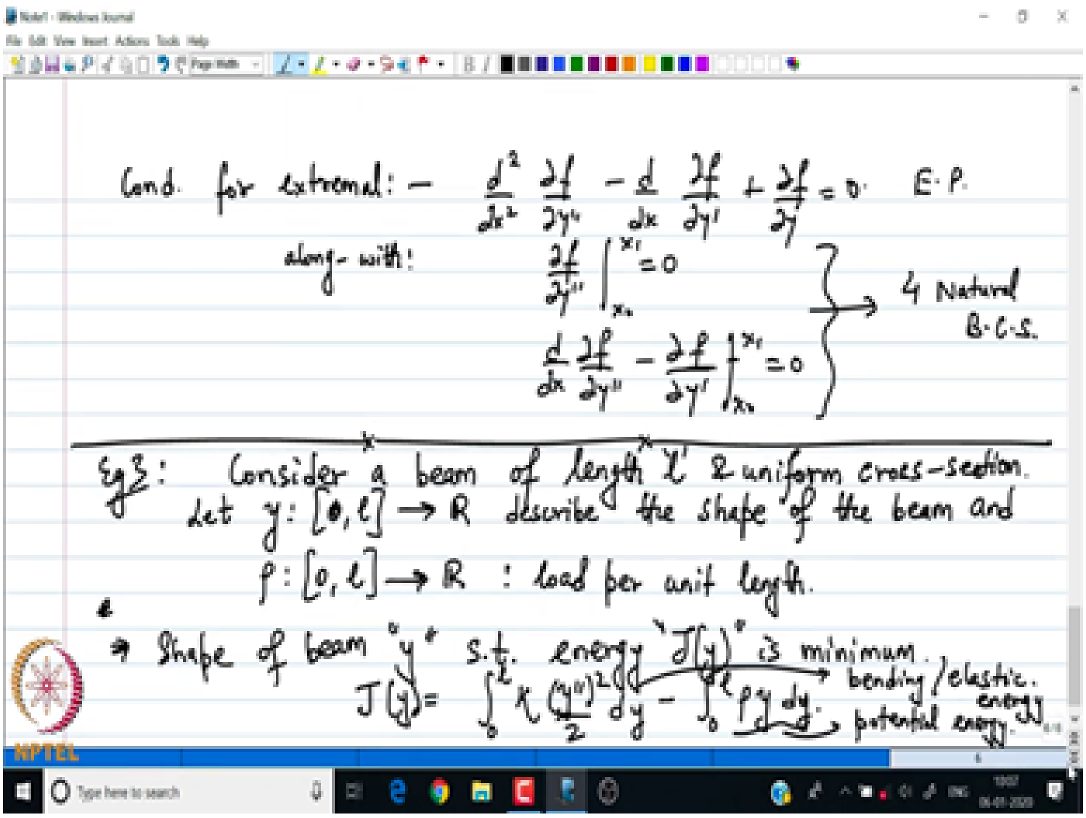
Variational Calculus and its Applications in Control Theory and Nano mechanics
Professor Sarthok Sircar
Department of Mathematics
Indraprastha Institute of Information Technology, Delhi
Lecture 29
Problems with Holonomic and Non-Holonomic Constraints, Variable Endpoints Part 5

(Refer Slide Time: 00:15)

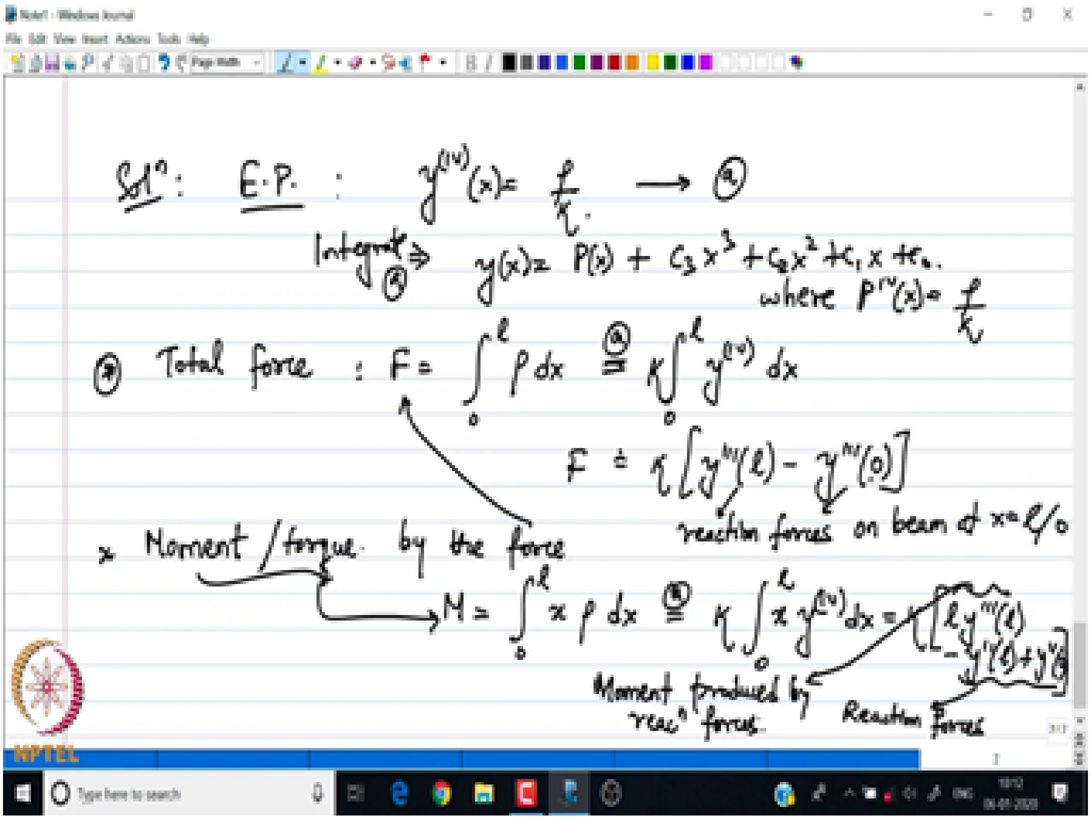


Example that I have in this case is of the beam mechanics. So, we consider a beam or a steel rod which is flexible and the beam is of length l and uniform cross section, The beam is of length l and uniform cross section. Let my function y: $[0, l] \to R$ describe the shape of the beam and let us say that $\rho : [0, l] \to R$ is a load per unit length or the mass per or unit length which is acting on the beam.

Then in that case the shape of the beam is governed by the functional which minimizes the energy of the beam, it turns out that the shape of the beam y(x) is such that the energy functional J(y) is minimum, the shape is such that the energy functional is minimum and the energy of the beam in general will be given by its some of elastic energy plus the potential energy.

We are ignoring the effect of the weight here. So, the elastic energy of the beam in the most simple rotation could be written as a constant, let $J(y) = \int_0^l \kappa \frac{y''}{2} dy - \int_0^l \rho y dy$ where this quantity in the integral is my bending energy or elastic energy of the beam and this quantity here is my potential energy of the beam due to the load that is acting on the beam. So, then we need to find the optimal shape of the beam such that it is going to first satisfy the Euler-Poisson equation.

(Refer Slide Time: 03:46)



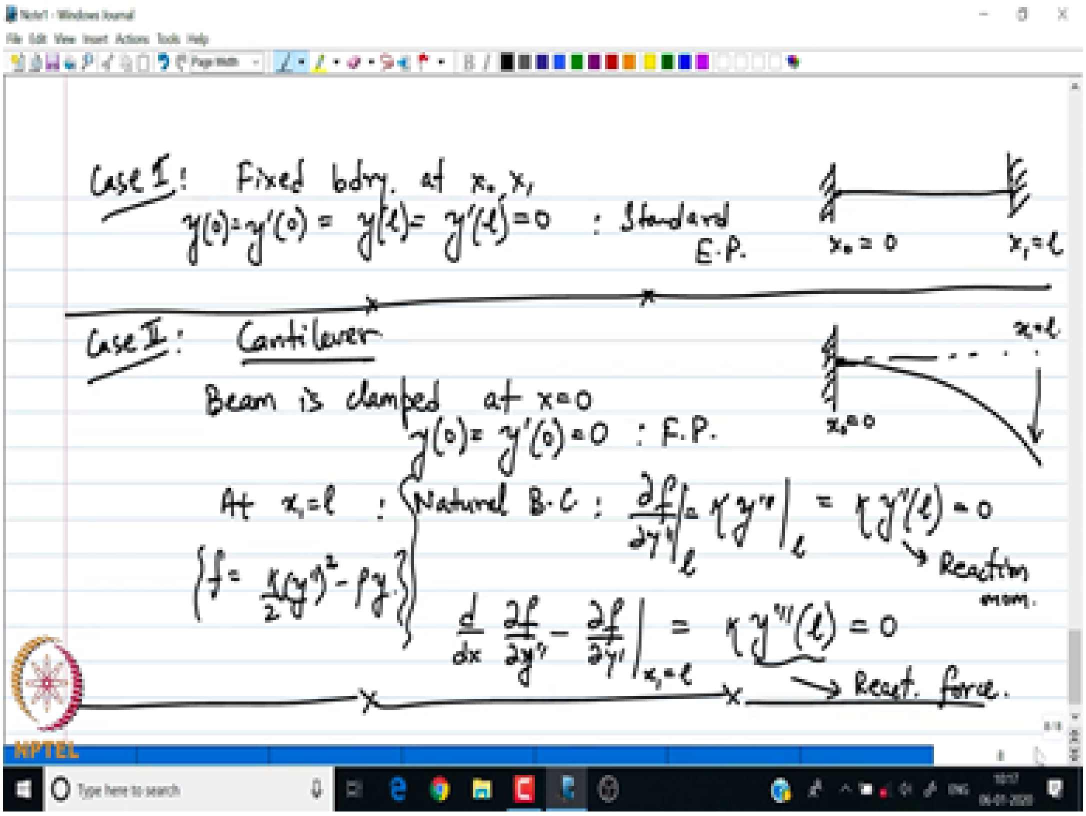
Euler-Poisson equation for this functional reduces to the ODE $y^{(IV)}(x) = \frac{\rho}{\kappa}$
once we integrate equation **a**, I get that $y(x) = P(x) + C_3x^3 + C_2x^2 + C_1x + C_o$ where $P^{(IV)}(x) = \frac{\rho}{\kappa}$
The other constants can be found by our 4 natural boundary conditions that we had just derived C_0 to C_3, Now, I am going to consider the different cases of these bent beam problems, let me introduce some notations in solid mechanics, we introduced certain physical terms or notations in solid mechanics.

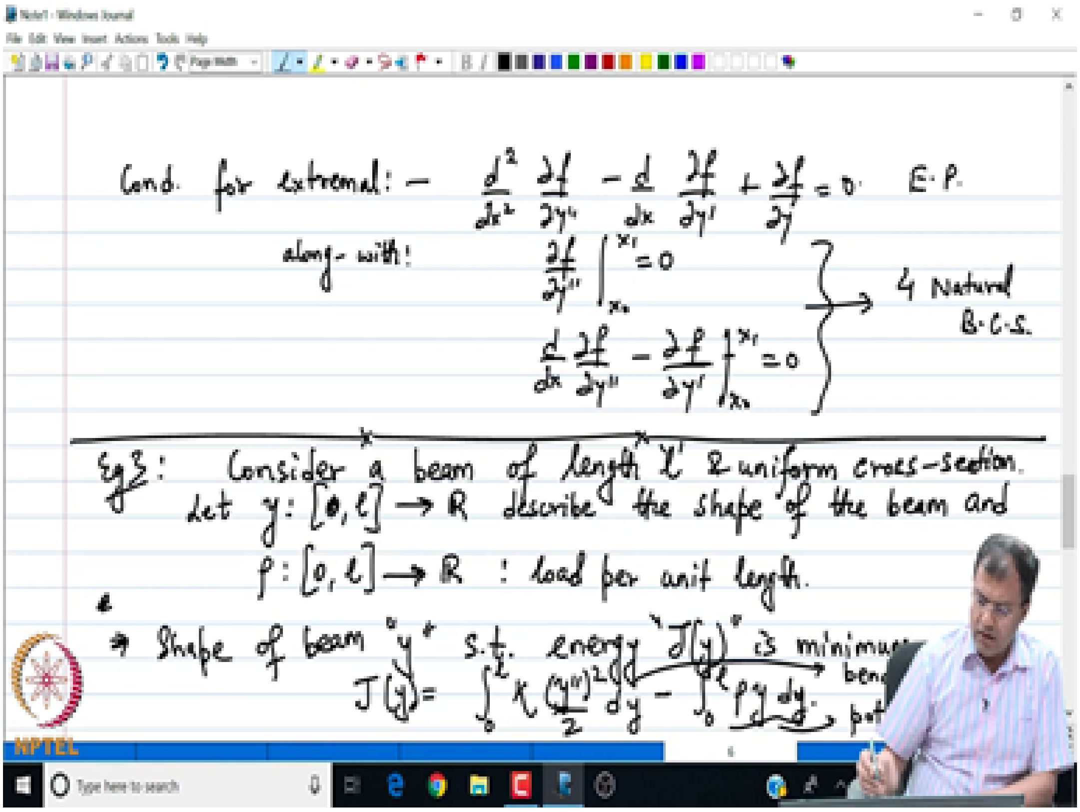
We introduced the term total force, the total force is defined as F = $\int_0^l \rho dx = \kappa \int_0^l y^{(IV)} dx$ and this is also equal to from **a**, then from here I can certainly integrate this is an exact differential, you see that this $F = \kappa \left[y^{(''')}(l) - y^{(''')}(0) \right]$

Where these quantities are known as the reaction forces on the beam at x = 0 and l, I introduced another term known as the moments as well as torque by the force above that we had just described. I denote these moments as M, where M = $\int_0^l x\rho dx = \kappa \int_0^l xy^{(IV)} dx$

We do the integration by parts and we are going to come to a fact that $M = \kappa \left[ly'''(l) - y''(l) + y''(0) \right]$
, again we have these quantities as my reaction forces and this quantity is the moment produced by the reaction forces.

(Refer Slide Time: 08:42)





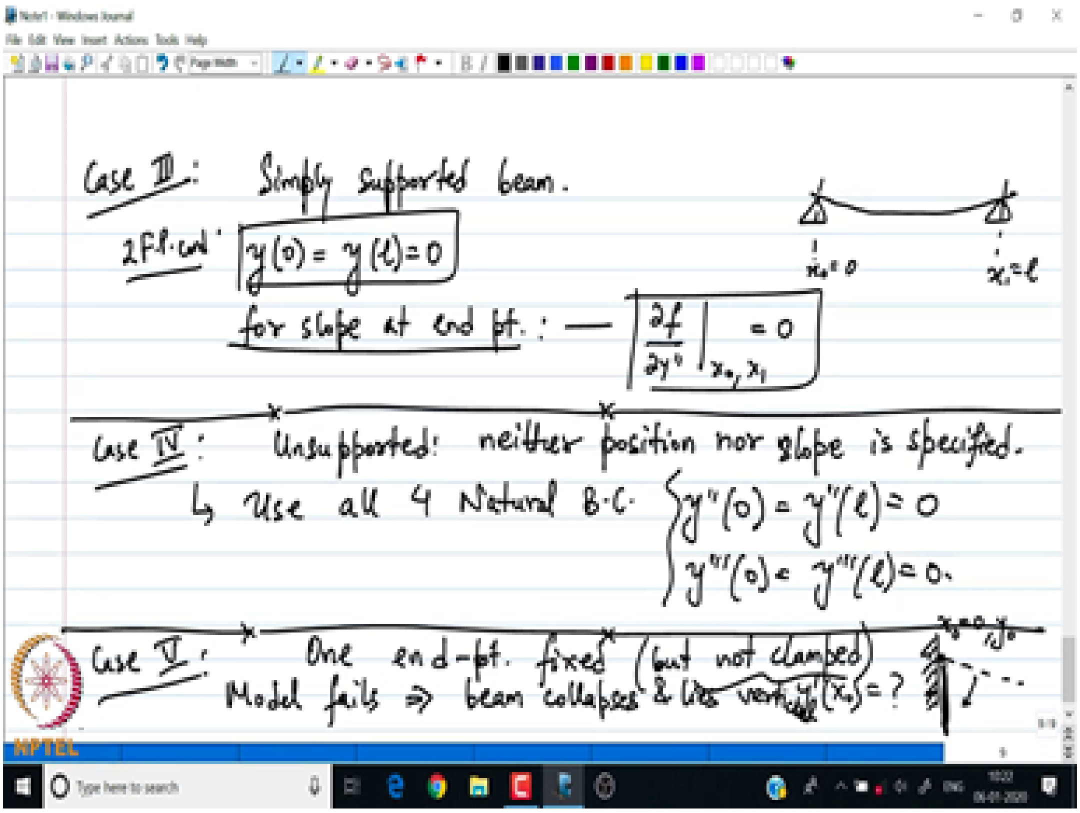
Now, let us now look at some special cases of these beam problems, the different examples where the beams are either clamped or freely moving. let us look at the simplest case. The first case we can have a beam which is doubly clamped or the beam is clamped at x_o and x_1, which means that we have a fixed boundary at x_o and x_1, this is my standard fixed point boundary condition problem. Let us say we have the boundary condition of the form $x_o = 0$ and $x_1 = l$, y(0) $= y^{'}(0) = y(l) = y^{'}(l) = 0$

We approach in a standard Euler-Poisson setup with only fixed point boundary conditions, that is already been done quite a few times. The second case we have in mind is the cantilever problem where we have a beam is able to bend at one of the end points.

So, this is my fixed boundary here at $x_o = 0$ the beam is fixed as well as its slope is defined but on the other end at $x_1 = l$ it is free to move, it means this is a situation where the beam is clamped at x= 0 or I have that y(0) $= y^{'}(0) = 0$, these are my 2 boundary conditions and the other 2 boundary conditions at x = l are natural conditions or at $x_1 = l$ I have the natural condition.

We impose the natural boundary condition that $\frac{\partial f}{\partial y'}|_l$ and $f = \frac{\kappa(y^{''})^2}{2} - \rho y \Rightarrow \frac{\partial f}{\partial y'}|_l = \kappa y^{''}|_l = \kappa y^{''}(l) = 0$ and $\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}|_l = \kappa y^{''}|_l = 0$
So, this is my reaction force, then we now have 4 boundary conditions and the problem now is fully defined. So, let us continue with another case.

(Refer Slide Time: 13:32)



Suppose we are given a beam which is just supported over a particular support. let us say this is my point $x_o = 0$ and $x_1 = l$ and suppose I have a beam which supported at these two points. Then, this question is how are we going to find the extremal shape of this beam.

The case is simply supported beam. So, how to approach for simply supported beam we will certainly have that, the fixed point condition y(0) and at y(l), they are set equal to 0. So, certainly we have 2 fixed point conditions given by y(0) = y(l) = 0 . However, note that in this case the slope may not be 0 because it is just a simply supported beam case. So, for the slope we are going to, instead of the fixed slope we are going to use the natural boundary condition to determine the slope at these end points.

So, the natural boundary condition would be that the reaction moments and reaction forces are set equal to 0. So, we have the following, for slope at the end points we instead replace with these two conditions. We have $\frac{\partial f}{\partial y''}|_{x_o,x_1} = 0$, those are my set of 4 boundary conditions for simply supported case.

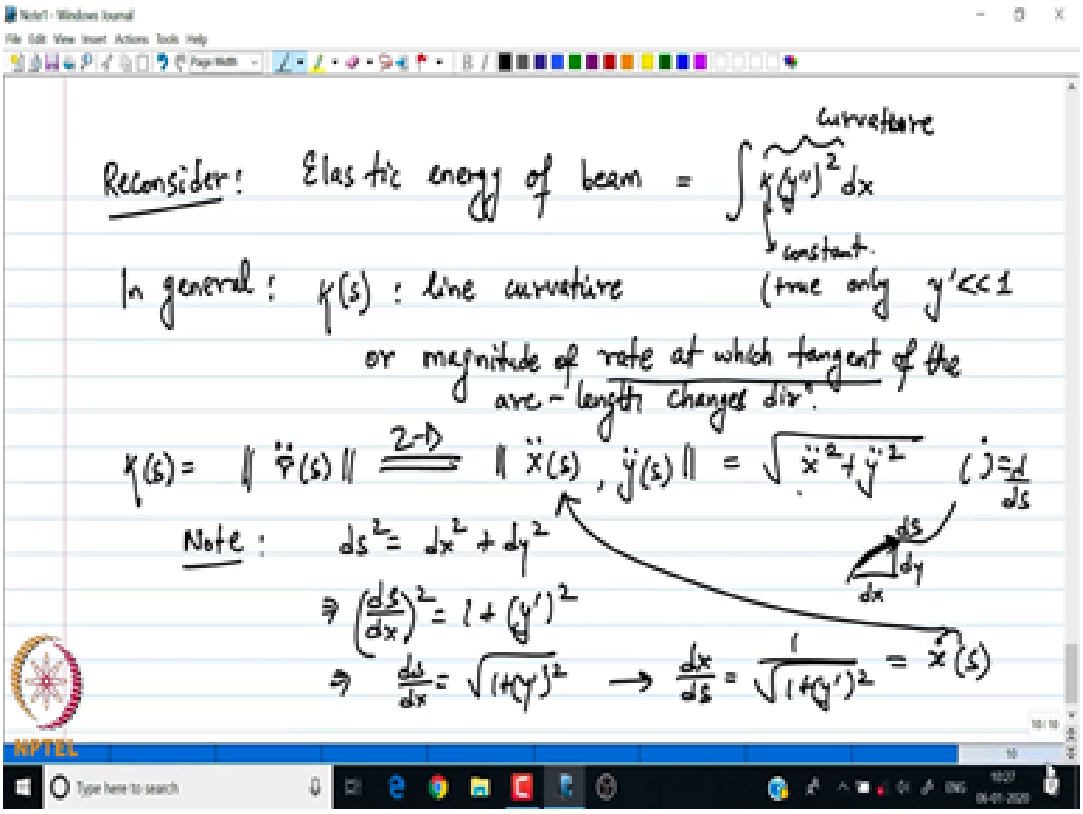
Then, finally we could also include a case which is the unsupported beam. The unsupported beam is where neither the position nor the slope is specified. So, in that case we can use all sets of 4 natural boundary condition which are the following, we have the reaction the reaction moment set equal to 0. So, $y^{''}(0) = y^{''}(l) = 0; y^{'''}(0) = y^{'''}(l) = 0$

That comes right from this set of condition here and my reaction forces are also, 0. So, those are my 4 sets of natural boundary conditions and that will complete our problem description. Finally, we have one more case and we will see that this is not solvable by the model that we have just described. The case that I have is with a beam having only one end point fixed. One end point fixed, but not clamped, but not clamped. So, what is this scenario? let us say the beam has one end point fixed $x_o = 0$ has a fixed coordinate y_o but the slope of the beam at this point is not defined.

In this case y prime at x_o also is unknown. So, what should we expect that the beam, what would be the optimal shape of the beam in this case? Well the beam is free to move except that the positon is fixed at one point. So, the physics tells us that the beam will just lie into a configuration which is parallel to the wall or to reduce the total energy of the beam and this is a case where the model has failed. The model cannot justify this configuration of the beam. It is purely coming out of the physics.

So, the model fails here because the beam collapses and lies vertical. So, there is no solution. So, let us look at another example.

(Refer Slide Time: 19:17)



Consider example we need to revisit our assumption on the elastic energy of the beam. So, let us reconsider our beam example and specially the elastic energy of the beam.

Notice that the elastic energy that we have chosen was of the following form $\int \kappa(y^{''})^2 dx$ where κ was assumed to be a constant. So, this was our elastic energy of the beam but this is when the beam bends, the bending of the beam is negligibly small. This is true only when the slope is very small.

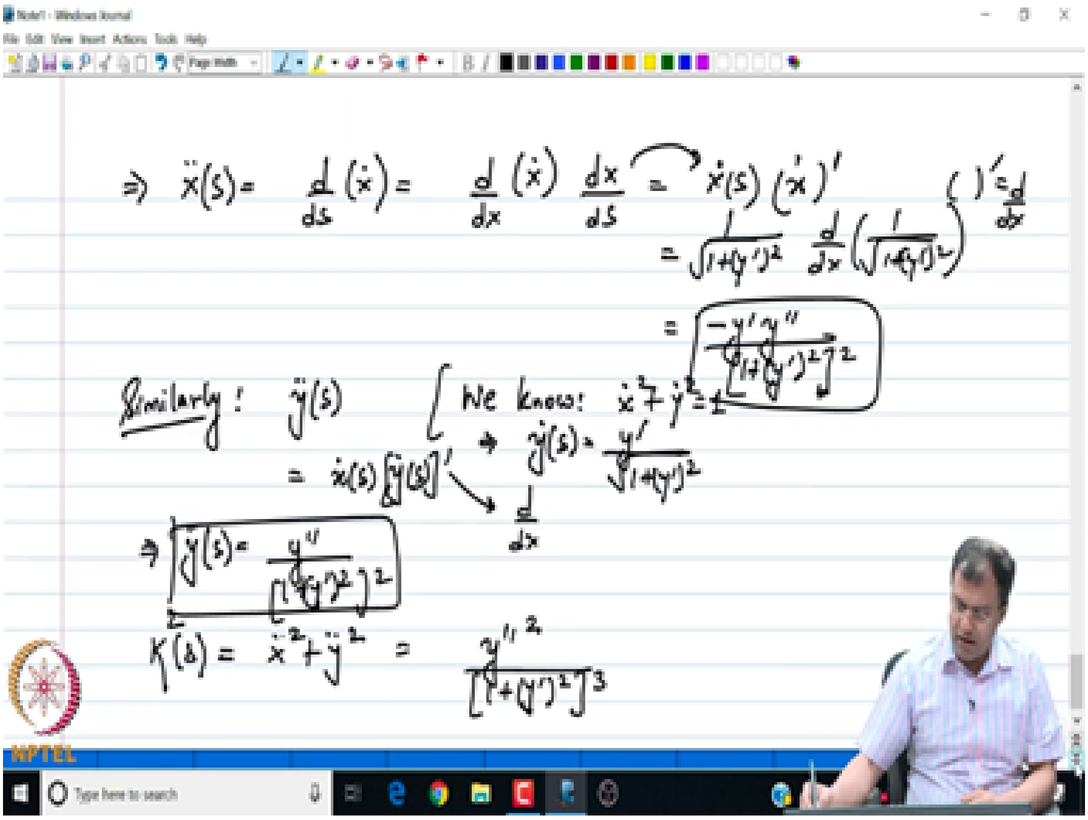
We will see once we derive the general, well, this is nothing but the curvature of the beam. The curvature decides the elastic energy. We will see that when we derive the general expression for the curvature. In general the curvature depends on the slope of the shape of the beam, and it again reduces to this following form when the slope of the beam is very small. So, in general, I have $\kappa(s)$ is the line curvature or the curvature of the curve of our interest and this is also equal to the magnitude of the rate at which the tangent of the arc length changes direction.

Also, the magnitude of the rate at which the tangent of the arc length changes direction. So, what exactly is $\kappa(s)$? So, $\kappa(s)$ is the magnitude of the rate of change of the tangent. The rate at which the tangent changes. So, the tangent of the curve is given by $\dot{r}(s)$ and the rate at which tangent changes is $\ddot{r}(s)$ and its magnitude.

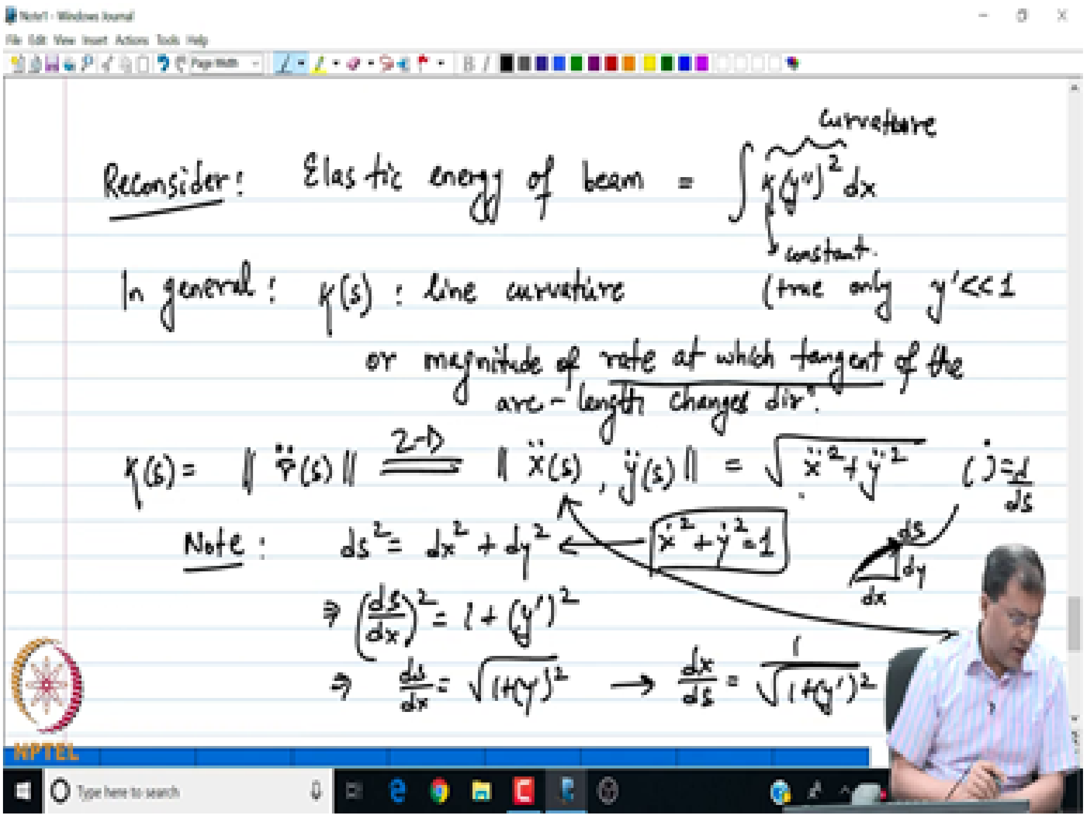
In 2-D this is nothing but the magnitude of $\|\ddot{x}(s), \ddot{y}(s)\| = \sqrt{\ddot{x}^2 + \ddot{y}^2}$ where $\dot{()} = \frac{d}{dx}$.

Notice that, note that my arc length of the curve is $ds^2 = dx^2 + dy^2$ or $\left(\frac{ds}{dx}\right)^2 = 1 + (y^{'})^2$ $\Rightarrow \frac{ds}{dx} = \sqrt{1 + (y^{'})^2} \Rightarrow \frac{dx}{ds} = \frac{1}{\sqrt{1+(y^{'})^2}} = x^{'}(s)$ from here I can find $\ddot{x}$ by differentiating this quantity one more time.

(Refer Slide Time: 24:04)



$\Rightarrow \ddot{x}(s) = \frac{d}{ds}(\dot{x}) = \frac{d}{dx}(\dot{x})\frac{dx}{ds} = \dot{x}(s)(\dot{x})'$
$(\ )' = \frac{d}{dx}$
$= \frac{1}{\sqrt{1+(y')^2}}\frac{d}{dx}\left(\frac{1}{\sqrt{1+(y')^2}}\right)$
$= \frac{-y'y''}{[1+(y')^2]^2}$
Similarly: $\ddot{y}(s)$
[We know: $\dot{x}^2+\dot{y}^2=1$
$\Rightarrow \dot{y}(s) = \frac{y'}{\sqrt{1+(y')^2}}$
$= \dot{x}(s)[\dot{y}(s)]'$
$\frac{d}{dx}$
$\Rightarrow \ddot{y}(s) = \frac{y''}{[1+(y')^2]^2}$
$\kappa(s) = \ddot{x}^2+\ddot{y}^2 = \frac{y''^2}{[1+(y')^2]^3}$
NPTEL



Reconsider: Elastic energy of beam $= \int k (y'')^2 dx$
curvature
constant
In general: $\kappa(s)$: line curvature (true only $y' \ll 1$
or magnitude of rate at which tangent of the arc-length changes dir^n.
$\kappa(s) = \|\ddot{\vec{r}}(s)\| \overset{2-D}{=} \|\ddot{x}(s), \ddot{y}(s)\| = \sqrt{\ddot{x}^2+\ddot{y}^2}$
$(\dot{\ }) = \frac{d}{ds}$
Note: $ds^2 = dx^2 + dy^2$
$\dot{x}^2+\dot{y}^2=1$
ds
dy
dx
$\Rightarrow \left(\frac{ds}{dx}\right)^2 = 1+(y')^2$
$\Rightarrow \frac{ds}{dx} = \sqrt{1+(y')^2} \rightarrow \frac{dx}{ds} = \frac{1}{\sqrt{1+(y')^2}}$
NPTEL

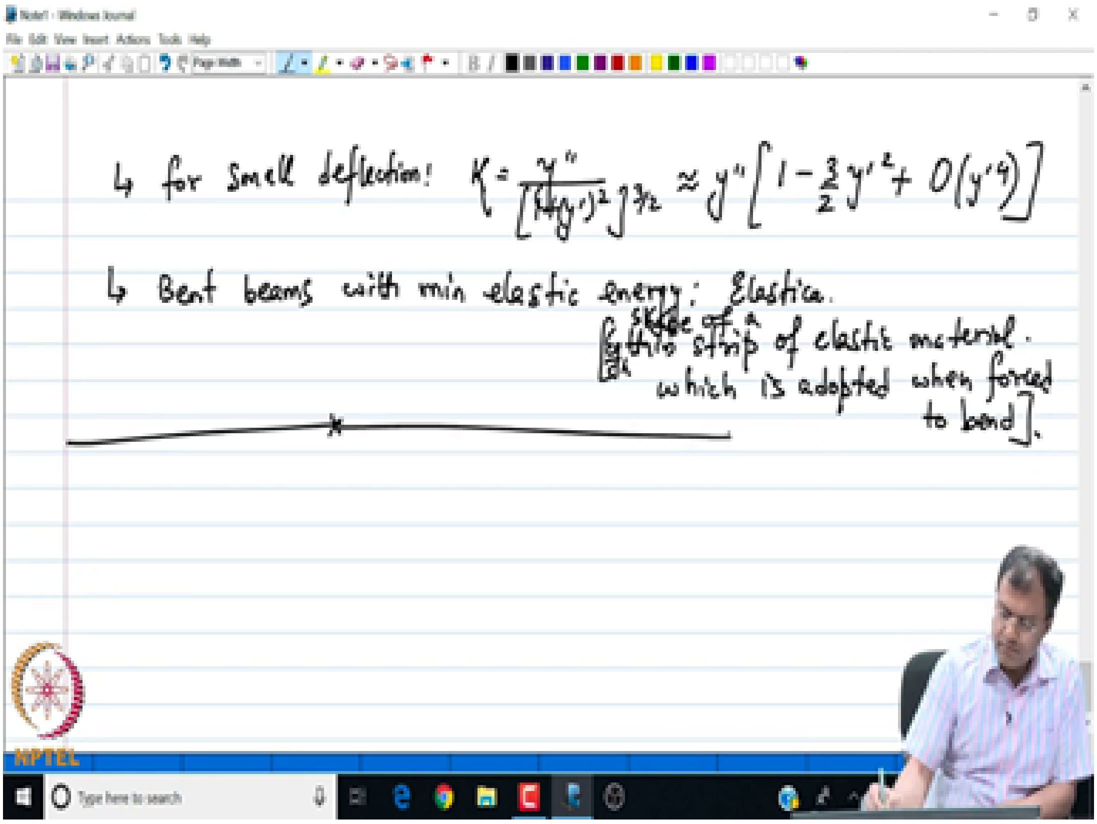
$$\Rightarrow \ddot{x}(s) = \frac{d}{ds}(\dot{x}) = \frac{d}{dx}(\dot{x})\frac{dx}{ds} = \dot{x}(s)(\dot{x})' = \frac{1}{\sqrt{1+(y')^2}}\frac{d}{dx}\left(\frac{1}{\sqrt{1+(y')^2}}\right) = \frac{-y'y''}{[1+(y')^2]^2}$$

Similarly I can find $\ddot{y}(s)$. How should we do that? We know that $\dot{x}^2 + \dot{y}^2 = 1 \Rightarrow \dot{y}(s) = \dot{x}(s)\dot{y}'(s)$

$\Rightarrow \ddot{y}(s) = \frac{y''}{[1+(y')^2]^2}$, finally my curvature $\kappa^2(s) = \ddot{x}^2 + \ddot{y}^2 = \frac{(y'')^2}{[1+(y')^2]^3}$

Notice that my curvature now depends on the slope of the curve y' and it reduces to $(y'')^2$ when the slope is negligibly small. So, what have I said is the following.

(Refer Slide Time: 28:18)



For small deflection $\kappa(s) = \frac{y''}{[1+(y')^2]^{\frac{3}{2}}} \approx y''\left[1 - \frac{3}{2}(y')^2 + O((y')^4)\right]$

We are going to look at the class of beam problems from now in this lecture where we minimize the elastic energy and those class of beam problems with minimum elastic energy are known as elastica.

So, we are going to look at the shape of elastica for a different setup, bent beams with minimum elastic energy are known as elastica, Well, typical example is a thin strip of elastic material which adopts a shape when forced to bend. So, we are going to look at two examples of elastica.