Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture 26
Problems with Holonomic and non-Holonomic Constraints (Part 02)

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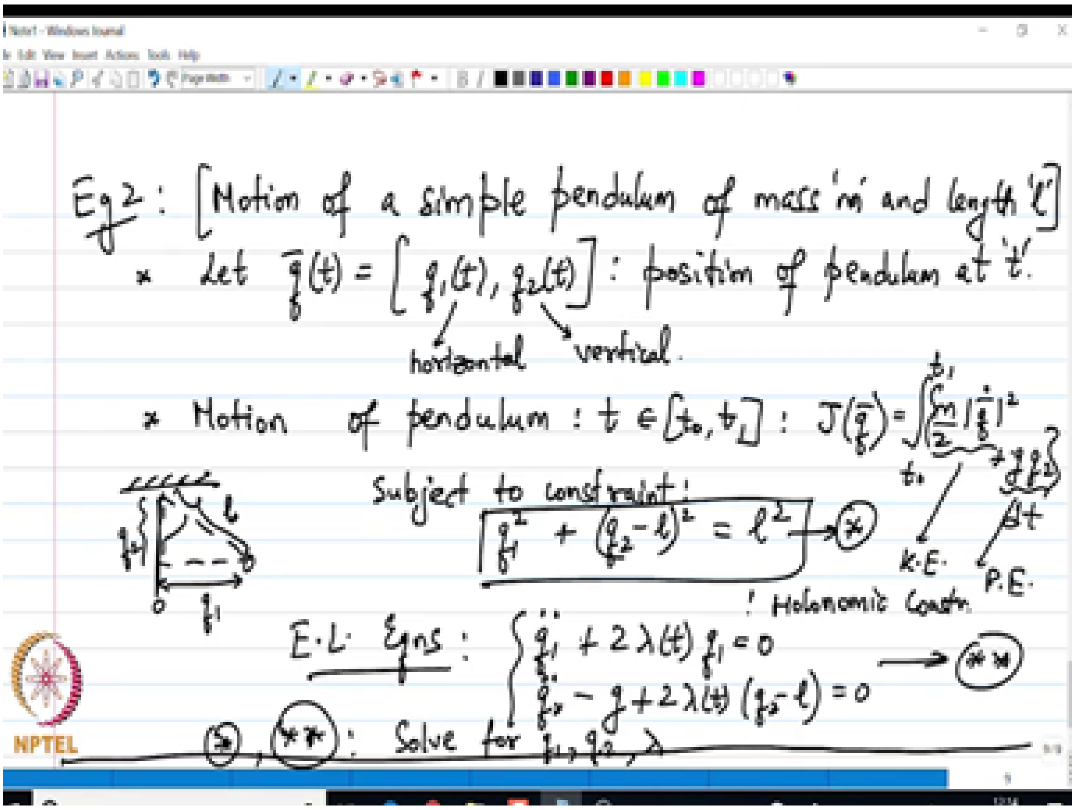

Let us look at Example 2: Motion of a simple pendulum in the form of a constraint problem with holonomic constraints of mass 'm' and length l.

Let $\bar{q}(t) = [q_1(t), q_2(t)]$ denote the position Vector of pendulum at time 't' denote q_1 denotes horizontal component and q_2 as vertical component of motion.

Further, we assume that the motion is restricted in a given parameter range, so the motion of the pendulum is $t \in [t_o, t_1]$: $J(\bar{q}) = \int_{t_o}^{t_1} \left\{ \frac{m}{2} |\dot{\bar{q}}|^2 + g q_2 \right\} dt$, Note that this particular quantity is kinetic energy of the system and depends on the vertical component of the position in the potential energy.

Here we have absorbed, it does not really matter where whether we have 'm' here or not the second scenario because 'm' is a constant of the problem so we can very well assume that 'm' or g as one of the constants, note that we have a pendulum which is rotating such that the total length of the pendulum is L.

The motion is subject to the constraint such that the sum of the square of the x component and the square of the y component which is $q_1^2 + (q_2 - l)^2 = l^2 \qquad *$
This is my holonomic constraints.

Further I'm ready with functional is an integrand containing two variables subject to this constraint of two variables. So I can find the necessary condition for an extremal by solving my Euler Lagrange equations for each of the components which are as follows.

$$\begin{cases} \ddot{q}_1 + 2\lambda(t)q_1 = 0 \\ \ddot{q}_2 - g + 2\lambda(t)(q_2 - l) = 0 \end{cases} \qquad **$$

By using * and ** solve for q_1, q_2, λ

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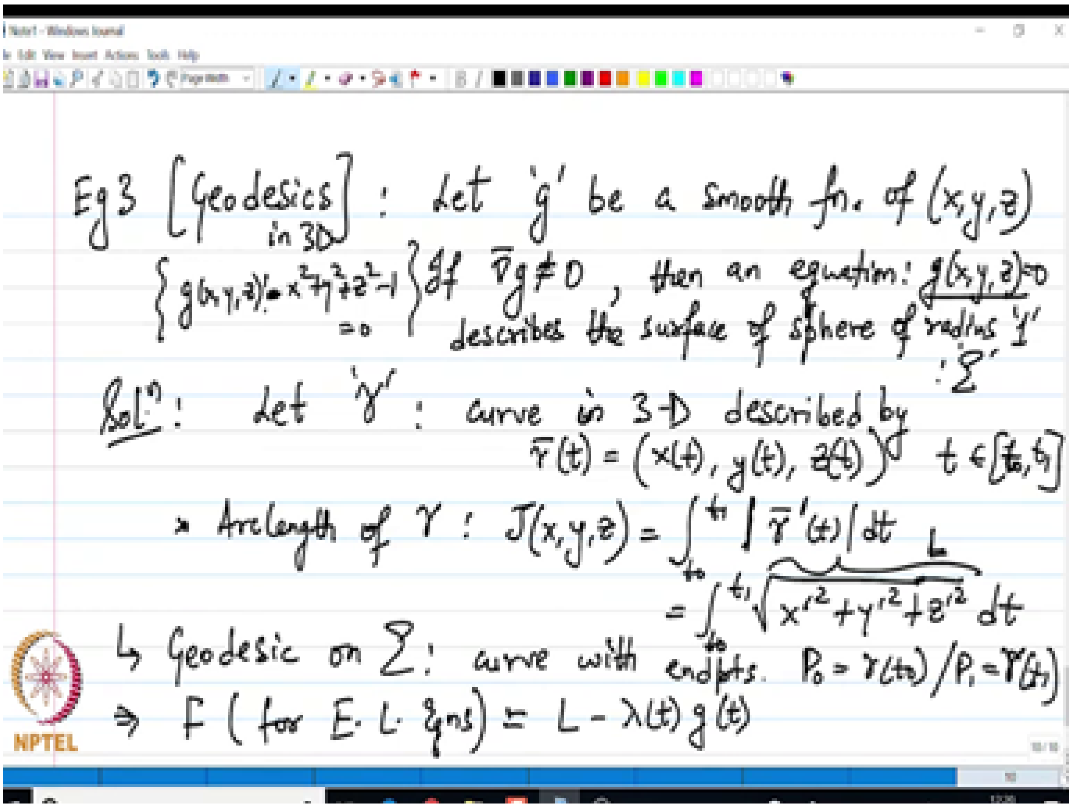

Let us look at Example 3: (Geodesics in 3D) Let 'g' be a smooth function of (x, y, z) [when I say smooth function I necessarily impose all the derivatives exists such that my Euler Lagrange equations are are solvable] and further if we assume that we are not in the scenario of abnormal problems, then an equation of the form g(x, y, z) describes the surface of sphere surface of sphere of radius radius 1: we denote it by 'Σ'.

Solution: Let assume that γ is the curve in 3-D described by $\bar{\gamma}(t) = (x(t), y(t), z(t)) \;\; t \in [t_o, t_1]$ and the arc length of γ is defined by the functional $J(x, y, z) = \int_{t_o}^{t_1} |\bar{\gamma}'(t)|dt = \int_{t_o}^{t_1} \sqrt{{x'}^2 + {y'}^2 + {z'}^2}dt$, so this is the arc length of the functional that we have to optimize.

Geodesic on Σ is going to be the extremal of this arc length functional. So geodesic on Σ is the curve with endpoints $P_o = \gamma(t_o)$ and $P_1 = \gamma(t_1)$, so I can directly go to the description of the extremal to solve the Euler Lagrange equation which means F(for my Euler LaGrange equation) $= L - \lambda(t)g(t)$

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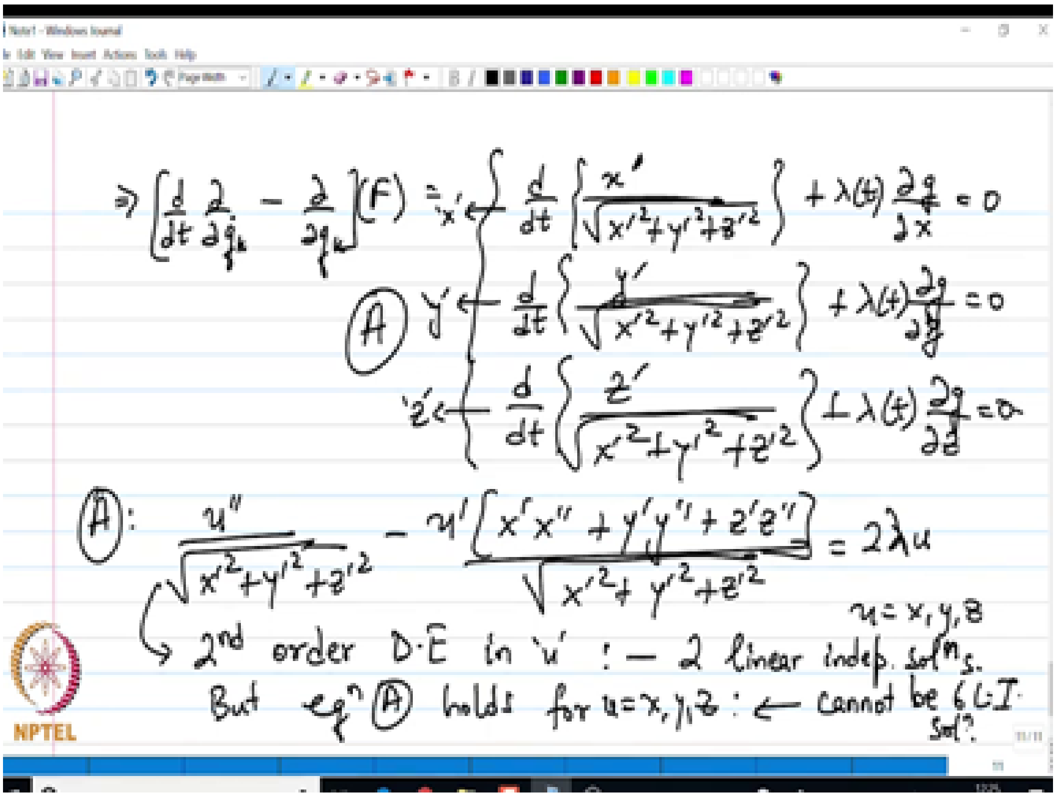

$$\Rightarrow \left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k} - \frac{\partial}{\partial q_k}\right](F) = \begin{cases} = \frac{d}{dt}\left\{\frac{x^{'}}{\sqrt{x^{'2}+y^{'2}+z^{'2}}}\right\} + \lambda(t)\frac{\partial g}{\partial x} = 0 \;\; (\text{with respect to } x^{'}) \\ = \frac{d}{dt}\left\{\frac{y^{'}}{\sqrt{x^{'2}+y^{'2}+z^{'2}}}\right\} + \lambda(t)\frac{\partial g}{\partial y} = 0 \;\; (\text{with respect to } y^{'}) \\ = \frac{d}{dt}\left\{\frac{z^{'}}{\sqrt{x^{'2}+y^{'2}+z^{'2}}}\right\} + \lambda(t)\frac{\partial g}{\partial z} = 0 \;\; (\text{with respect to } z^{'}) \end{cases} \qquad \mathbf{A}$$

Now, we see that we have three unknowns x y z and the fourth unknown is the λ and we have three equations plus the isoperimetric holonomic constraints. However, we can see that the solution is extremely complicated. So let me let me club this set as A can be written in a more general format.

$$\frac{u^{''}}{\sqrt{x^{'2}+y^{'2}+z^{'2}}} - \frac{u^{'}[x^{'}x^{''}+y^{'}y^{''}+z^{'}z^{''}]}{\sqrt{x^{'2}+y^{'2}+z^{'2}}} = 2\lambda u$$

Where where u = x, y, z . Plugging in the values of u respectively will help me to generate each of these equations described by the set **A**. Note that for each 'u' this is a second order differential equation which means it has two linearly independent solutions but however there are 3 equation overall for u = x, y, z .

So are there six linearly independent Solutions? The answer is no, we have four variables. How can there be six linearly independent Solutions? So there cannot there cannot be six linearly independent solutions, which means that there will be a relation which relates at least three variables together and then the other three variables are separate.

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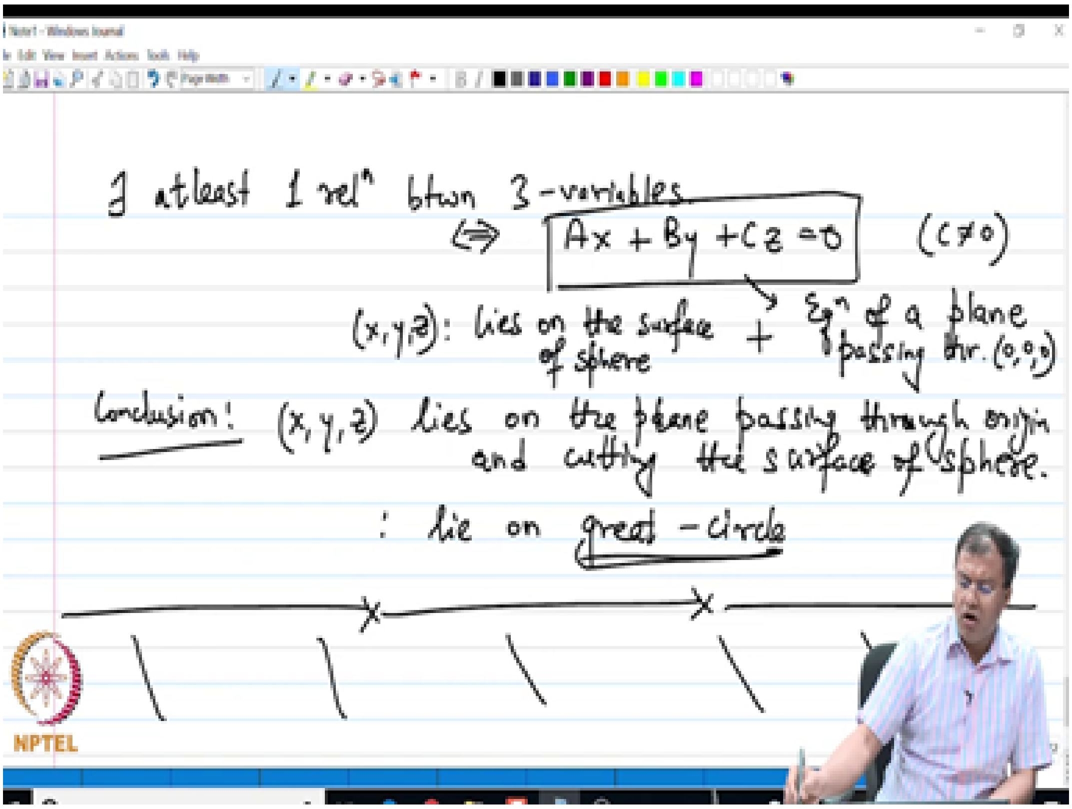

So at least there will be one relation, which means if three variables are x, y, z can be expressed in the form of x and y then x, y, z are linearly dependent, right?
So given the fact that there cannot be six linearly independent, there will be at least one relation between three variables so that the system is properly defined. So let me call those three variables without loss of generality could be x, y, z.

which means that Ax + By + Cz = 0 ($C \neq 0$), Notice that this relation is an equation of a plane. So far we are trying to solve our system of second order differential equation purely by reasoning and and geometric arguments.

We see that this equation is linear dependence relation is nothing but the equation of a plane which passes through origin passing through (0, 0, 0) plus (x, y, z) satisfies the holonomic constraints or it lies on the surface of sphere which passes through the origin

So the conclusion is that the points (x, y, z) lies on the intersection of the sphere and the plane passing through the origin or (x, y, z) lie on the plane passing through origin and cutting the surface of the sphere the surface of the sphere, and that is only possible if (x, y, z) lie on the great circle, which is exactly the result that we have looked at in the previous discussion of geodesics in 3D.

So, this is another approach of solving this problem by using a holonomic constraints. I end by discussion on holonomic constraints and then move on to our discussion of non-holonomic constraints of optimization, we will see that non-holonomic constraints optimization is a more General situation.

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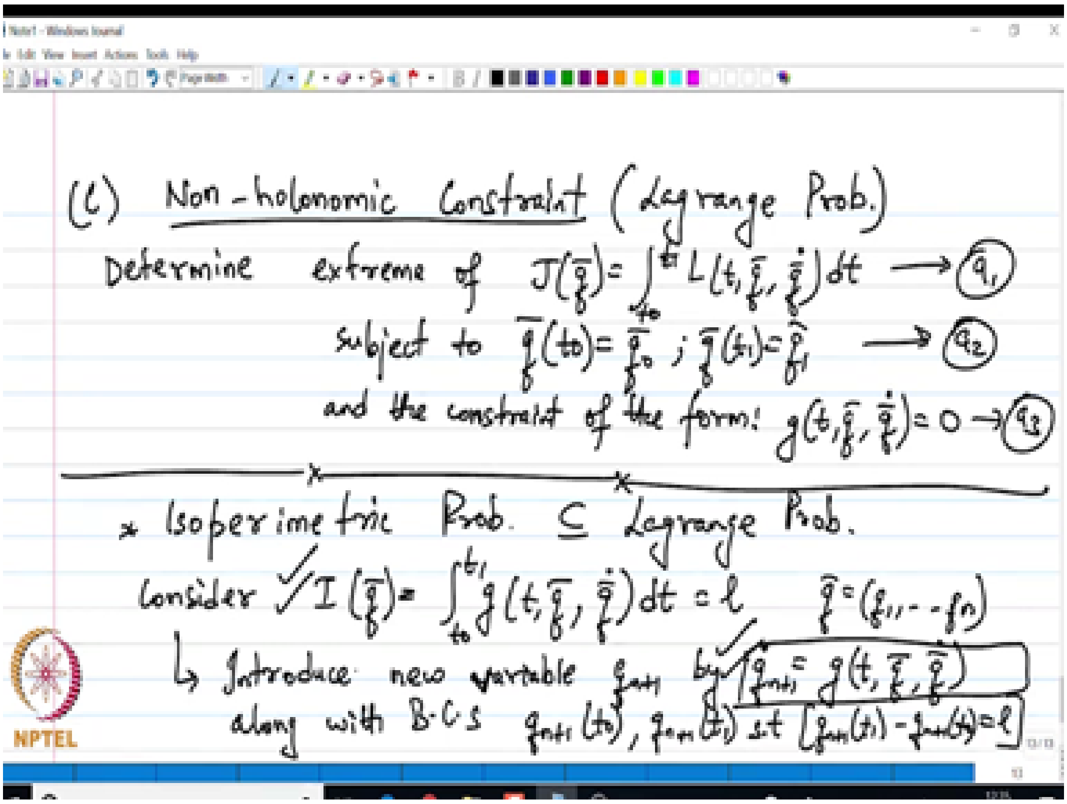

Non-holonomic constraint (class of problem as the Lagrange problems) We want to determine the extrema of $J(\bar{q}) = \int_{t_o}^{t_1} L(t, \bar{q}, \dot{\bar{q}})dt$ **a_1**
Subject to the boundary condition $\bar{q}(t_o) = \bar{q}_o; \bar{q}(t_1) = \bar{q}_1$ **a_2**
and the constraint of the form $g(t, \bar{q}, \dot{\bar{q}}) = 0$ **a_3**
Now I am going to show that this class of constraint problems is a very general class. In fact, the the class of isoperimetric problem is a subset of this class of problems where we look at the generalized case of Euler Lagrange equations with functions having higher order derivative , so this class is quite general. In fact, our class of isoperimetric problems could also be posed as a class of non-holonomic problems, so isoperimetric problems problems are a subset of Lagrange problems.

Consider the constraint of the form $I(\bar{q}) = \int_{t_o}^{t_1} g(t, \bar{q}, \dot{\bar{q}})dt = l$, where $\bar{q} = (q_1, ..., q_n)$ and to pose this integral constraint as a differential constraint we introduce a new variable q_{n+1} such that $\dot{q}_{n+1} = g(t, \bar{q}, \dot{\bar{q}})$ along with boundary condition $q_{n+1}(t_o), q_{n+1}(t_1)$ such that $q_{n+1}(t_1) - q_{n+1}(t_o) = l$

So now instead of using this constraint if we use this particular constraint, then the problem is identical. So the first constraint is an integral consent and the second constraint is a differential constraint. So the isoperimetric problems are is subset of Lagrange problems.

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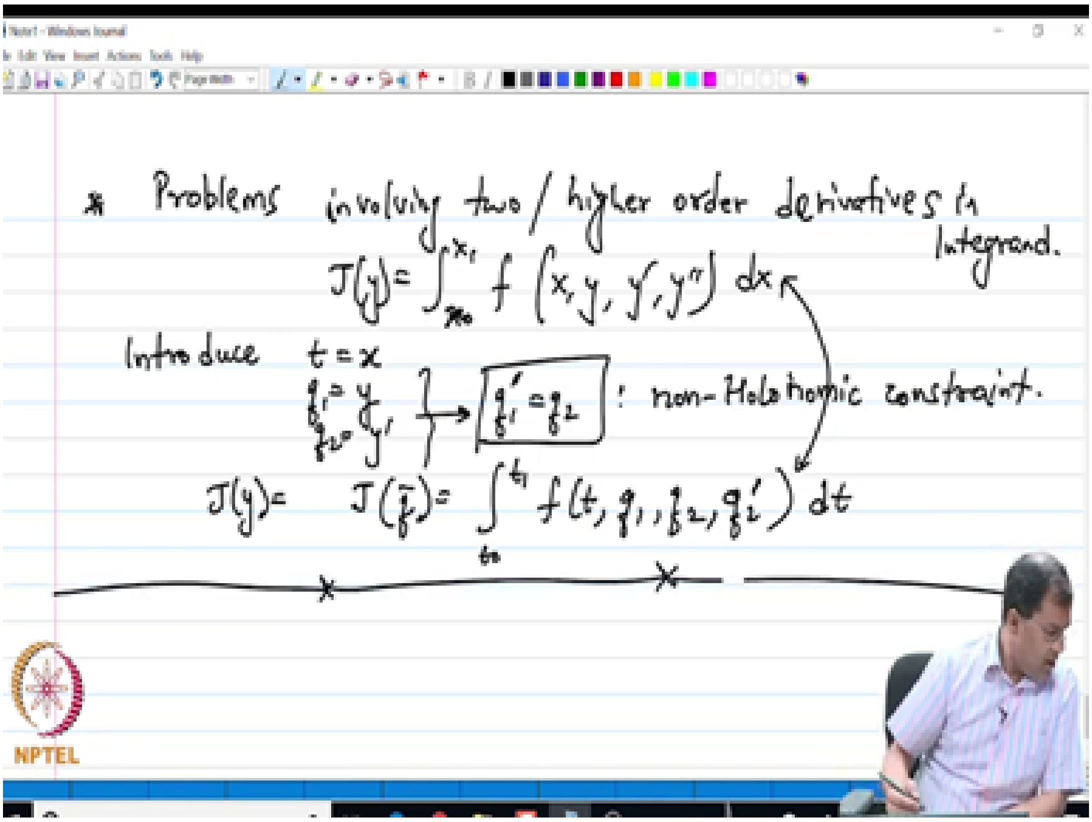

We could have the Lagrange problem discussions all classes of isoperimetric problems, Next, let us also look at the problems involving higher order derivatives, problems involving two or more higher order derivatives in the integrand. We are talking about the functional of the following form $J(y) = \int_{x_o}^{x_1} f(x, y, y^{'}, y^{''})dx$

If we introduce new sets of variables of the form t = x, $q_1 = y, q_2 = y^{'} \Rightarrow q_1^{'} = q_2$, so this is a condition which is our non-holonomic constraint.

Notice how integral functional reduces to the functional $J(y) = J(\bar{q}) = \int_{t_o}^{t_1} f(t, q_1, q_2, q_2^{'})dt$

Originally it dependent on functions up to second derivative and now it dependent on functions up to first derivative with the additional non-holonomic constraints.