Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 41
Hamilton-Jacobi Equations Part 5

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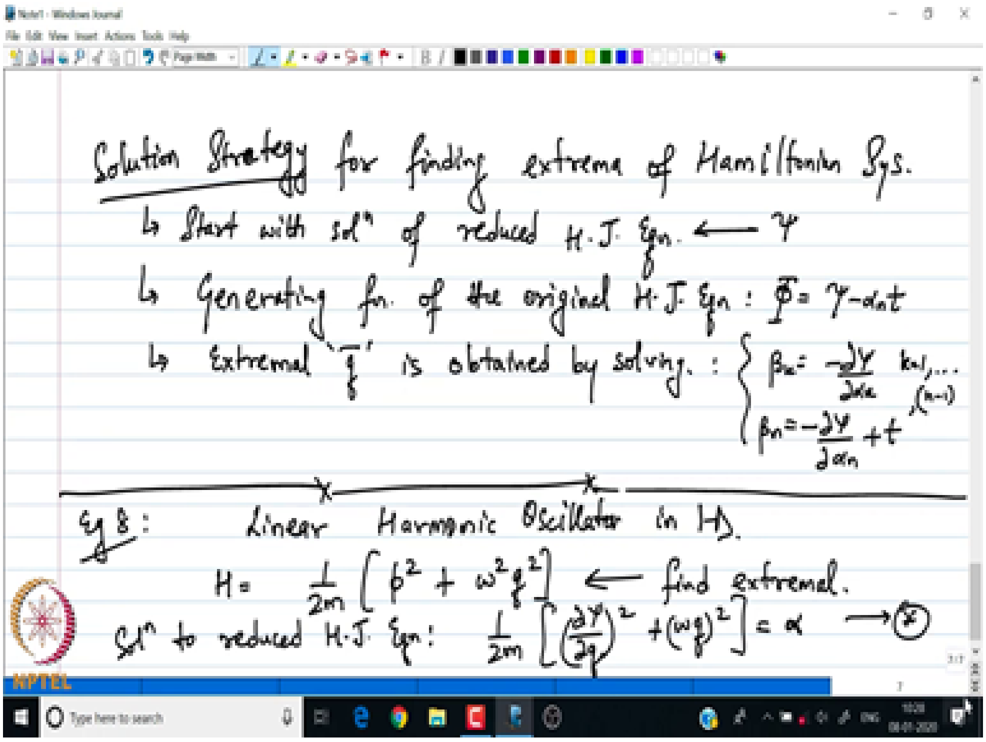

So, the example I have is, example number 8, is that of the Linear Harmonic Oscillator in 1-D, one dimension. So,my H, the Hamiltonian given in this problem is the following:

$$H = \frac{1}{2m}\left[p^2 + \omega^2 q^2\right]$$

And we need to find extremal. Clearly, the H is independent of the variable t which is the independent variable.

So we look at the solution to reduced, Hamilton-Jacobi equation, which is nothing but, the reduced Hamilton-Jacobi equation will be

$$H = \frac{1}{2m}\left[\left(\frac{\partial \psi}{\partial q}\right)^2 + (\omega q)^2\right] = \alpha \qquad (*)$$

So, we have to solve this equation, so, to find ψ. So let me call this as, as my star.

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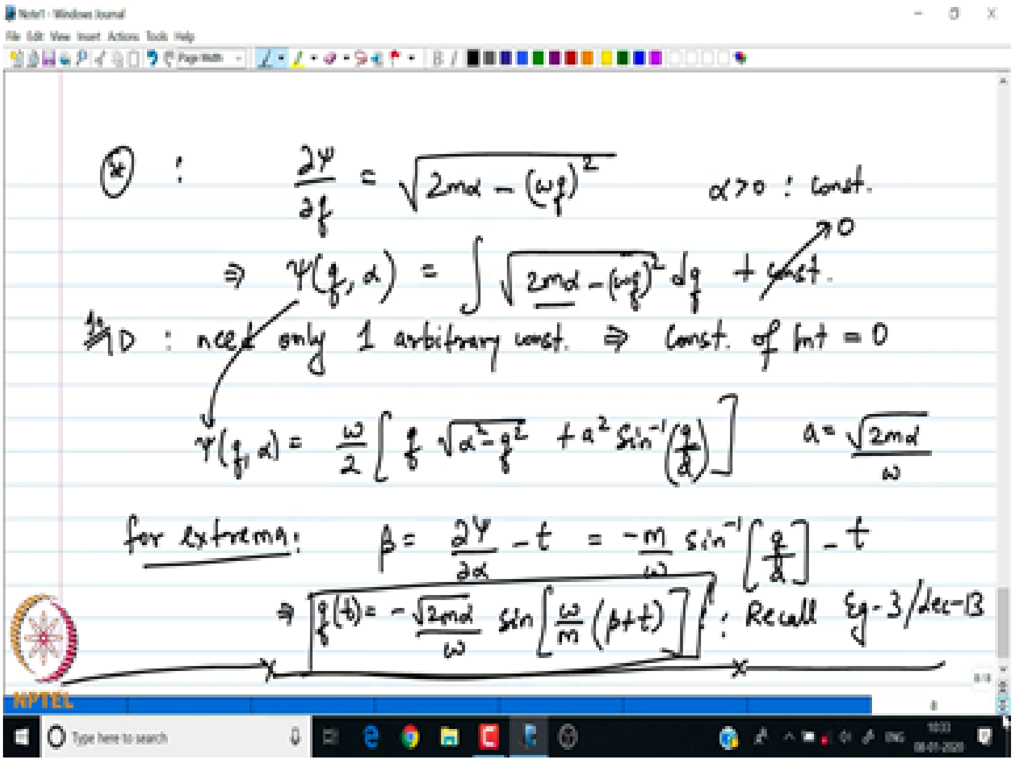

So from star, I see that

$$\frac{\partial\psi}{\partial q} = \sqrt{2m\alpha - (\omega q)^2}$$

where alpha is positive constant. And then, the next step involves integrating this equation.

$$\psi(q,\alpha) = \int \sqrt{2m\alpha - (\omega q)^2}\partial q + constant$$

So,this is without loss of generality, because even if we keep more constants, later on in the solution, we can always club them to reduce the number of constants. So that is why, what we do is, we take the constant of integration to be 0. So constant of integration, we take it equal to 0. So this is 0, because we already have one constant sitting here. So, then, we can (direct), I am going to directly write down the solution to this integral equation

$$\psi(q,\alpha) = \frac{\omega}{2}\left[q\sqrt{\alpha^2 - q^2} + a^2 \arcsin\left(\frac{q}{a}\right)\right]$$

$$where \qquad a = \frac{\sqrt{2m\alpha}}{\omega}$$

And,for extrema, I just need to solve one equation that I highlighted in my solution strategy

$$\beta = \frac{\partial\psi}{\partial\alpha} - t = -\frac{m}{\omega}\arcsin\left[\frac{q}{\alpha}\right] - t$$

$$\Rightarrow \qquad q(t) = -\frac{\sqrt{2m\alpha}}{\omega}\sin\left[\frac{\omega}{m}(\beta + t)\right]$$

And the solution is similar to, recall that we had looked at this problem of geometric optics in more than one occasion. Specially we had seen the solution to this system in our previous lecture.

So recall, I have example 3 in my lecture 13, the previous lecture series. So, that completes this example and let us now, look at how to solve, we have right now just shown the solution to this special class of Hamiltonian system, namely conservative system. The next set of discussion will be involving how to solve, this Hamilton-Jacobi equation for this conservative system class.

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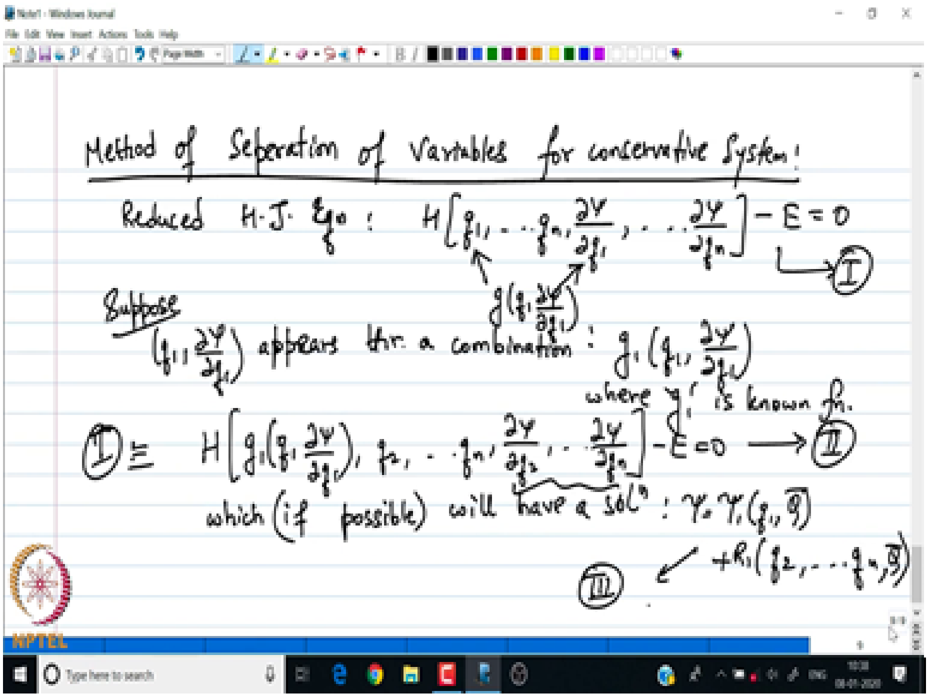

Namely, we are going to look at the method of separation of variables. So, method of separation, which we promised few minutes back, that I am going to talk about it. So let us look at this, this method in the most general form. Consider the reduced Hamilton-Jacobi equation for the conservative class ,which is equal to a constant , let me call the constant as E.So,I am going to rewrite this by saying that this is equal to 0.

$$H\left[q_1, q_2..., q_n, \frac{\partial\psi}{\partial q_1}, ..., \frac{\partial\psi}{\partial q_n}\right] - E = 0 \qquad \textbf{(1)}$$

Now suppose, we are in a situation where, all the functions of q_1,q_2,..q_n they can be clubbed together in the form of $g\left(q_1, \frac{\partial\psi}{\partial q_1}\right)$ Suppose we are able to separate out one of the variables, that is q_1 and write it in a specific form. Let us say the function to be g of that variable.

So what I said is the following. Suppose $\left(q_1, \frac{\partial\psi}{\partial q_1}\right)$ appears through a combination of the form $g_1\left(q_1, \frac{\partial\psi}{\partial q_1}\right)$ where g_1 is a known function. Now equation 1 is identical to the following form

$$H\left[g_1\left(q_1, \frac{\partial\psi}{\partial q_1}\right), q_2..., q_n, \frac{\partial\psi}{\partial q_2}, ..., \frac{\partial\psi}{\partial q_n}\right] - E = 0 \qquad \textbf{(2)}$$

Now, I, notice that since we have separated out the variable q_1, possibly in this situation, we can have a solution in which the variable q_1 can be separated. So, if possible,let me call this expression be 2, will have a solution,of the form

$$\psi = \psi_1(q_1, Q) + R_1(q_2, ..., q_n, Q) \qquad \textbf{(3)}$$

So, we have separated out, you know, the solution in which the variable q_1, appears, where capital Q is the generalized coordinate, the coordinate in the new Hamiltonian system, so, that can also appear here. So, I call this set of equation to be my 3. So notice, what have we got here? So, notice that since my variables q_2 to q_n have been separated out from q_1, so then $\frac{\partial\psi}{\partial q_2}$ will be nothing but $\frac{\partial R_1}{\partial q_2}$ and so on forth. So if I were to use 3 in 2, then some of these derivatives can be simplified.

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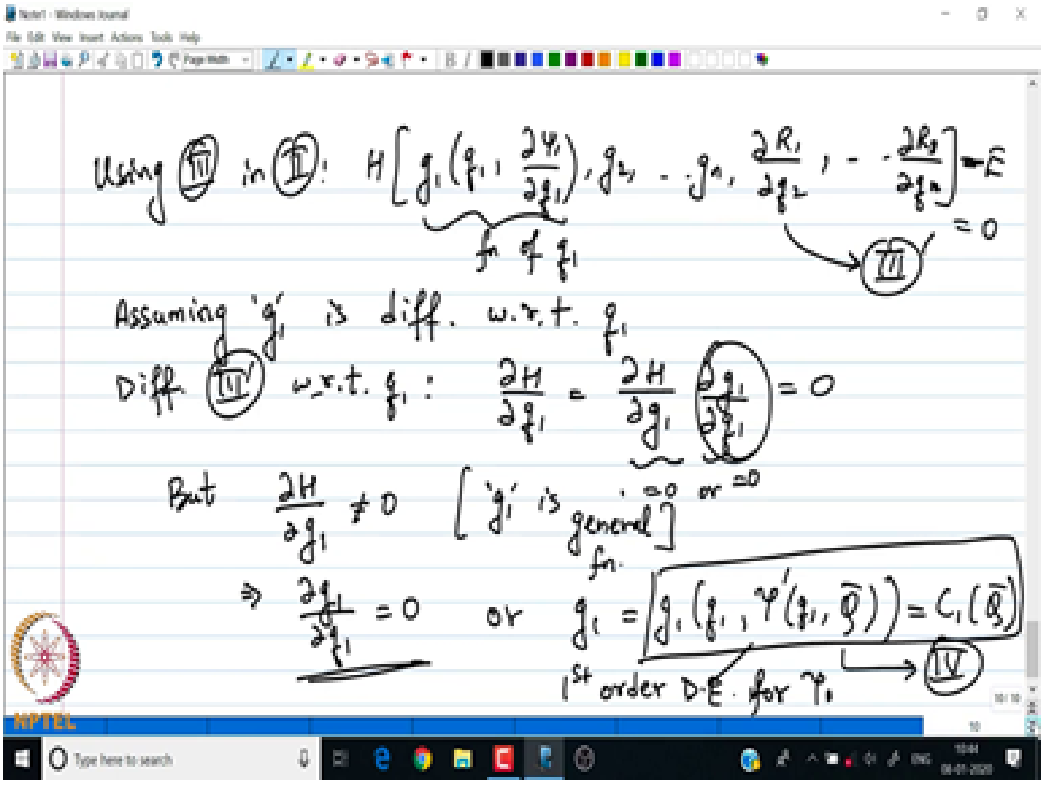

So using expression 3 in 2, then I see that

$$H\left[g_1\left(q_1, \frac{\partial \psi}{\partial q_1}\right), q_2..., q_n, \frac{\partial R_1}{\partial q_2}, ..., \frac{\partial R_n}{\partial q_n}\right] - E = 0 \qquad \textbf{(3')}$$

So, what we got? Notice that g_1 is only a function of, q_1

Suppose I have found this function g_1, such that g_1 is differentiable with respect to q_1. So, assuming g_1 is differentiable with respect to q_1, So we differentiate $3'$ with respect to q_1. I see the following

$$\frac{\partial H}{\partial q_1} = \frac{\partial H}{\partial g_1}\frac{\partial g_1}{\partial q_1} = 0$$

Now, this equation tells us that either $\frac{\partial H}{\partial g_1} = 0$ or $\frac{\partial g_1}{\partial q_1} = 0$. But $\frac{\partial H}{\partial g_1}$ cannot be 0, because g_1 is a general function.We cannot, all the time have that $\frac{\partial H}{\partial g_1} = 0$. For certain class of functions g_1, it may be the case but not for all of functions g_1. So, which means that

$$\frac{\partial g_1}{\partial q_1} = 0 \;\; or \;\; g_1 = g_1\left(q_1, \psi'\left(q_1, Q\right)\right) = C_1\left(Q\right) \qquad \textbf{(4)}$$

So, which means that this condition here, the condition 4 is actually a first order differential equation for my unknown ψ_1 . ψ_1 is the only function which depends on q_1. So, from here I can find out my function ψ_1. So, similarly, we can continue this process for ψ_2 , ψ_3 ,ψ_4 and ψ_n and so on.

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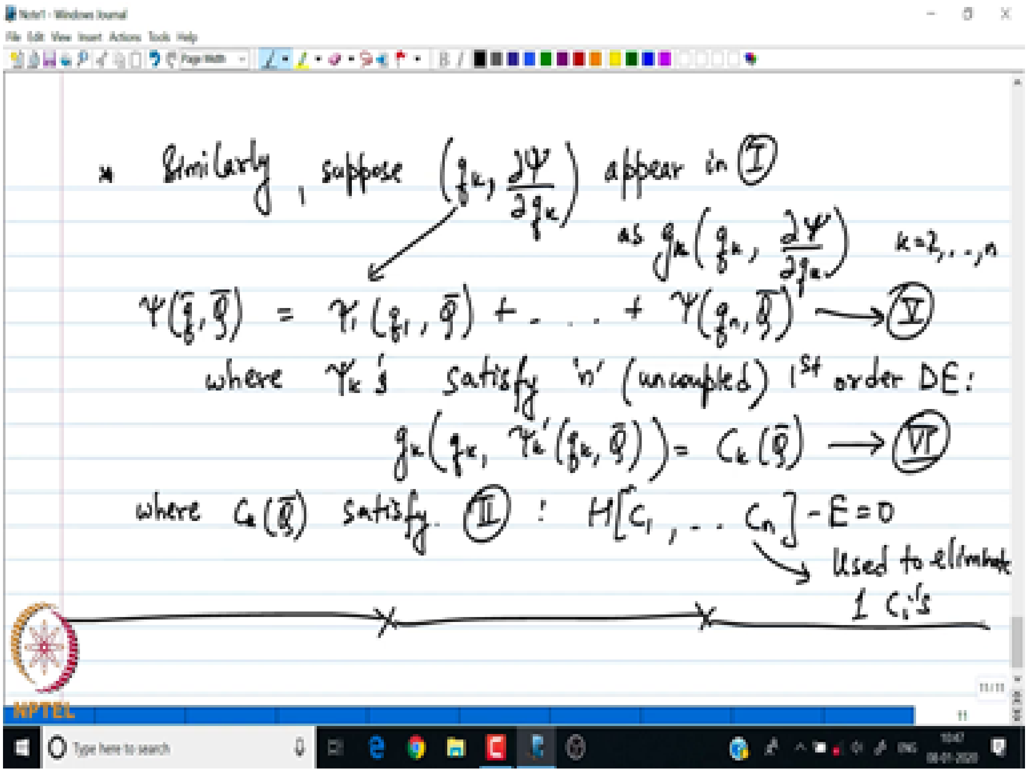

Similarly,what I have is, suppose $\left(q_k, \frac{\partial \psi}{\partial q_k}\right)$ they appear in 1 as the functional form $g_k\left(q_k, \frac{\partial \psi}{\partial q_k}\right)$ where g_k's are all known function from k 2 ,.., n, for each of the cases and from here I can deduce that

$$\psi\left(\bar{q}, Q\right) + \ldots + \psi_n\left(q_n, Q\right) \qquad \textbf{(5)}$$

Where my function ψ_k's, satisfy n uncoupled first order differential equations, of the form

$$g_k\left(q_k, {\psi_k}'\left(q_k, Q\right)\right) = C_k\left(Q\right) \qquad \textbf{(6)}$$

In the similar fashion, we will have the first order differential equation for each component, like we did it in the first component,where Ck's are not independent

So C_k's, the constants, they satisfy, the reduced Hamilton-Jacobi equation. So they satisfy 2. So what is 2? So instead of g_k's, I replace the g_k's by C_k's. So they satisfy 2, which means that

$$H\left[C_1, \ldots, C_n\right] - E = 0$$

Essentially, this equation is used to eliminate, 1 of the Ci's. So which means thatC_i's are not completely linearly independent of each other, but this is one linear dependence, in which we can eliminate one of the constants. So, far I have described the general methodology of the separation of variables for conservative systems.

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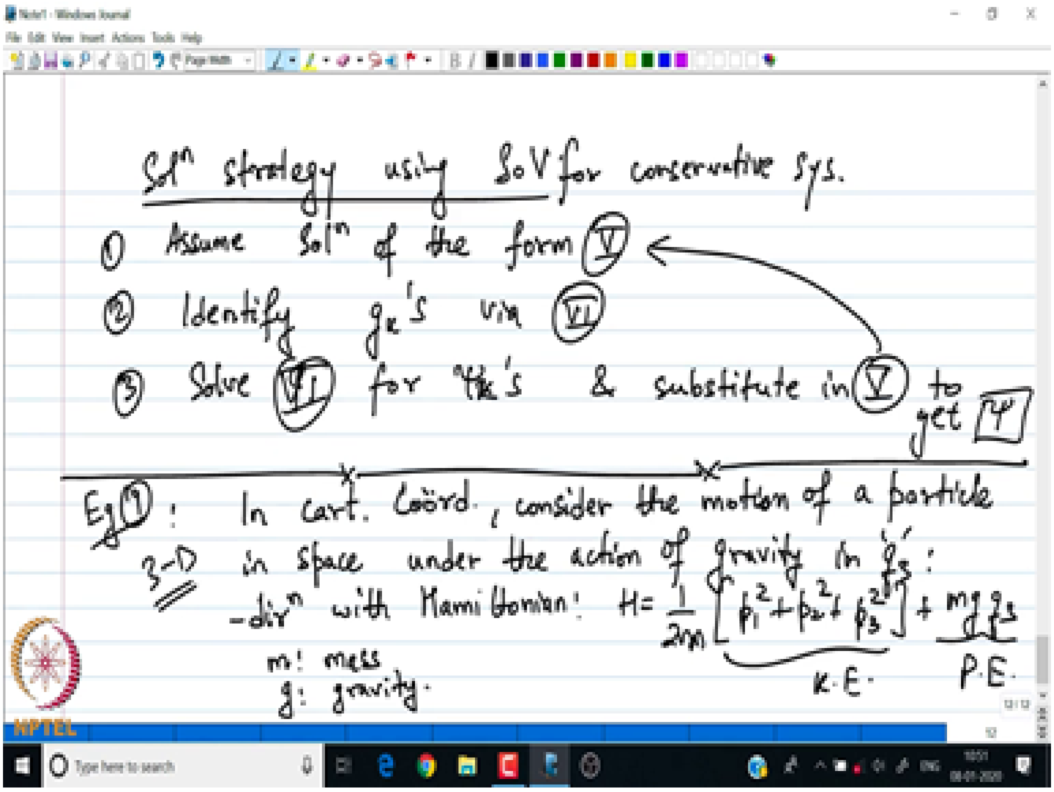

So, let me summarize.solution strategy using separation of variables for conservative systems. So let us say we assume solution of the form 5. then identify the g_k's the functional dependence of q_i's via 6.

So, again going back, , we solve 6 for ψk's, and substitute, in our equation 5 to get our generating function ψ. And which is what we are after, which is the solution of the reduced Hamilton-Jacobi. So let us look at one example which involves this solution strategy. So the example is numbered as 9, following our sequence

So the example is, we are solving the problem in Cartesian coordinates. we consider the motion of a particle in space under the action of gravity in, and the gravity only acts in one of the coordinates q_3, direction with Hamiltonian as follows :

$$H = \frac{1}{2m}\left[{p_1}^2 + {p_2}^2 + {p_3}^2\right] + mgq_3$$

where my mass is m, g is my gravitation constant and q_1, q_2, q_3 makes their own sense. Now we write down our reduced Hamilton-Jacobi equation.

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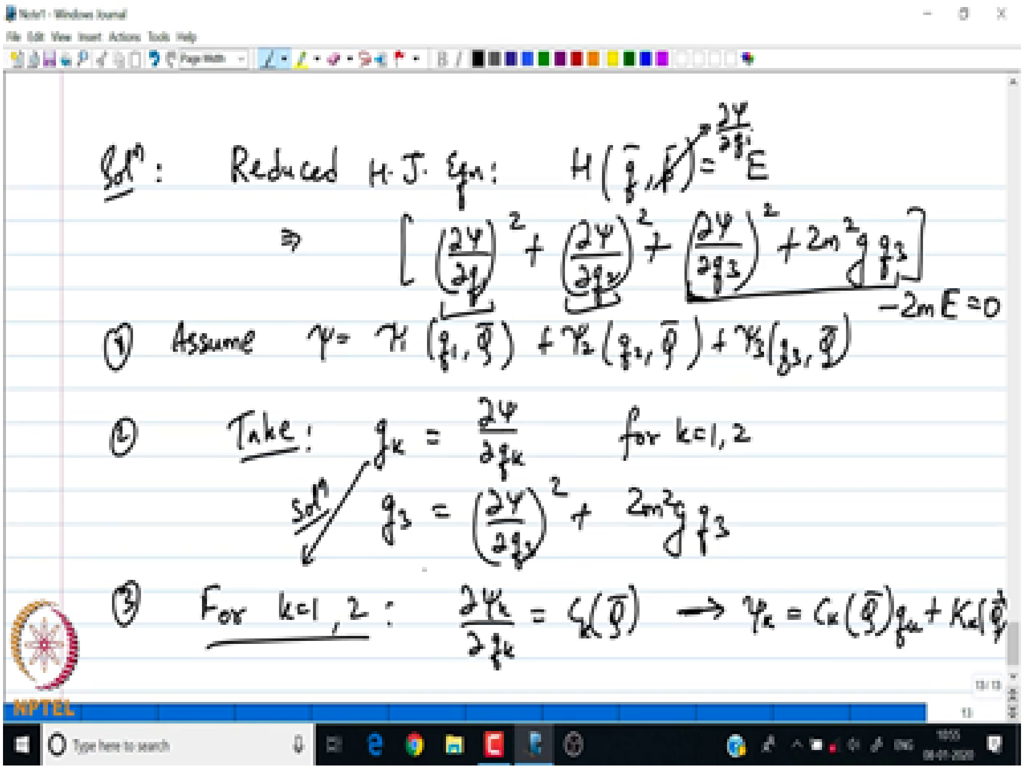

So, my reduced Hamilton-Jacobi equation is as follows:

$$H\left(\bar{q},\bar{p}\right)=E$$

Where E is a constant then from here, I replace p by $\frac{\partial\psi}{\partial q_i}$ and from here I know the form of the Hamiltonian. So I get the equation as follows:

$$\left[\left(\frac{\partial\psi}{\partial q_1}\right)^2+\left(\frac{\partial\psi}{\partial q_2}\right)^2+\left(\frac{\partial\psi}{\partial q_3}\right)^2+2m^2gq_3\right]-2mE=0$$

Now I have to solve this system, we have to separate variables or the first step involves identifying the functions g_k's, in which the variables q_1, q_2, q_3 can be clubbed together. First of all, the step 1 involves, assume a separable solution. So assume $\psi=\psi_1\left(q_1,\bar{Q}\right)+\psi_2\left(q_2,\bar{Q}\right)+\psi_3\left(q_3,\bar{Q}\right)$ The second stage is identifying g_k's. So we can take $g_k=\frac{\partial\psi}{\partial q_k}$ for my first two components k =1, 2. Notice that they are quite similar.

But for the third component I have to take my g_k to be

$$g_3=\left(\frac{\partial\psi}{\partial q_3}\right)^2+2m^2gq_3$$

Note that I have not taken a square in g_1 and g_2 because if if I had take a square, the solution involves g_k set equal to the constant C_k. So, we could have taken square roots. So taking square does not matter in the first two components.

So then, the third stage involves the solution to these to find ψ_k's. So note that, for k = 1 and 2, the solution methodology is identical and we see that $\frac{\partial\psi_k}{\partial q_k}=C_k\left(\bar{Q}\right)$ or from here I see that $\psi_k=C_k\left(\bar{Q}\right)q_k+K_k\left(\bar{Q}\right)$. I have just integrated this. We see that the ψ_k's for k equal to 1 and 2, they are linear functions of the respective position coordinates q_1 and q_2.

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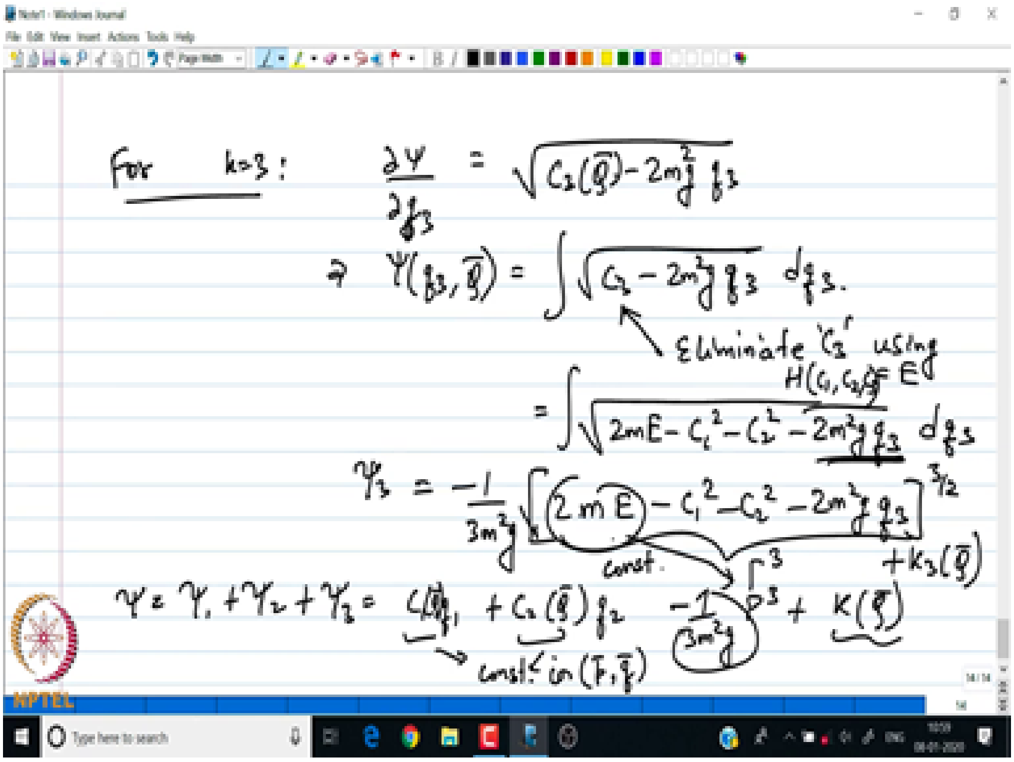

For the third case g_3 for k =3, I have that

$$\frac{\partial\psi}{\partial q_3} = \sqrt{C_3\left(\bar{Q}\right) - 2m^2 g q_3}$$

So now I have to integrate this equation

$$\psi\left(q_3, \bar{Q}\right) = \int\left(\sqrt{C_3\left(\bar{Q}\right) - 2m^2 g q_3}\right) dq_3$$

So this is integral with respect to q3, but we can eliminate C3 in terms of C1 and C2 using my H = E, using that expression.

So we eliminate one of the constants. As I said, we have one constraint H is equal to E, so H is a function now of C_1, C_2, C_3. From here, I get that

$$\psi\left(q_3, \bar{Q}\right) = \int\left(\sqrt{2mE - C_1^2 - C_2^2 - 2m^2 g q_3}\right) dq_3$$

Now see that this integration is with respect to q_3

So, I am going to write down the answer

$$\psi_3 = -\frac{1}{3m^2 g}\left[2mE - C_1^2 - C_2^2 - 2m^2 g q_3\right]^{\frac{3}{2}} + K_3\left(\bar{Q}\right)$$

Now, let me call the quantity inside the power 3 by 2 as γ^3, where gamma is the square root of this quantity here, which is underlined.

So finally, my solution is

$$\psi = \psi_1 + \psi_2 + \psi_3 = C_1\left(\bar{Q}\right) q_1 + C_2\left(\bar{Q}\right) q_2 - \frac{1}{3m^2 g}\gamma^3 + K\left(\bar{Q}\right)$$

Now, notice that, C_1 and C_2, these two are my constants. These are my constants in the framework $(\bar{p}, \bar{q})$. These are my constants in the original reference frame .

Well, we also have a constant, which is inside gamma.And finally, we are saying, we are introducing another constant, so we have introduced 4 constants. As I said previously in an n dimensional problem, we try to keep n constants. So we take K such that K is already absorbed in the third constant, 2mE, so we only have a 3 constant problem.

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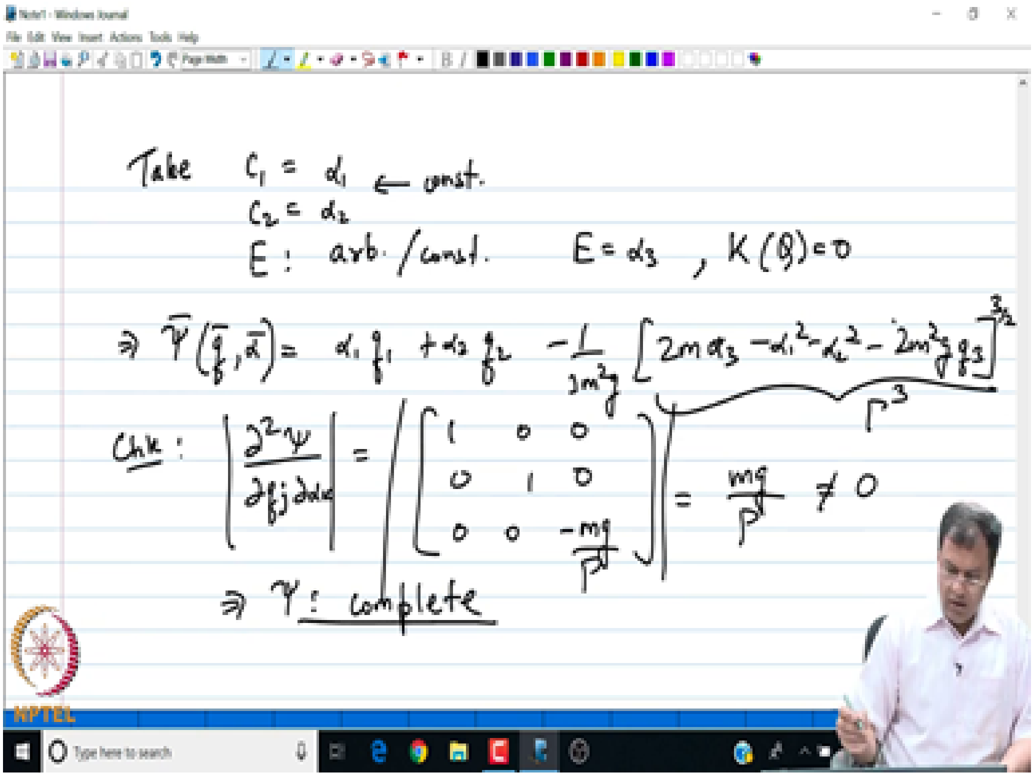

So, what have we done is, we take our C_1 to be α_1 and C_2 to be α_2. These are my constants in my original reference frame my E is arbitrary/constant, so that, I take my E to be α_3 and I take my $K(Q) = 0$, which means essentially, I have absorbed one constant.

So now my solution of the reduced Hamilton-Jacobi looks like the following.

$$\bar{\psi}(\bar{q}, \bar{\alpha}) = \alpha_1 q_1 + \alpha_2 q_2 - \frac{1}{3m^2 g}\left[2m\alpha_3 - \alpha_1^2 - \alpha_2^2 - 2m^2 g q_3\right]^{\frac{3}{2}}.$$

So all it needs to see is whether this solution is complete or not and only then we are able to find the extremal q.

So check

$$\frac{\partial^2 \psi}{\partial q_j \partial \alpha_k} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{mg}{\gamma} \end{pmatrix}$$

So the determinant of this matrix is certainly not equal to 0, provided $|gamma$ is defined. So the conclusion is, ψ is complete, which means, now we are able to find the extremal.

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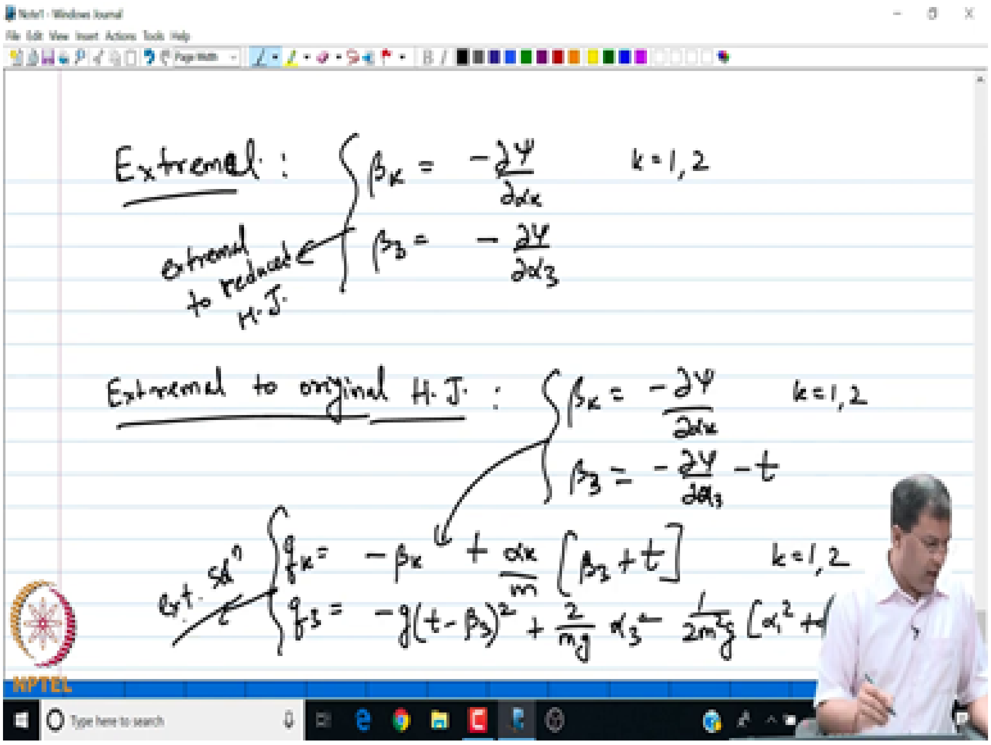

The extremal is given by the solution to the following set of equations. I see that

$$\beta_k = -\frac{\partial \psi}{\partial \alpha_k} \qquad k = 1, 2$$
$$\beta_3 = -\frac{\partial \psi}{\partial \alpha_3}$$

This will give my extremalto the reduced, to the reduced Hamilton-Jacobi equation. Now, if you were to seek the extremal to the original Hamilton-Jacobi equation .So what is the relation between the original and the reduced?

We had the dependence of t. So here, the, the solution to the original Hamilton-Jacobi will

$$\beta_k = -\frac{\partial \psi}{\partial \alpha_k} \qquad k = 1, 2$$
$$\beta_3 = -\frac{\partial \psi}{\partial \alpha_3} - t$$

And from here, I can find out that my extremals are such that

$$q_k = -\beta_k + \frac{\alpha_k}{m}\left[\beta_3 + t\right] \qquad k = 1, 2$$
$$q_3 = -g\left(t - \beta_3\right)^2 + \frac{2}{mg}\alpha_3 - \frac{1}{2m^2 g}\left[\alpha_1^2 + \alpha_2^2\right]$$

So, my original Hamilton-Jacobi is where the time, the independent variable appears. So, once we have the solution to the reduced Hamilton-Jacobi, we have shown that all we had to do was to change our last coordinate. And that, from here we can get the extremals. So this is my extremal solution.