Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture 20 – Isoperimetric Problems Part 2

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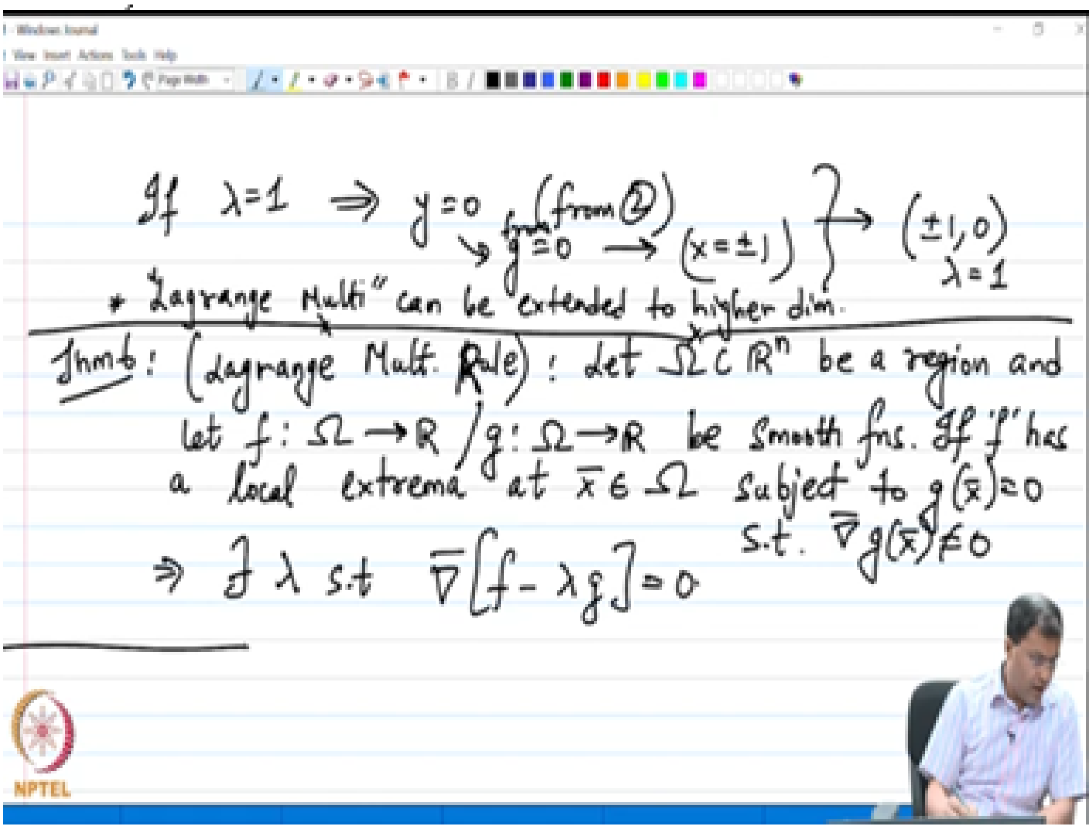

Theorem 6 (Theorem of the Lagrange, Lagrange Multiplier Rule) let $\Omega \in \mathbb{R}^n$ be a region and let $f : \Omega \to \mathbb{R}$ or $g : \Omega \to \mathbb{R}$ be smooth functions and if f has a local extrema's at $\bar{x} \in \Omega$ subject to $g(\bar{x}) = 0$ s.t $\bar{\bigtriangledown} g(\bar{x}) \neq 0$

$$\Rightarrow \ \exists \ \lambda \text{ s.t } \ \bar{\bigtriangledown}\left[f - \lambda g\right] = 0$$

this rule is true for $\mathbb{R}^n$ where n is any integer. So what about the scenario when we have multiple constraint, more than one constraint? It turns out that Lagrange Multiplier works there as well.

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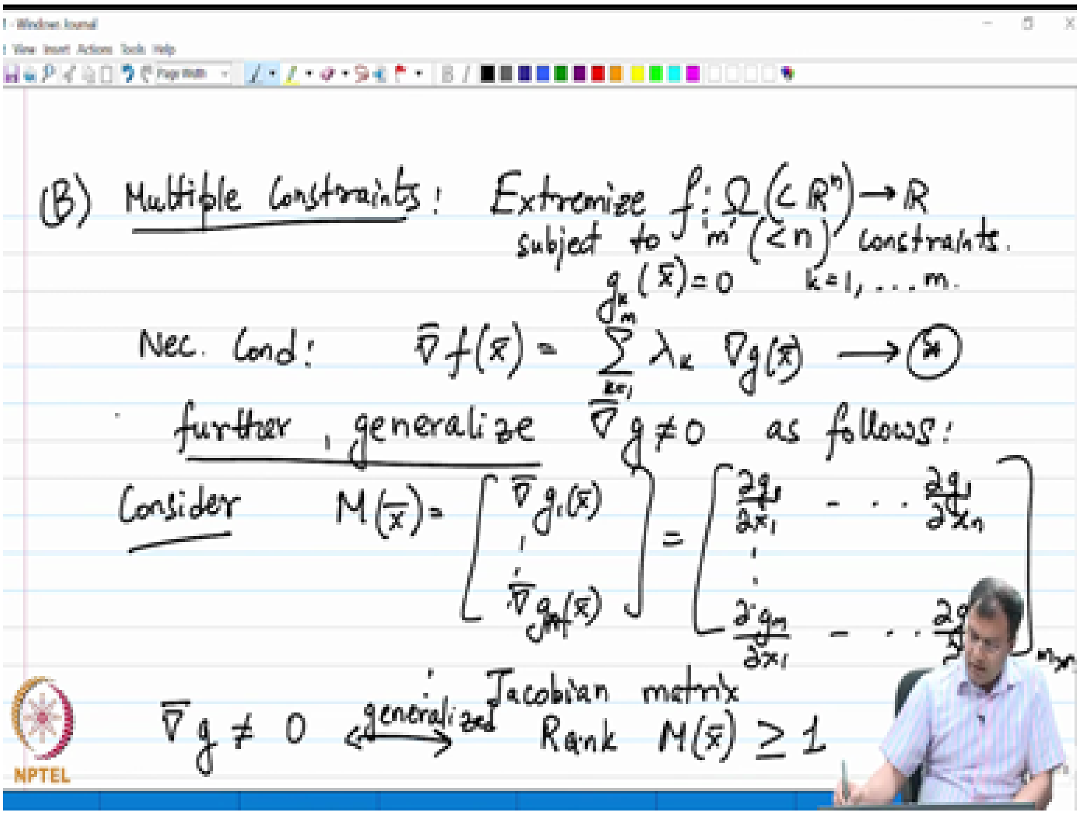

let me highlight with an example, in the case of multiple constraints, we are going to extremize $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ subject to m which is typically less than n or the dimension of the space m constraints, So $g_k(\bar{x}) = 0 \ \ k = 1, 2, ...m.$

Necessary condition in this case given by Lagrange reduces to the following
$\bar{\Omega} f(\bar{x}) = \sum_{k=1}^{n} \lambda_k \bigtriangledown g(\bar{x})$ $\qquad *$
further we need to generalize the condition that the $\bar{\bigtriangledown} g \neq 0$ as follows.

To look at the fact, now we are talking about problem having multiple constraints and it could be a problem in $\mathbb{R}^n$ let us say the general situation where x is a vector, so we have multiple constraint and multiple dimensions, so to generalize this condition that $\bar{\bigtriangledown} g \neq 0$, let us consider the following matrix. So, consider this matrix

$$M(\bar{x}) = \begin{bmatrix} \bar{\bigtriangledown} g_1(\bar{x}) \\ . \\ . \\ \bar{\bigtriangledown} g_m(\bar{x}) \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & . & . & \frac{\partial g_1}{\partial x_n} \\ . & . & . & . \\ . & . & . & . \\ \frac{\partial g_m}{\partial x_1} & . & . & \frac{\partial g_m}{\partial x_n} \end{bmatrix}_{m \times n}$$

known as the augmented and also known as the Jacobian matrix, students who have done bit of multi-variate calculus can immediately tell me that generalizing the condition $\bigtriangledown = 0$ would lead to the fact that the Jacobian will be such that the determinant is nonzero.

Not only that, we need something more. So, first of all, let us say, to generalize a condition that $\bar{\bigtriangledown} g \neq 0$, it is equivalent to saying that the rank $M(\bar{x}) \geq 1$, which means at least one component of $\bar{\bigtriangledown} g \neq 0$. So there is at least nonzero row of this Jacobian matrix.

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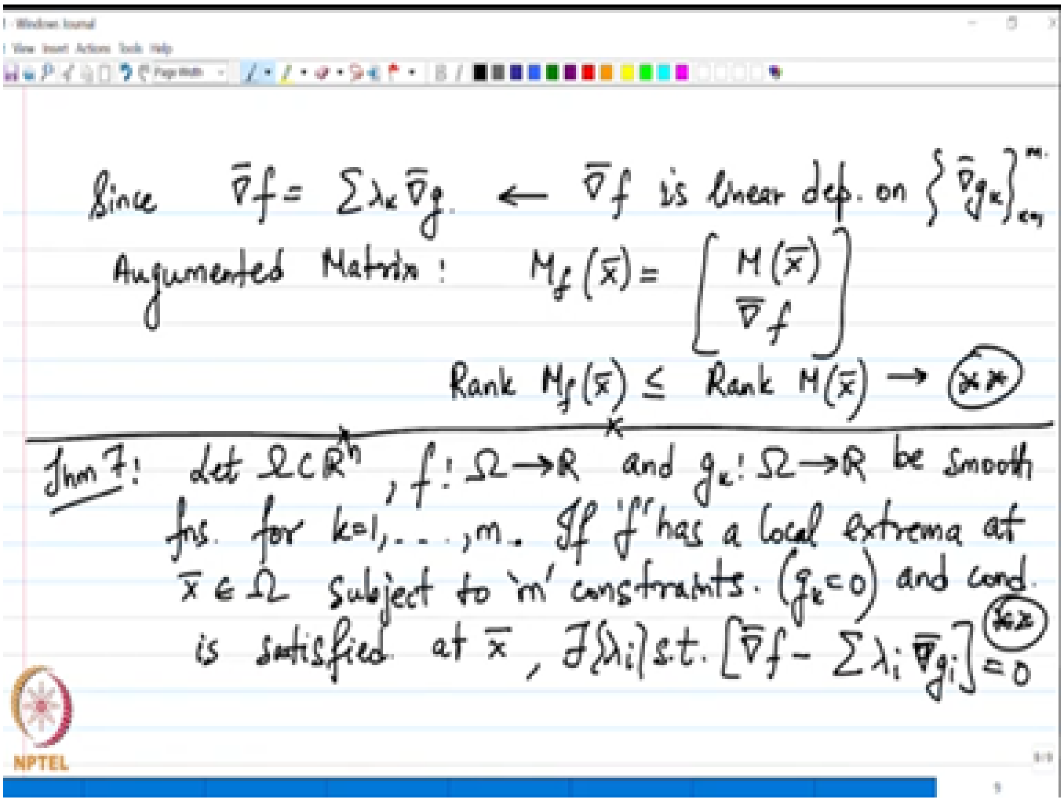

Since $\bar{\bigtriangledown} f = \sum \lambda_k \bar{\bigtriangledown} g$ so that is why the Lagrange Multiplier Method, it also tells me that $\bar{\bigtriangledown} f$ is linearly dependent on the family $\{\bar{\bigtriangledown} g_k\}_{k=1}^{m}$. It is linearly dependent on the set of vectors which means that look at this augmented matrix: $M_f(\bar{x}) = \begin{bmatrix} M(\bar{x}) \\ \bar{\bigtriangledown} f \end{bmatrix}$

Now what can we say for this augmented matrix is that since the last row is linearly depended on all the above rows, which means that the rank $M_f(\bar{x}) \leq \text{Rank} \;\; M(\bar{x}) \qquad\qquad **$ because the last row is linearly dependent, so, we are ready to state the result for Lagrange Multiplier with multiple constraints.

Theorem : Let $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ be smooth functions for k = 1,.....m. If f has local extrema at $\bar{x} \in \Omega$ Subject to m constraints which are given by $g_k = 0$ and the condition ** is satisfied at $\bar{x}$, then $\exists \{\lambda_i\}$ such that $\left[\bar{\bigtriangledown} f - \sum \lambda_i \bar{\bigtriangledown} g_i\right] = 0$

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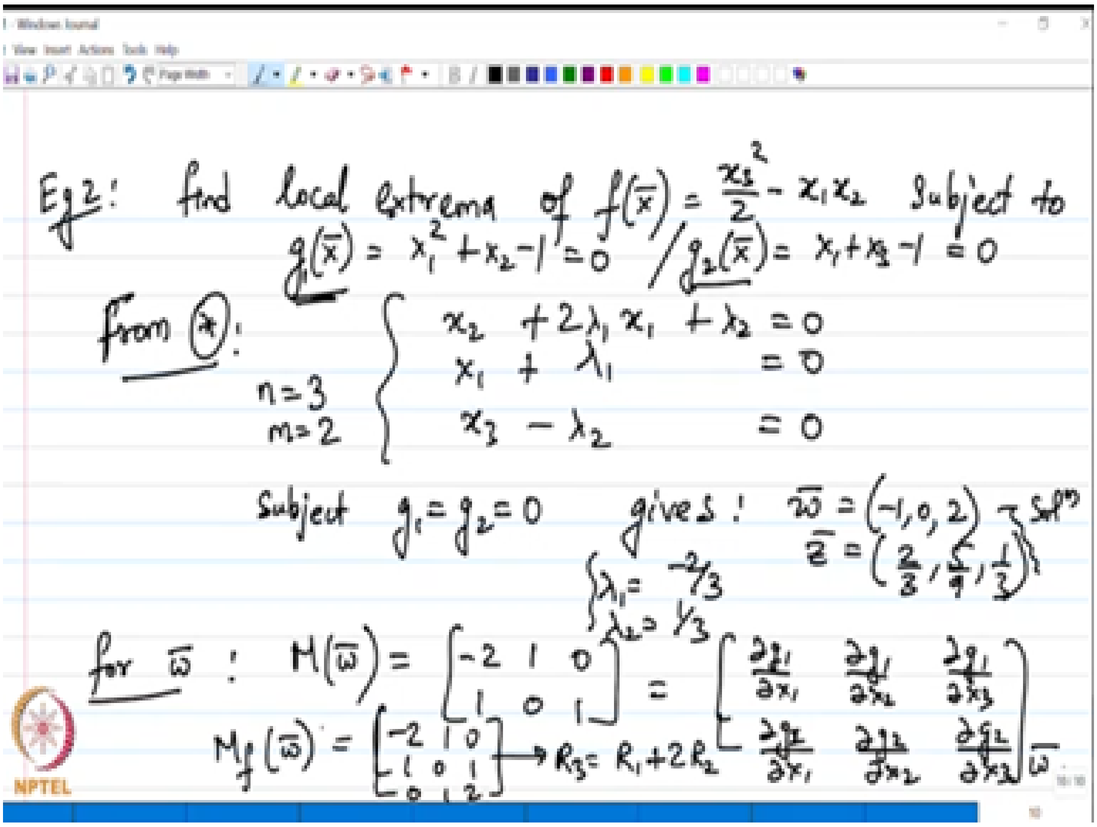
Eg 2: find local extrema of f(x̄) = x3²/2 − x1x2 subject to
g(x̄) = x1² + x2 − 1 = 0 / g2(x̄) = x1 + x3 − 1 = 0
From (*):
n = 3
m = 2
x2 + 2λ1x1 + λ2 = 0
x1 + λ1 = 0
x3 − λ2 = 0
Subject g1 = g2 = 0 gives: w̄ = (−1, 0, 2) Solⁿ
z̄ = (2/3, 5/9, 1/3)
λ1 = −2/3
λ2 = 1/3
for w̄: M(w̄) = [−2 1 0; 1 0 1] = [∂g1/∂x1 ∂g1/∂x2 ∂g1/∂x3; ∂g2/∂x1 ∂g2/∂x2 ∂g2/∂x3] w̄
Mf(w̄) = [−2 1 0; 1 0 1; 0 1 2] → R3 = R1 + 2R2
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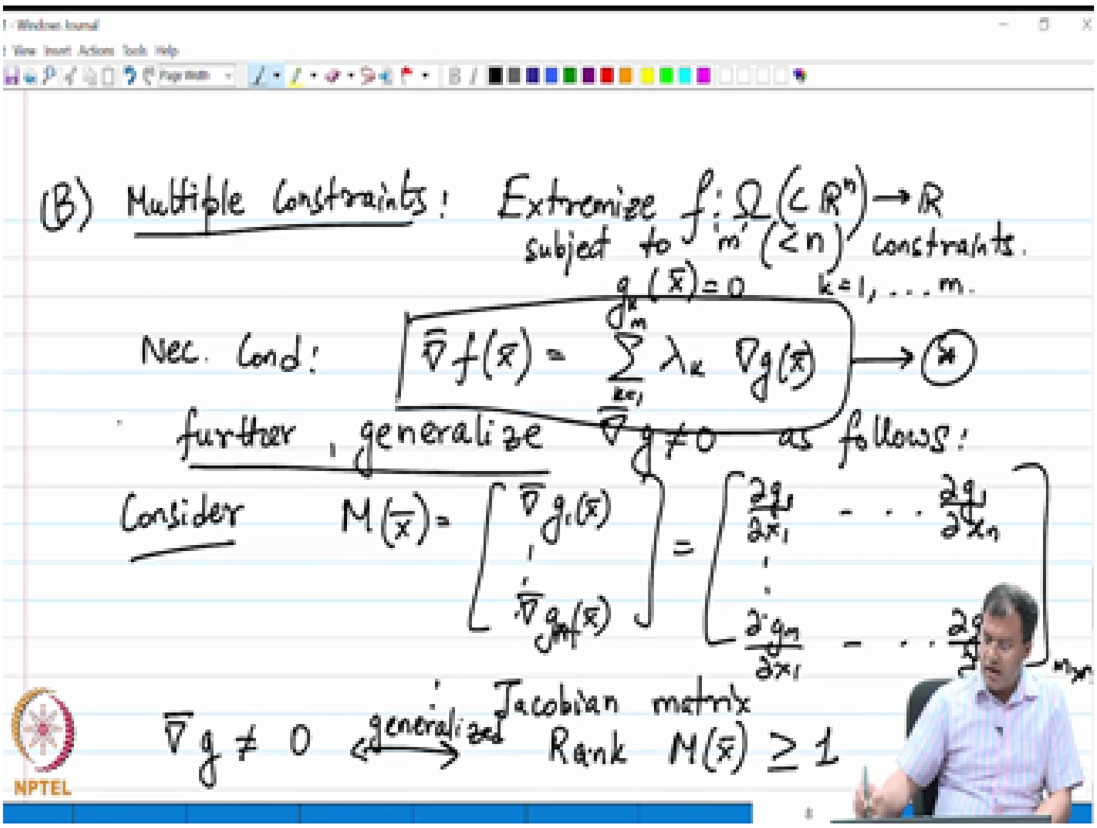
(B) Multiple Constraints: Extremize f: Ω (⊂ ℝⁿ) → ℝ
subject to 'm' (< n) constraints.
gk(x̄) = 0 k = 1, . . . m.
Nec. Cond: ∇̄f(x̄) = Σ_{k=1}^{m} λk ∇gk(x̄) → (*)
further, generalize ∇g ≠ 0 as follows:
Consider M(x̄) = [∇̄g1(x̄); ⋮; ∇̄gm(x̄)] = [∂g1/∂x1 . . . ∂g1/∂xn; ⋮; ∂gm/∂x1 . . . ∂gm/∂xn] m×n
Jacobian matrix
∇g ≠ 0 ⟺ generalized Rank M(x̄) ≥ 1
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let us look at this situation of multiple constraints with an Example 2: Find the local extrema of $f(\bar{x}) = \frac{x_3^2}{2} - x_1 x_2$ subject to $g_1(\bar{x}) = x_1^2 + x_2 - 1 = 0/g_2(\bar{x}) = x_1 + x_2 - 1 = 0$ Find the local extrema of this subject to these two conditions, so how are we going to approach this equation? We will just use our Lagrange multiplier constraints to begin with I have defined my Lagrange multiplier constraint as *

From * I see that I can directly find, so here I have that n, the total number of variables are three. So n is 3 and the number of conditions m is 2. So the equations that I get from the Lagrange multiplier methods will be 3 in total from the gradient condition and we have two constraints, so, I am directly writing my conditions coming out from *. So these are my three relations, which can be checked very easily

$$\begin{cases} x_2 + 2\lambda_1 x_1 + \lambda_2 = 0 \\ x_1 + \lambda_1 = 0 \\ x_2 - \lambda_2 = 0 \end{cases}$$

Subject to the condition $g_1 = g_2 = 0$, I will immediately be able to solve this system. It gives two points, the first point I am directly writing the answer, the students are asked to check that this is indeed the solution. So, points are $\bar{\omega} = (-1, 0, 2)$ and $\bar{z} = (\frac{2}{3}, \frac{5}{9}, \frac{1}{3})$ these are the solutions and we get $\lambda_1 = -\frac{2}{3}, \lambda = \frac{1}{3}$

For $\bar{\omega}$ let us now check about the uniqueness of the solution that we have got. Or so let us see what happens to the rank of the Jacobian matrix and the augmented matrix for these two solutions. Now, for $\bar{\omega}$, if we calculate the Jacobian matrix

$$M(\bar{\omega}) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \frac{\partial g_1}{\partial x_3} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{bmatrix}_{\bar{\omega}}$$

Similarly, if we check the augmented matrix $M_f(\bar{\omega}) = \begin{bmatrix} -2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$ notice that in this case

$R_3 = R_1 + 2R_2$, so the last row is linearly dependent as expected $\text{Rank}(M_f) \leq \text{Rank}\ (M)$ **a** or in this case exactly equal to the rank of the Jacobian matrix, so this condition is certainly true which guarantees that $\bar{\nabla} g \neq 0$

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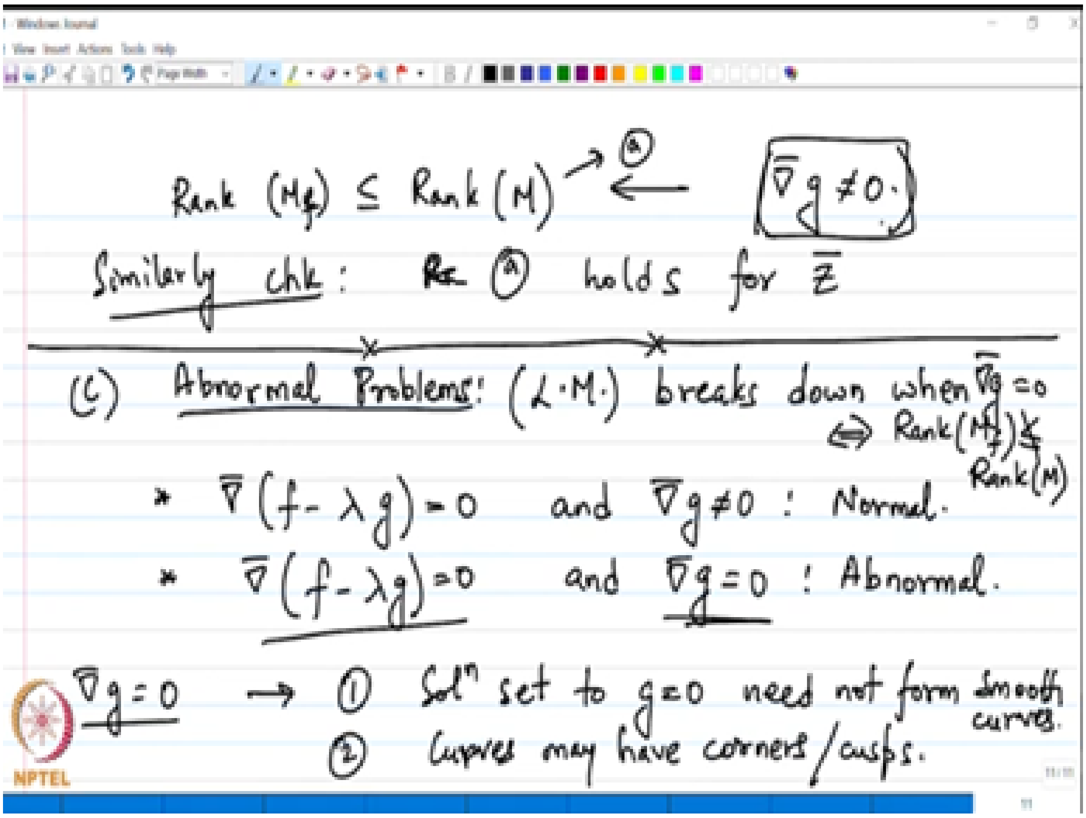

Similarly, the students are asked to check that this rank **a** holds for $\bar{z}$ so, which means that $\bar{\omega}$ and $\bar{z}$ both are extrema's or local extrema's that are found using Lagrange Multiplier, so, it seems that Lagrange Multiplier is the way to go, it presents very rosy picture and allows us to solve almost all different class of problems.

Now I am going to show another class of problems where the Lagrange Multiplier creates some problems or completely fails, so those class of problems are going to be denoted by abnormal problems. Now, what are this class of problems? Of course, we have seen that throughout our lecture discourse, we have assumed this following constraint, this is what we have assumed so far. How about when this is not true? That is $\bar{\bigtriangledown} g = 0$ and then we will see that we run into some troubles.

It means the Lagrange method breaks down when $\bar{\bigtriangledown} g = 0 \Leftrightarrow \text{Rank}(M_f) \leq \text{Rank}\ (M)$, when this inequality is not satisfied, we will run into trouble. So, which means we have two class of problems, so the first class of problem is that we have this Lagrange setup$\bar{\bigtriangledown}(f - \lambda g) = 0$ and $\bar{\bigtriangledown} g \neq 0$ (Normal)
$\Rightarrow \bar{\bigtriangledown}(f - \lambda g) = 0$ and $\bar{\bigtriangledown} g = 0$ (Abnormal)

that is the points that we find from Lagrange setup does not satisfy the gradient criteria. Now what is the difficulty about gradient vanishing? It turns out that gradient vanishing, even if gradient vanishes, the gradient of the constraint vanishes, we may still be able to find the solution but we run into trouble like the derivative of, the derivative vanishes or the derivative does not exist of the objective functions and similar such problems.

$\bar{\bigtriangledown} g = 0$ may lead to problems of the sort that the solution set g = 0 need not form smooth curves and the second is that the curves may have corners or cusps and our next set of few examples are going to exactly show these difficulties.

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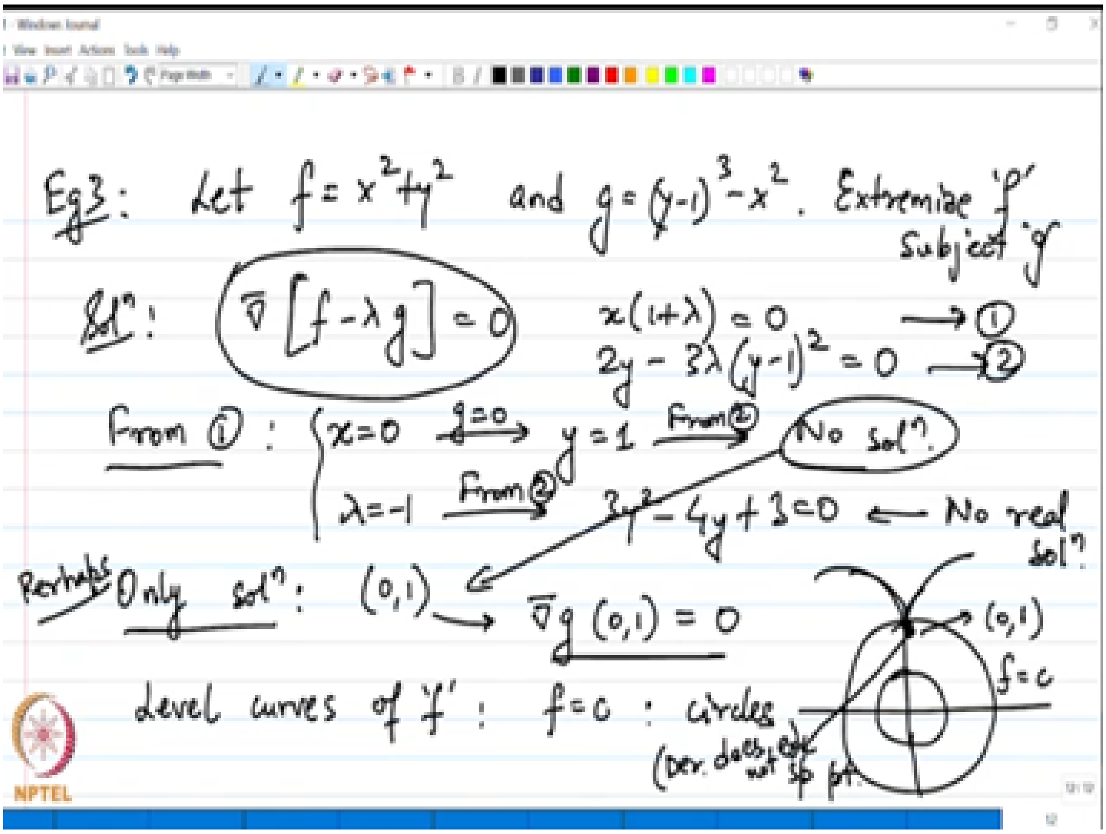

let me show these class of problems that we face via some Examples
Let $f = x^2 + y^2$ and $g = (y-1)^3 - x^2$, Extremize f subject to g

When we do that, again we start solving by setting up the Lagrange constraint $\bar{\bigtriangledown}[f - \lambda g] = 0$ and we are going to get set of two equations

$x(1 + \lambda) = 0$ **1**

$2y - 3\lambda(y - 1)^2 = 0$ **2**

Now, we have three unknowns x, y and λ and of course we have the constraint g, it seems everything is nice but notice the from **1** x = 0 $\overset{g=0}{\rightarrow}$ $y = 1$. However From **2** which is independent of x when I plug y = 1 we have no solution, y =1 does not provide us any solution.

Further we should perhaps explore the other condition $\lambda = -1$then from **2** , I get $3y^2 - 4y + 3 = 0$ It turns out that there is no real solution to this problem. So, we are stuck, it seems that there is no extrema, and the Lagrange method completely breaks down. Now, but something still can be done. Notice that the only solution that we found although it did not satisfy the constraint, perhaps the only solution that we found was (0, 1)

I should definitely write the word perhaps because we have shown that there is some problem here, so, perhaps the only solution, the only promise that we were getting was at point (0, 1). So, notice that, at this point $\bar{\bigtriangledown}g(0, 1) = 0$, so the $\bar{\bigtriangledown}g$ vanishes, so this is certainly an abnormal problem. So the idea here is that the abnormal problem will not, cannot be approached by the Lagrange multiplier method. Now, geometrically let us see what is the problem happening, what is so problematic about this example. Now, if we were to look at the level curves of f, let us say the level curves of f = constant, I know that these are circles. So $x^2 + y^2 = constant$ are circles which are having its center at the origin.

If I were to plot, so this is my level curves f = c, g = o , we will see that it looks as if it is the following, so this point is (0, 1), at x = 0, we see that it forms a cusp point, so the derivative vanishes as has been readily shown here. So, which means that this sort of a method where we are taking the derivative of the function will not hold because the derivative vanishes for the condition, for the constraint. That is why the Lagrange multiplier fails because we are assuming that the derivative is available for us. Well, it is not vanishes but the derivative does not exist, so we will have a problem at this point.

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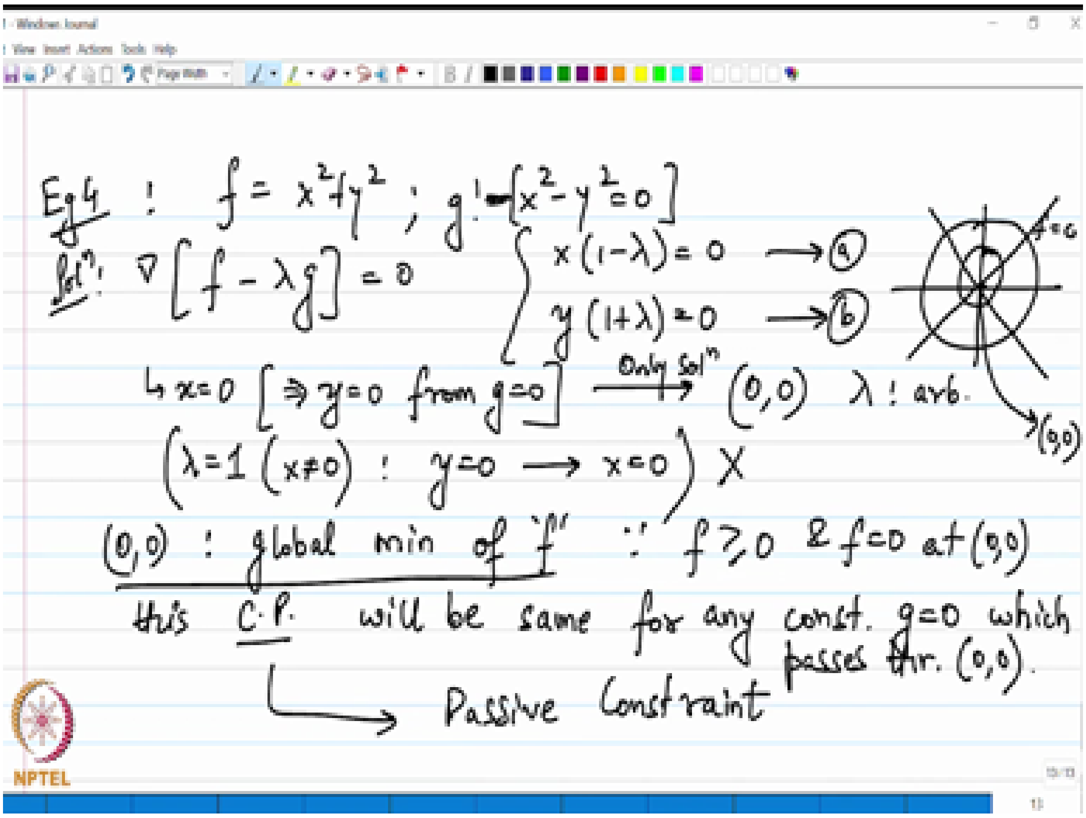

let us look at another example $f = x^2 + y^2, g = x^2 - y^2 = 0$

$$\bigtriangledown [f - \lambda g] = 0 \text{ and } \begin{cases} x(1-\lambda) = 0 & \textbf{a} \\ y(1+\lambda) = 0 & \textbf{b} \end{cases}$$

If I chose x = 0 then I get the solution y = 0 from the condition g = 0.

and if I choose $\lambda = 1$ or $x \neq 0$ then I still get y = 0 but y = 0 gives me x = 0, this setup is not possible which means that the only solution that we get is (0, 0) and λ is arbitrary i.e any value of λ will satisfy for this point (0, 0). Now further notice the objective function, is a real valued function with square of its variable and it is easy to see that the minimum of the function will be obtained at x = 0 and y = 0.

So, the point that we got is minimizing the objective function, so it turns out that (0, 0) is the global minima of f because ≥ 0 and f = 0 at (0, 0). It is a global minima and let us do the same exercise, try to draw f and g. So, g is the constraint which is this. So if I were to draw this level curves, notice the level curves of f are again as following function, so I were to draw these level curves, we see that the level curves are as follows and if I were to solve this constraint g, I see that the constrain is $y = \pm x$, these are pairs of two straight lines which are passing through the origin, which means that the only level curve that satisfies g will be the point at the origin, so what I have just now showed is the following, so f (0, 0) is the global minimum of f and this critical point (0, 0) will be the same for any constraint which passes through (0, 0) and will be the same for any constraint g = 0 which passes through the origin.

As I do not care about what the constraint is here, in fact constraint plays almost minimal role, as long as the constraint passes through the origin, it is quite clear that the critical point (0, 0) will be an extremal to this constraint optimization. So, in this case, the constraint is playing a passive role, or this is a case of passive constraint problem, the constraint plays a lower role than the objective function.