Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 19 Isoperimetric Problems Part 1

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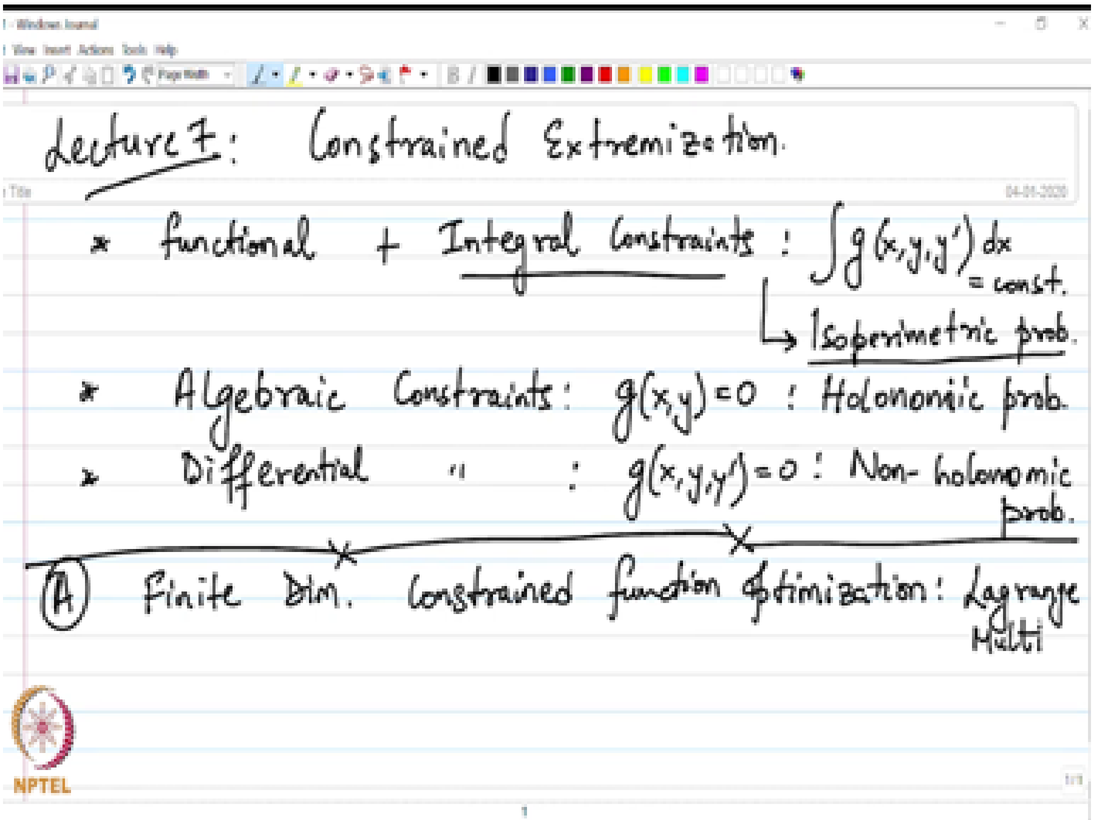

In this lecture series, I am going to talk about the topic of constrained extremization. So, namely I am going to introduce one such constrain namely the isoperimetric constrain. So, before I do that, let me just talk about the different type of constrained extremization that we are going to talk over the different lectures of this course.

We could either have a functional that we want to extremize plus we also have integral constraints of the form $\int g(x, y, y^{'})dx$ = constant, I call this class of problem as my Isoperimetric problem and then I also have functionals plus some problems having algebraic constraints and in that case, let us say we have the constraint of the form g(x, y) = 0.

The problems of this class we will denote them by the holonomic problems. The problems having constraints which are purely algebraic are the holonomic problems and finally the problems in which we have differential constraints, we call them as the non-holonomic problems and today in this lecture course, I am going to talk almost exclusively on isoperimetric problems or problems with functional optimization with integral constraints.

But before I talk about isoperimetric problem, I need to introduce some basics, we have to go back to our finite dimensional or multi variate calculus. So, let us now revise some of the basics and look at the various concepts that have already been discussed well part of them are already been discussed but part of them are now going to be discussed in finite dimensional calculus. So, finite dimensional constrained

constrained function optimization.

So, now people who are familiar with this finite dimensional constrained function optimization, they know that one of the standard way to perform this optimization is via the method of Lagrange multiplier. So, we are going to revise this method and to do that, let us start with the very basic constraint problem, namely problem with one constraint or single constraint problems. So, what I am saying is the following.

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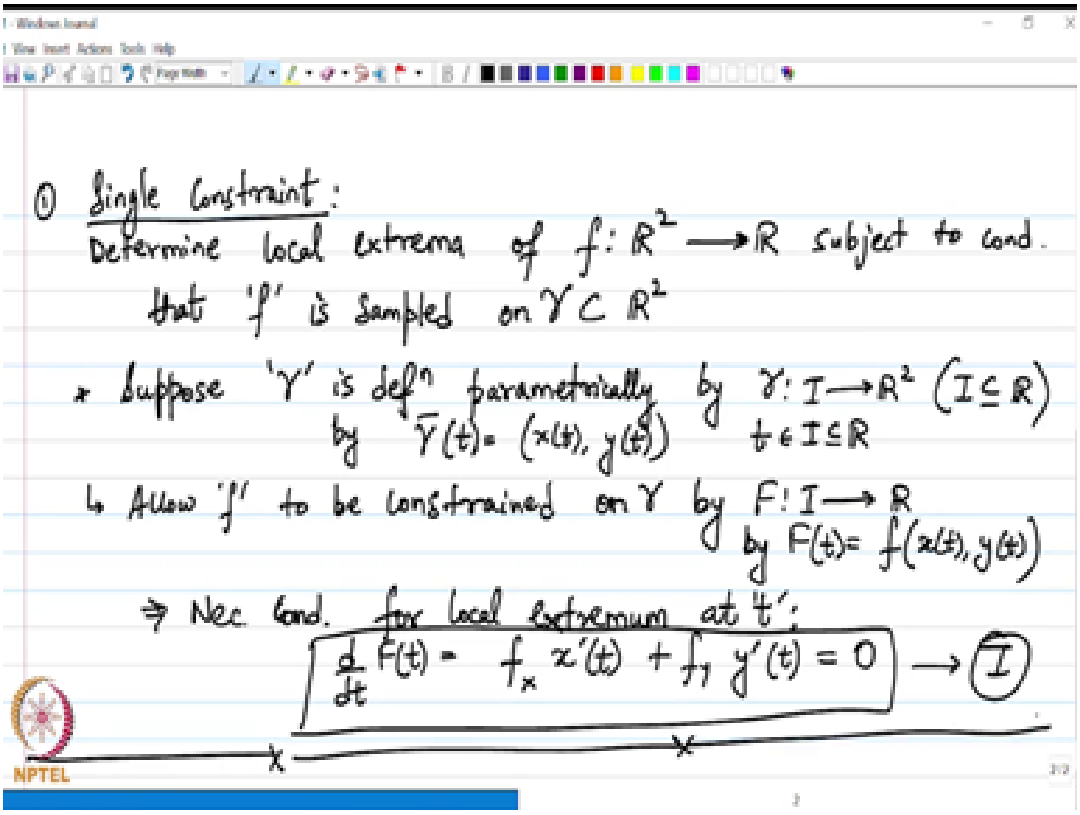

Let us start with our discussion with the single constraint problems, the problem here is to determine local extrema of the function f : $\mathbb{R}^2 \to \mathbb{R}$ subject to the condition that f is sampled on $\gamma \subset \mathbb{R}^2$

Suppose that my gamma can be defined with respect to some parameter by $\gamma : I \to \mathbb{R}^2$ $(I \subseteq \mathbb{R})$, I is a subset of the real axis, so my all parameter are real numbers. let us say by $\bar{\gamma}(t) = (x(t), y(t))$ $t \in I \subseteq \mathbb{R}$.

Then we allow in this problem, we allow f our objective function from $\mathbb{R}^2$ to $\mathbb{R}$ to be constraint on γ by the following function capital F : $I \to \mathbb{R}$ by the function F(t) = f(x(t), y(t)).

The moment we say that f is constrained with the condition γ, then all the argument points of this objective function small f has to be picked from γ. So, then my necessary condition for local extremum at point t is given by $\frac{d}{dt}F(t) = f_x x^{'}(t) + f_y y^{'}(t) = 0$ $\mathbb{I}$
Just used the chain rule to see that the derivate of F. To find the local extremum, this must be set equal to 0. So, that is the basic criteria for the constrained optimization where the points are picked from the constraints set γ and then the problem then reduces to that of the unconstraint version.

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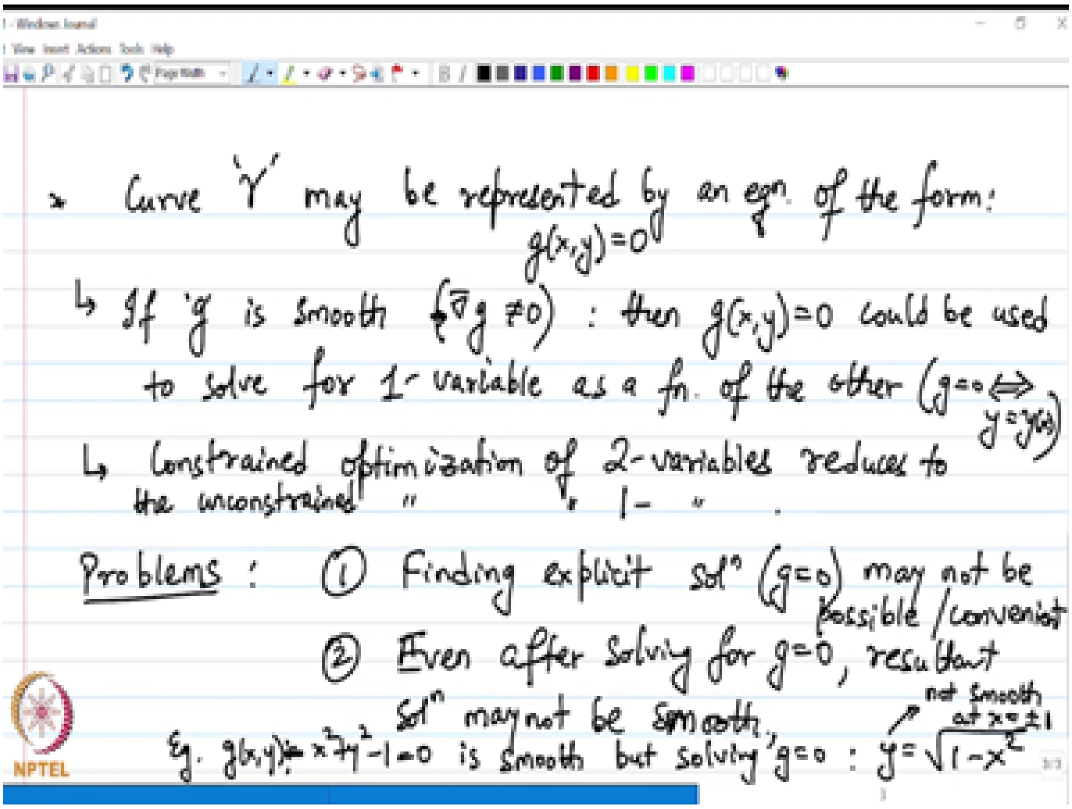

let us say we represent the curve γ is a constrained curve by some function, represented by an equation of the form g(x, y) = 0 which means that if g is smooth, when I say smooth then it means that the derivative exists and they are non-zero i.e $\bar{\nabla} g \neq 0$, that is what we are assuming, then my condition g(x, y) could possibly be solved to represent one variable y with respect to the other variable x, so the idea is, it seems that my constrained problem can be reduced to an unconstrained problem through the use of the constraint g(x, y) and I am just writing some of these steps but the question is, is it really true?

Can we really represent or change a constrained problem into an unconstrained problem without suffering any setbacks. So, this question is something that we will answer over the course of time. let us say now g(x, y) could be used to solve for one variable as a function of the other.

Using g = 0, I can very well express y as a function of x that is something I am assuming and if that is the case if that is the case then the constrained optimization of two variables reduces to the unconstrained optimization of one variable.

It seems the picture is very rosy. There is no point in talking about constrained optimization at least in finite dimensional calculus. But certainly it is not true. We can encounter many problems. so let me list some problems in this in this methodology, well the first problem is obvious not necessarily we could solve g(x, y) = 0 to represent y as a function of x. So, finding explicit solution of g = 0 may not may not be possible or convenient I would say convenient, at least analytically and then the second problem is even after solving for g = 0, the resultant solution may not be smooth.

Let me just highlight this particular case this problem example let us take g(x, y) $= x^2 + y^2 - 1 = 0$ so, let us take this function and this constraint as follows. So, this constraint is all the point x and y which lies on the unit circle or the boundary of the unit circle although this constraint is smooth that is we can find the derivatives of g with respect to x and y and they do not vanish on the boundary of the circle.

But but solving for g so what happens when we solve g = 0. we see that, we definitely get y as a function of x, I am just showing the positive branch we see that y as a function of x is not smooth at certain points. So, this is not smooth at $x \pm 1$ So, the alternative is the method of Lagrange multiplier.

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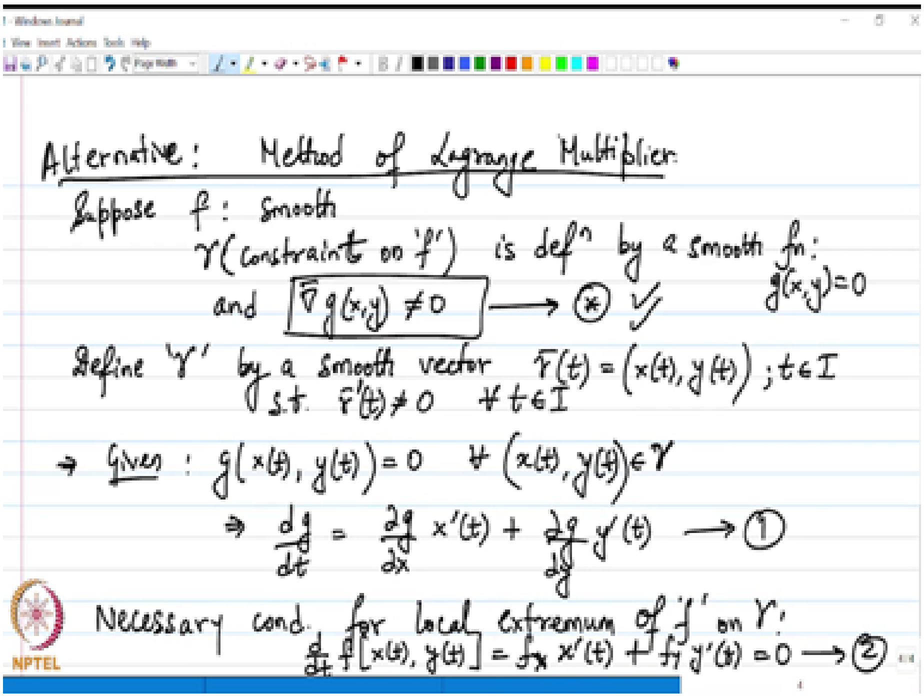

The alternative to avoid all this problems is to use the method of Lagrange Multiplier. Well we will see that even this method encounters some problem and we are going to segregate those class of problems also known as the abnormal problems. But let us look at the normal problems first.

Suppose f is smooth and the curve γ which is the constraint on f is defined by a smooth function given by g(x, y) = 0 and I have that further $\bar{\nabla} g(x, y) \neq 0$ $\quad *$
we need that otherwise we will run into trouble, we will see later on.

let me also further define γ by a smooth vector, I define $\bar{\gamma}(t) = (x(t), y(t)); t \in I$ such that $\bar{\gamma}^{'}(t) \neq 0 \; t \in I$ because if it is zero for some t which means both x(t), $x^{'}(t)$ and $y^{'}(t)$ are 0 and then we will not satisfy this stared condition that is the $\bar{\nabla} g(x, y) \neq 0$ since it is given that I have the constraint g is the constraint. So, I am representing the constraint with respect to the parameter t, it is given that this constraint is zero $\forall \; (x(t), y(t)) \in \gamma$

$$\Rightarrow \frac{dg}{dt} = \frac{\partial g}{\partial x} x^{'}(t) + \frac{\partial g}{\partial y} y^{'} \qquad \mathbf{1}$$

So, we also know that the necessary condition for local extremum of f on γ gives us the following condition
$\frac{d}{dt} f(x(t), y(t)) = f_x x^{'}(t) + f_y y^{'}(t) = 0$ $\quad$ **2**
that is my necessary condition for the extremum to occur. Now, we know that the $\bar{\nabla} g$ from * does not vanish.

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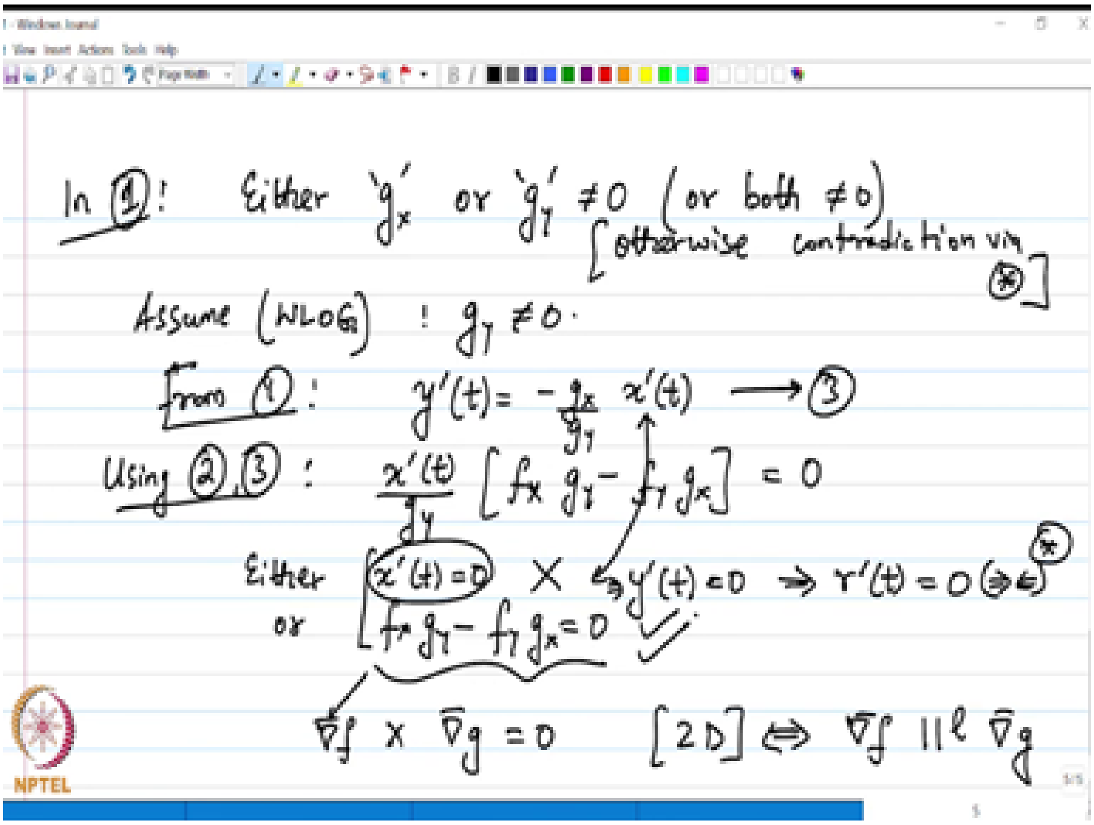

Look at look at this quantity **1**, I have used both the derivative of g with respect to x and y. So, we assume that either g(x) or g(y) $\neq 0$ or otherwise we will have a contradiction via the * i.e the $\bar{\bigtriangledown} g \neq 0$

So, either one of them or both of them are non 0. So, let us assume without loss of generality that one of them is non 0. So, let us say that $g_y \neq 0$ which means from **1** $y^{'}(t) = \frac{-g_x}{g_y} x^{'}(t)$ **3**
I have represented one derivative with respect to the other derivative.

Using **2** and **3**: $\frac{x^{'}}{g_y}\left[f_x g_y - f_y g_x\right] = 0$ so, either we have that $x^{'}(t) = 0$, I am sure it is clear that this is not possible. This condition is not possible because if $x^{'}(t) = 0$, then from this relation I can immediately see that $y^{'}(t) = 0 \Rightarrow \gamma^{'}(t) = 0$ that leads to a contradiction via our result *, where we say that the gradient of the constraint is non-zero. So, we lead to a contradiction. So, certainly only this is possible.

Now what is this condition? This condition is nothing but $\bar{\bigtriangledown} f \times \bar{\bigtriangledown} g = 0$, we are talking in 2D or in other words it is equivalent to saying that $\bigtriangledown f \parallel \bar{\bigtriangledown} g$

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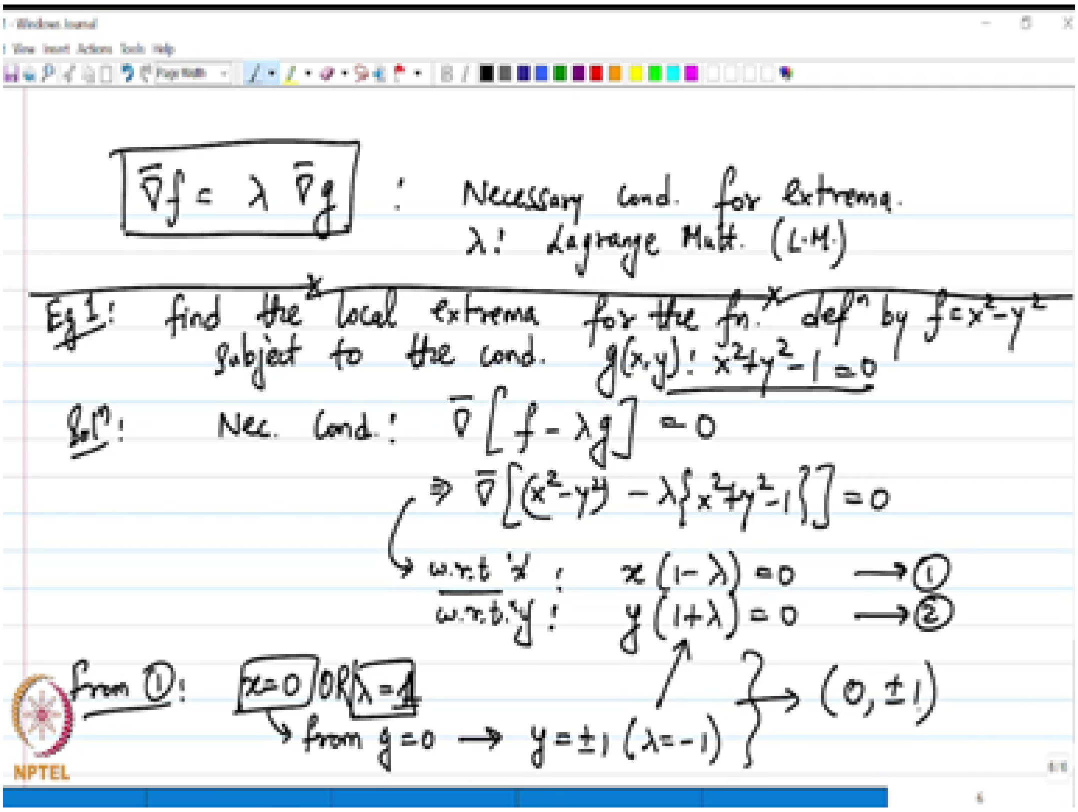

What I am saying is that the vector $\bar{\bigtriangledown} f = \lambda \bar{\bigtriangledown} g$ and this is the primary statement of the Lagrange multiplier for extrema, where λ constant of proportionality is Lagrange multiplier or the LM method, then let us look at an example to see how this method works

Notice that the Lagrange multiplier has introduced another constant, that is an unknown constant λ. So, how to evaluate this extra constant? Well the answer lies that we have also the constraint g = 0 and that will take care of this extra unknown.
let us look at an Example: Find the local extrema for the function defined by f = $x^2 - y^2$ subject to the condition $g(x, y) = x^2 + y^2 - 1 = 0$

So, we have to extremize this function subject to the fact that x and y lies on the boundary of the unit circle. Well we can directly start with the Lagrange condition, the necessary condition is the grad will boil down to the 2 dimensional derivative $\quad \bigtriangledown[f - \lambda g] = 0$
$\Rightarrow \bar{\bigtriangledown} \left[(x^2 - y^2) - \lambda(x^2 + y^2 - 1)\right] = 0$

When we take the necessary derivatives with respect to x we get $x(1 - \lambda) = 0$ **1**
and with respect to y we get $y(1 + \lambda) = 0$ **2**
now we have 3 unknowns x y and λ and then the third equation is this constraint which solves for the third unknown.

To find the solution, notice that from **1** either x = 0 or or $\lambda = 1$, if x = 0 I can plug the value of x= 0 in the constraint. So, from the condition g= 0, I get that $y = \pm 1$ Notice that we get value of x and y but we have not spoken about what is the value of lambda?

If $y = \pm 1$ I get from the second equation, I get that $\lambda = -1$ So, one such point or two such points that we have found from this exercise is $(0, \pm 1)$ and the other equation that can be found is by assuming $\lambda = 1$.

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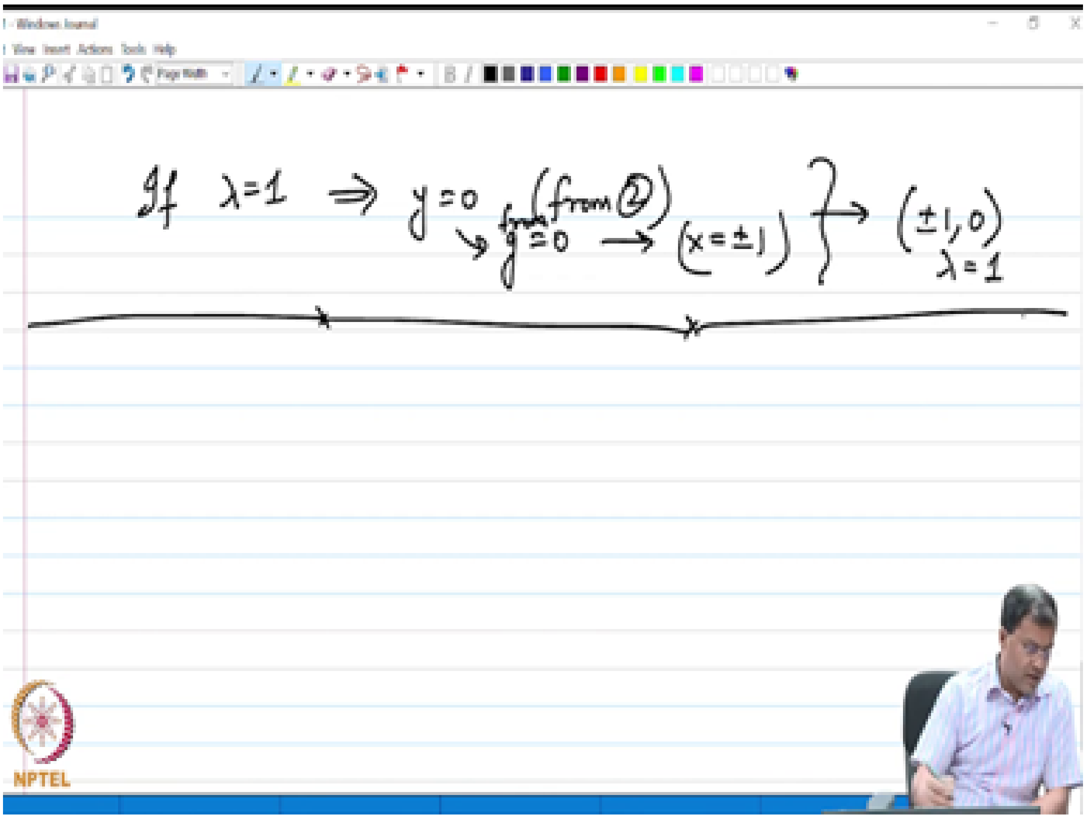

If $\lambda = 1$ then we must have that the second equation is satisfied only when y = 0. So, we get it implies that y = 0 from from **2** and putting this in the constraint, so from my constraint I get that x = ±1 and If y = 0, $x = \pm 1$ and the second set of solutions that I am getting from this exercise is $(\pm 1, 0)$ with $\lambda = 1$..

This example clearly highlights the applicability of Lagrange multiplier method. let us now state this result of Lagrange multiplier in the form of a theorem that we are going to use frequently. By the way Lagrange multiplier can be extended to higher dimensions.