}{2}\left[\dot{q_1}^2 + \dot{q_2}^2 + \dot{q_3}^2\right] + V(\bar{q}) = \text{Constant}$$

Notice that the Beltrami Identity has very beautifully resulted into the conservation of energy. So, this statement is nothing but the conservation of energy. It says that the sum of the kinetic plus the potential energy is constant or conserved.

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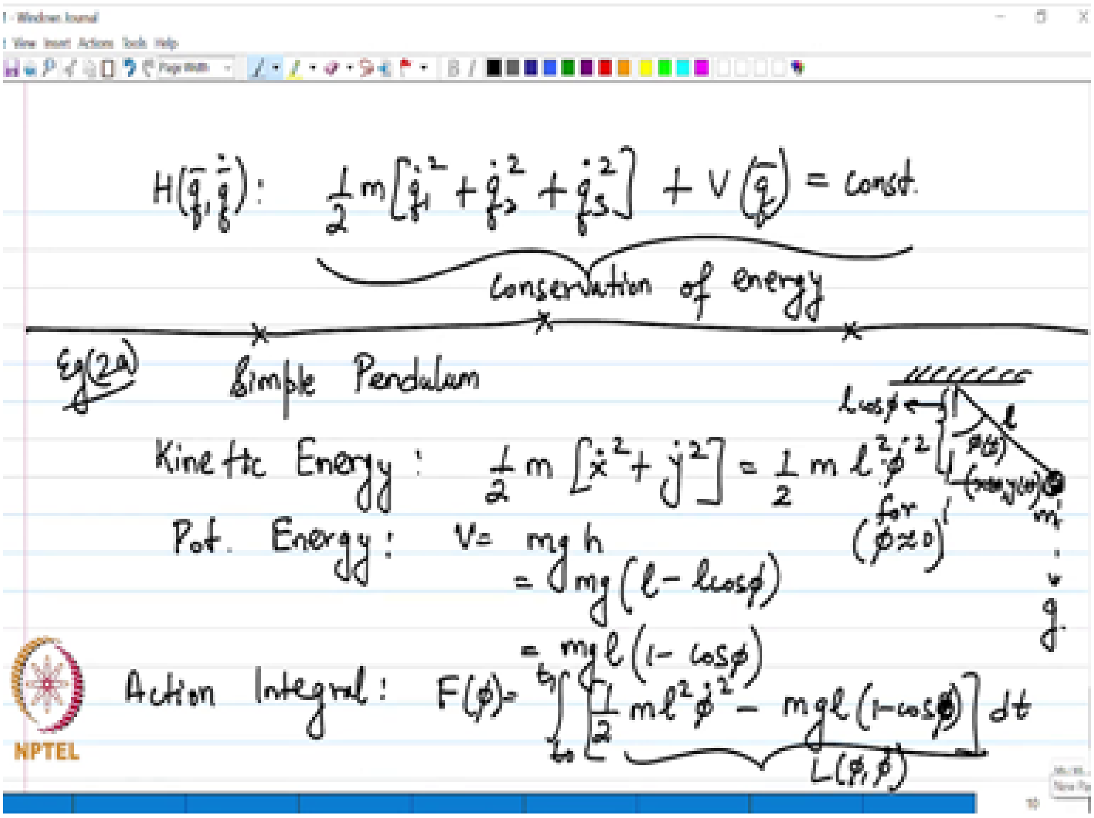


let us look at an related example of a not really a freely moving particle but a particle performing a certain motion

**Example (2a)**: we are going to look at the motion of a simple pendulum, so let us say I have a blob of mass m and it is hanging by a rope of length l, the rope has almost 0 mass, and the blob is under the influence of the gravity and further the coordinates of the blob are X(t), y(t) this is my coordinate of the blob

We see that in this case the motion of the simple pendulum, so we need to write all the different forms of energy, the kinetic energy $= \frac{1}{2}\left[\dot{x}^2 + \dot{y}^2\right]$.

we are assuming very small oscillation, well how did we get that, let me just briefly mention that, so if my $\phi$ is very small, if I represent x and y by its polar coordinates, I see that x $= L\sin\phi, y = L\cos\phi$ and for small $\phi$, $\sin\phi$ becomes $\phi$ and $\cos\phi$ becomes 1.

when we plug it in this expression we get kinetic energy $= \frac{1}{2}ml^2\dot{\phi}^2$, for small angle of oscillation. So, then my potential energy V = mgh $= mgl(1-\cos\phi)$, now I can write down my action integral $F(\phi) = \int_{t_o}^{t_1}\left[\frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)\right]dt$, so there is no explicit appearance of t in this Lagrangian

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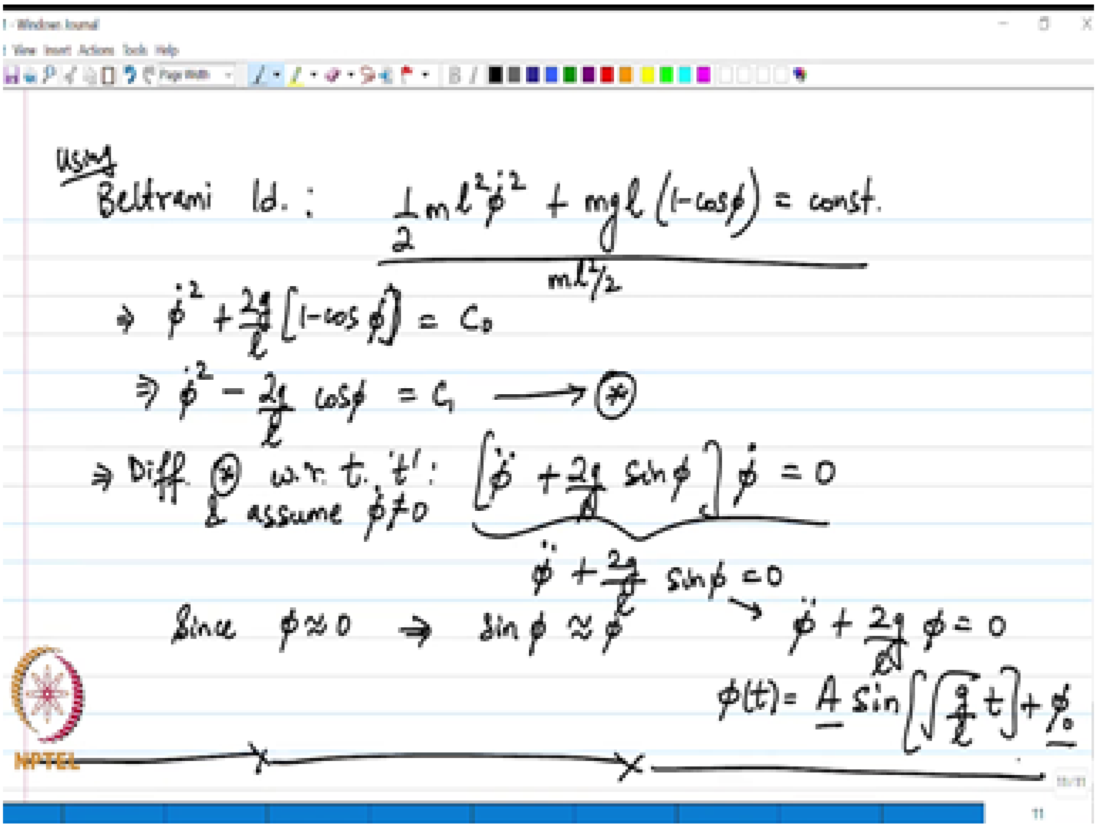


By using Beltrami Identity I see that $\frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi)$ = Constant, so that is my Beltrami Identity which is the sum of the kinetic energy plus the potential will be a constant or the conservation of energy. So, what it says is now that the extremal $\phi$ is going to satisfy this conservation of energy.

So, the next step involves trying to remove the constants as much as possible, so we divide throughout by $ml^2$ and we get the following expression $\dot{\phi}^2 + \frac{2g}{l}[1-\cos\phi] = C_o$ but again $\frac{2g}{l}$ is another constant, so $\dot{\phi}^2 + \frac{2g}{l}\cos\phi = C_1$ $\qquad$ *

We differentiate * with respect to Independent parameter t, we get $\left[\ddot{\phi} + \frac{2g}{l}\sin\phi\right]\dot{\phi} = 0$

we assume that $\dot{\phi} \neq 0$, otherwise we are going to get a trivial solution $\phi$ is a constant and that is something we are not after, so which means that from here we get that the solution reduces to the following equation $\ddot{\phi} + \frac{2g}{l}\sin\phi = 0$

since $\phi$ the oscillation angle is small, so since phi is very-very close to 0, we can reduce equation further by saying that $\sin\phi = \phi$ and then in this case the equation reduces to $\ddot{\phi} + \frac{2g}{l}\phi = 0$

So, I can write down the the solution $\phi(t) = Av\sin\sqrt{\frac{g}{l}}t + \phi_o$ So, it is a solution with two unknowns A and $\phi_o$ and these two unknowns are determined by the boundary conditions and that is where we can stop our discussion on this example.

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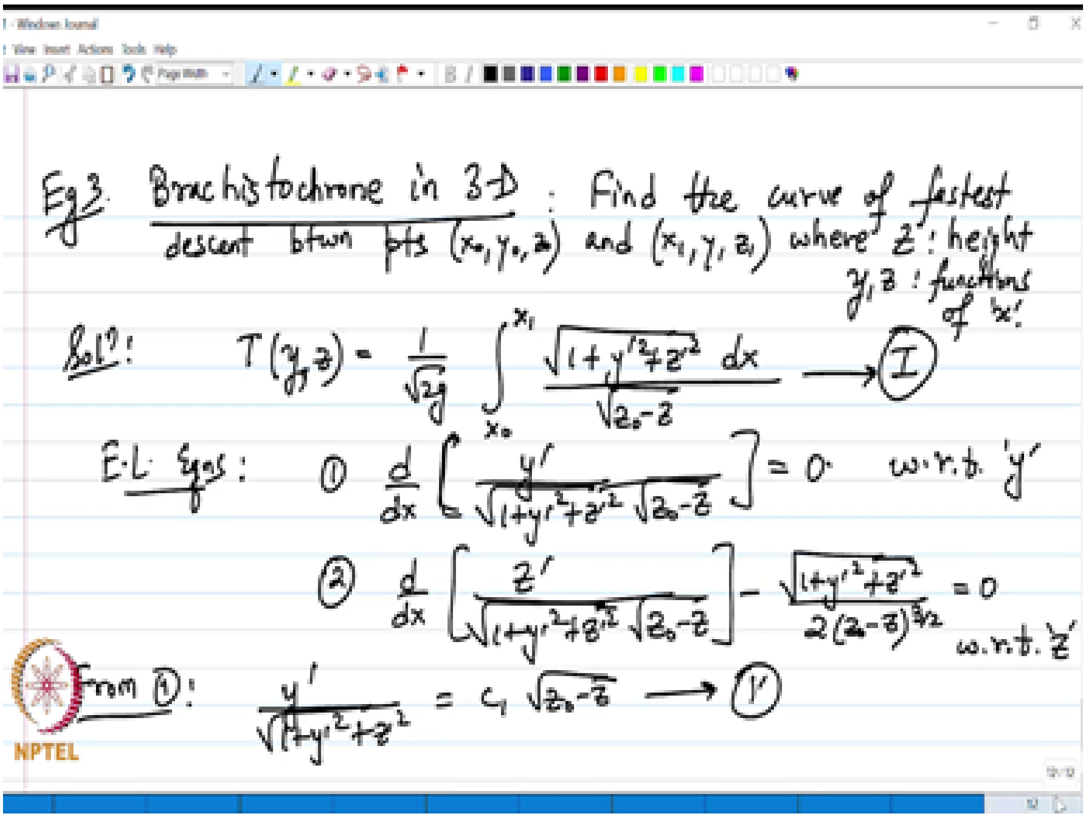


So, next we look at another example, namely the Brachistochrone . so, again what is the problem, the problem is to find the optimal curve in 3D, so that a bead sliding without friction reaches at the bottom of the curve in the minimum possible time, but this time the curve is in 3D, so the problem is find the points or the curve of fastest descent between the points $(x_o, y_o, z_o)$ and $(x_1, y_1, z_1)$ where my z is a height of the curve and I have 2 dependent variables y, z, they are functions of x, So, what I have is the following, so we see that to figure out the solution of the Brachistochrone in 3D we again write down the time functional

$$T(y,z) = \frac{1}{\sqrt{2g}} \int_{x_o}^{x_1} \frac{\sqrt{1+y'^2+z'^2}dx}{\sqrt{z_o - z}} \qquad \mathbf{I}$$

Then, well, of course, now we have two variables, two independent, two dependent variables and both are dependent on x, so which means that we are going to have two, sets of two Euler Lagrange equation, one for y, one for z. So, which means we have the following system of equation to solve. So, the first system is for y , we see that this integrant is only a function of $y'$.

So, which means that our Euler Lagrange equation is simple which is of this form

$$\text{w.r.t to y} \qquad \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2+z'^2}\sqrt{z_o - z}}\right] = 0 \qquad \mathbf{1}$$

$$\text{w.r.t to z} \qquad \frac{d}{dx}\left[\frac{z'}{\sqrt{1+y'^2+z'^2}\sqrt{z_o - z}}\right] - \frac{\sqrt{1+y'2+z'2}}{2(z_o - z)^{\frac{3}{2}}} = 0 \qquad \mathbf{2}$$

$$\text{From } \mathbf{1} \qquad \frac{y'}{\sqrt{1+y'^2+z'^2}} = C_1\sqrt{z_o - z}$$

Notice that **2** is a mess and it is not going to be easy but we note that **I** is also independent of x so we can definitely use Beltrami Identity for 2 variables, so what I just said is the following

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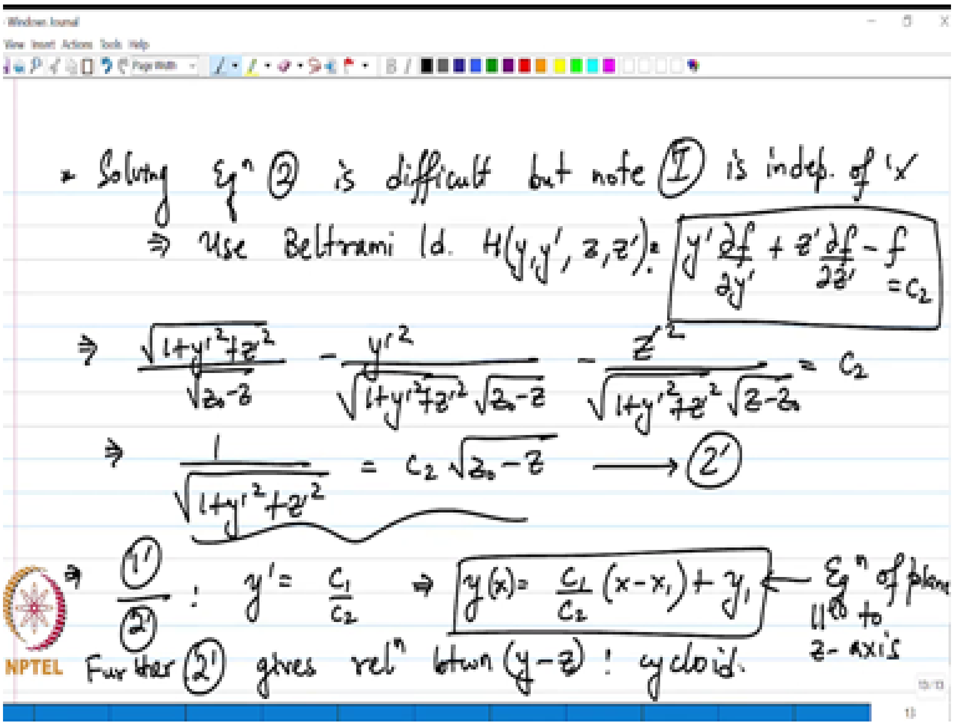


So, in this case we have the identity with respect to two variables, y, $y^{'}$, z, $z^{'}$, i.e

$$H(y, y^{'}, z, z^{'}) = y^{'}\frac{\partial f}{\partial y'} + z^{'}\frac{\partial f}{\partial z'} - f = C_2$$

Now we can directly substitute our f we see the following

$$\Rightarrow \frac{\sqrt{1 + y^{'2} + z^{'2}}}{\sqrt{z_o - z}} - \frac{y^{'2}}{\sqrt{1 + y^{'2} + z^{'2}}\sqrt{z_o - z}} - \frac{z^{'2}}{\sqrt{1 + y^{'2} + z^{'2}}\sqrt{z_o - z}} = C_2$$

We get the following expression, a huge expression which we eventually simplify, this is also equal to the constant $C_2$, so when we simplify after taking the necessary LCM and common denominator, the simplified version reduces to the following form

$$\Rightarrow \frac{1}{\sqrt{1 + y^{'2} + z^{'2}}} = C_2\sqrt{z_o - z} \qquad \mathbf{2}^{'}$$

$$\Rightarrow \frac{\mathbf{1}^{'}}{\mathbf{2}^{'}} : \frac{C_1}{C_2} \Rightarrow y(x) = \frac{C_1}{C_2}(x - x_1) + y_1$$

So, we get, for the variable y we again get the equation of a straight line, however, it does not contain any z which means z can take any value from $-\infty$ to $\infty$ So, what I am trying to say here is that this new equation is not the equation of a straight line because there is the variable z as well, so this is an equation of a plane parallel to the z axis, it could contain z axis but not necessarily.

So, this vertical or parallel to the z axis and then the further the further solution can be found by solving $\mathbf{2}'$ and we will see that the further solution will be a cycloid, this equation is nothing but the equation for a cycloid, so which means combining this solution which is in the box with the motion of the cycloid we conclude that the Brachistochrone in 3D leads to an extremal which is a cycloid in a plane parallel to the z axis, so what I said is the following. The further 2 prime gives relation between y and z which is going to be cycloid.

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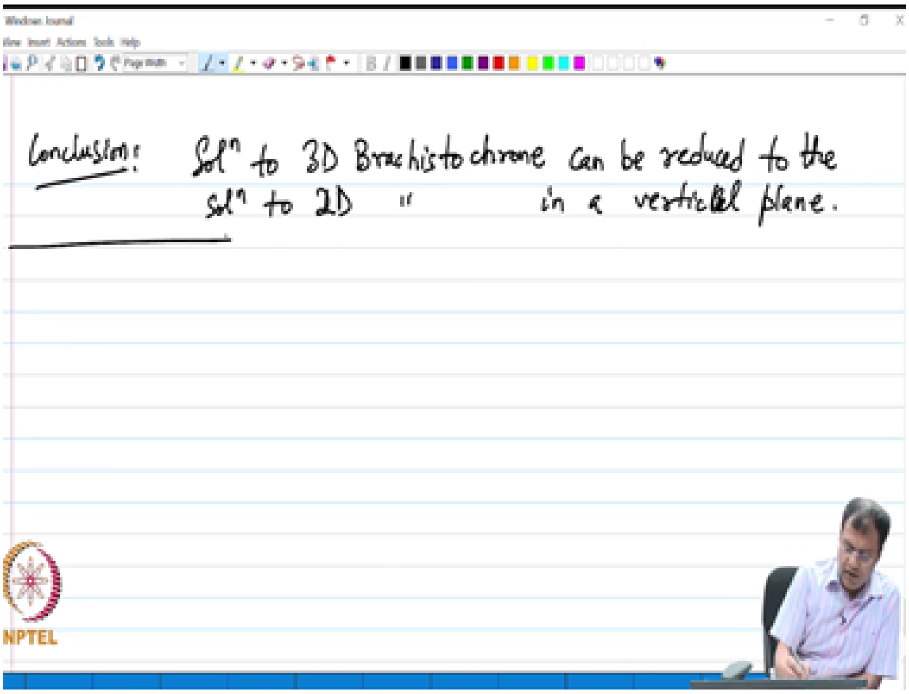


And so the conclusion, the following conclusion can be drawn that the solution to the 3D Brachistochrone can be reduced to the solution to the 2D Brechistochrone in a vertical plane or a plane parallel to the z axis.