Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 46
Noether's Theorem, Introduction to Second Variation Part 4

In today's lecture, I am going to talk about the final topic on the necessary condition for finding the extremal of the functional, namely on how to find the transformation or the variational symmetry that leads to the conservation law. Remember, in my previous lecture, we had spoken about Noether's theorem, which relates the variational symmetries with the conservation law. So all it remains now is how to find these variational symmetries

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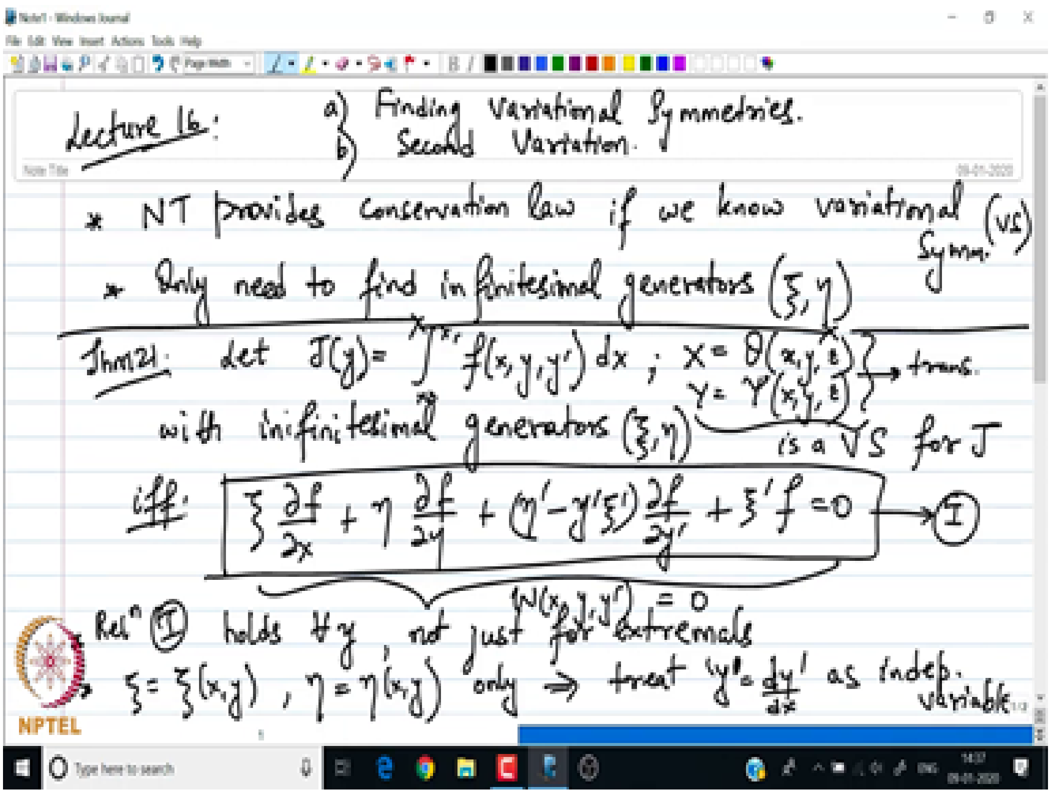

So, then towards the latter half of this discussion, I am also going to touch upon the new topic of finding the sufficient condition for an extremal, namely the topic on second variation. So, the topic we will cover today have two parts. First, we will discuss about how to find the transformation or the variational symmetries that lead to the conservation laws. And the second half of this discussion will involve finding the second variation of the functional, what is the use and why do we need to see the second variation.

So, as I had mentioned in my previous lecture, Noether's theorem provides me the conservation law if we know the variational symmetry. So,the key is to find these variational symmetries, let me denoted it by VS. In other words, all we need to find are the infinitesimal generators of these transformations. So, only thing that we need to find infinitesimal generators (ξ, η) that we saw in our previous lecture.

So,I am going to right away provide the results on how to find these infinitesimal generators and the result is in the form of a theorem. We will use this theorem in some of our examples which will follow next to look at how to apply this result. So, let us say that we are given a functional of the form

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')\, dx$$

And we are given the transformations

$$X = \theta(x, y, \epsilon)$$

$$Y = \psi(x, y, \epsilon)$$

So, these are transformationswith infinitesimal generators (ξ, η). So, we say that this transformation is a variational symmetry for J if and only if the following differential equation is satisfied.

$$\xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + (\eta' - y'\xi') \frac{\partial f}{\partial y'} + \xi' f = 0 \qquad \textbf{(1)}$$

So, if and only if, the moment we satisfy this differential equation, we are guaranteed to say that (ξ, η) is the infinitesimal generator are leading to our variational symmetries. The key thing is that equation 1 does not necessarily holds only for extremal y. It will hold for all (x, y) in the given domain, which means in order to find out (ξ, η) from this equation, we can essentially equate the coefficients of various powers of y', keeping x and y fixed.

So what I just said is the following relation(1) holds $\forall y$, not just for extremals. And the other issue that I highlighted is $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ they are both functions of x, y. So we can assume, since they are only functions of x and y, it implies we can treat $y' = \frac{\mathrm{d}y}{\mathrm{d}x}$ as independent variable.

So they do not appear in the expression for (ξ, η) the infinitesimal generator. So to get (ξ, η) we have to equate the coefficients of various powers of y'. noNtice that I am going to use a shorthand notationin particular the expression in the left hand side of relation (1) is said to be a function $W(x, y, y')$

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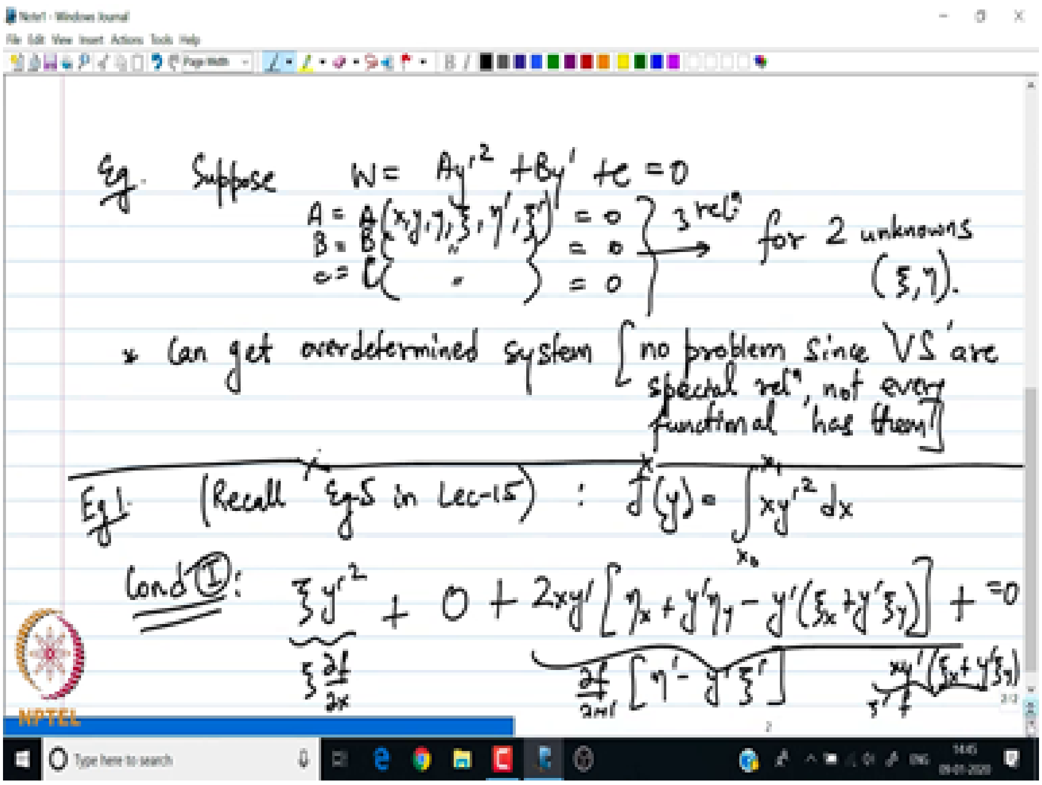

In particular suppose W is of this form $W = Ay'^2 + By' + C = 0,$. Now typically

$$A = A\left(x, y, \eta, \xi, \eta', \xi'\right)$$
$$B = B\left(x, y, \eta, \xi, \eta', \xi'\right)$$
$$C = C\left(x, y, \eta, \xi, \eta', \xi'\right)$$

are all functions of $(x, y, \eta, \xi, \eta', \xi')$. So these are all the variables that we can get which means, all I just said is to find out the relation η *and* ξ, we just equate coefficients of powers of y' to 0.

So, we get three relations for two unknowns which are η *and* ξ, which means it seems that the system is over determined. But we will see that is not the case becauseη *and* ξ they correspond to the infinitesimal generators of the variational symmetry. So variational symmetries are very special, they will not be found for every functional. So this, although it seems the system is over determined, we will see generally that will not be a problem.

So,what I just said is that we can get over determined system and that is not a problem since variational symmetries are special relations and not every functional has them. We will see these ideas through some examples.

So the example that I have in today's lecture as this is a new sequence. So I start with example number 1. So the first example is we recall our example 5 in lecture 15, our previous lecture we said we took a functional of the form $J(y) = \int_{x_0}^{x_1} xy'^2 dx$.

So condition (1) gives us the following equation:

$$\xi y'^2 + 0 + 2xy' \left[\eta_x + y'\eta_y - y'\left(\xi_x + y'\xi_y\right)\right] + xy'\left(\xi_x + y'\xi_y\right) = 0$$

So, all that remains in this expression .Now we equate the different powers of y'.

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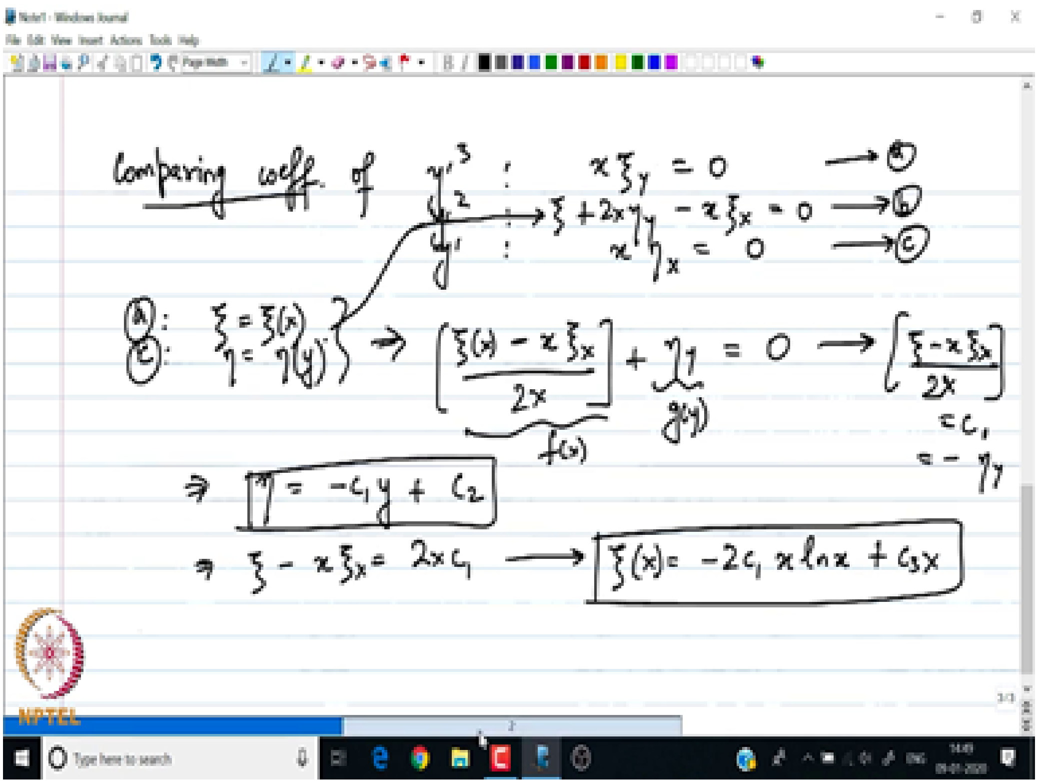

So when we do that, comparing the coefficients of y'^3 , y'^2 , y' we get the following expressions

$$x\xi_y = 0 \qquad \textbf{(a)}$$

$$\xi + 2x\eta_y - x\xi_x = 0 \qquad \textbf{(b)}$$

$$x\eta_x = 0 \qquad \textbf{(c)}$$

So, from (a), I am going to get that $\xi = \xi(x)$ is a function of x purely, because x cannot be 0 in general. And from (c), I get that $\eta = \eta(y)$ must be a function of y. Now we plug all this result into our expression (b).

And what we get is

$$\left[\frac{\xi(x) - x\xi_x}{2x}\right] + \eta_y = 0$$

Now notice that this bracketed quantity is purely a function of x and this quantity is purely a function of y and they are negative equal to each other, provided both of them are constants.

We are getting

$$\left[\frac{\xi(x) - x\xi_x}{2x}\right] = c_1 = -\eta_y$$

From here I get

$$\eta = -c_1 y + c_2$$

So, that is functional dependence of y on η and also solving the first equation. So I have

$$\xi - x\xi_x = 2xc_1$$

Let me write down the solution. We have to just integrate once. And the solution I get from here is

$$\xi(x) = -2c_1 x \ln x + c_3 x$$

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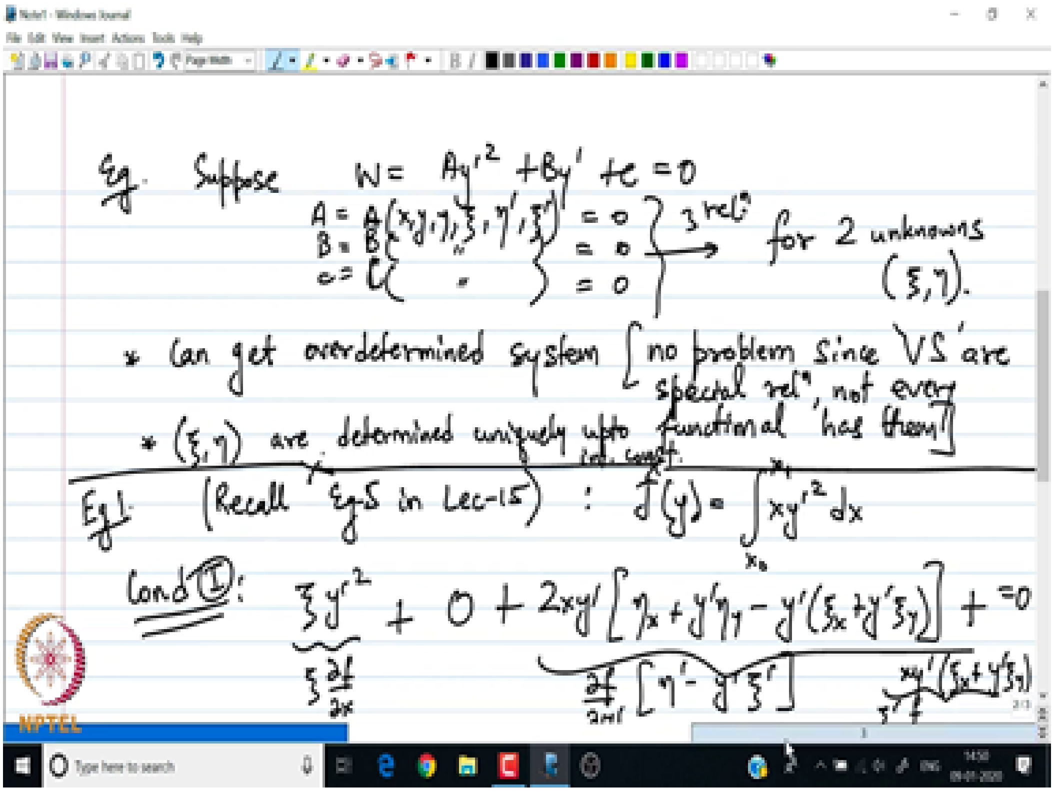

Now, we have got the infinitesimal generators. But notice that these generators are unique up to a family of constants. So that is something I wanted to mention. I should have mentioned a little bit earlier. So let us go back. My infinitesimal generators, ξ *and* η are determined uniquely up to integration constants.

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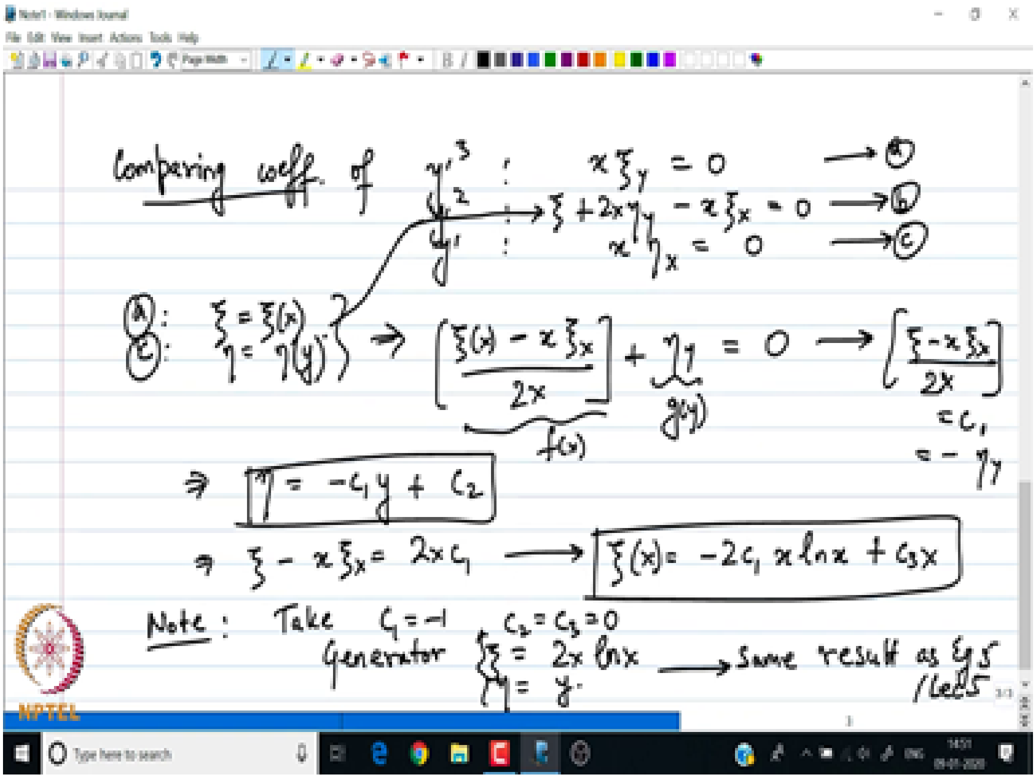

And, so let us come back to the example we are discussing. Suppose, we take $c_1 = -1$, $c_2 = c_3 = 0$. So I get my generators

$$\xi = 2x \ln x$$

$$\eta = y$$

And this is the same result that we saw as in example 5, lecture 5. We saw that this was one of the infinitesimal generators and we have recovered that through specific set of these constants.

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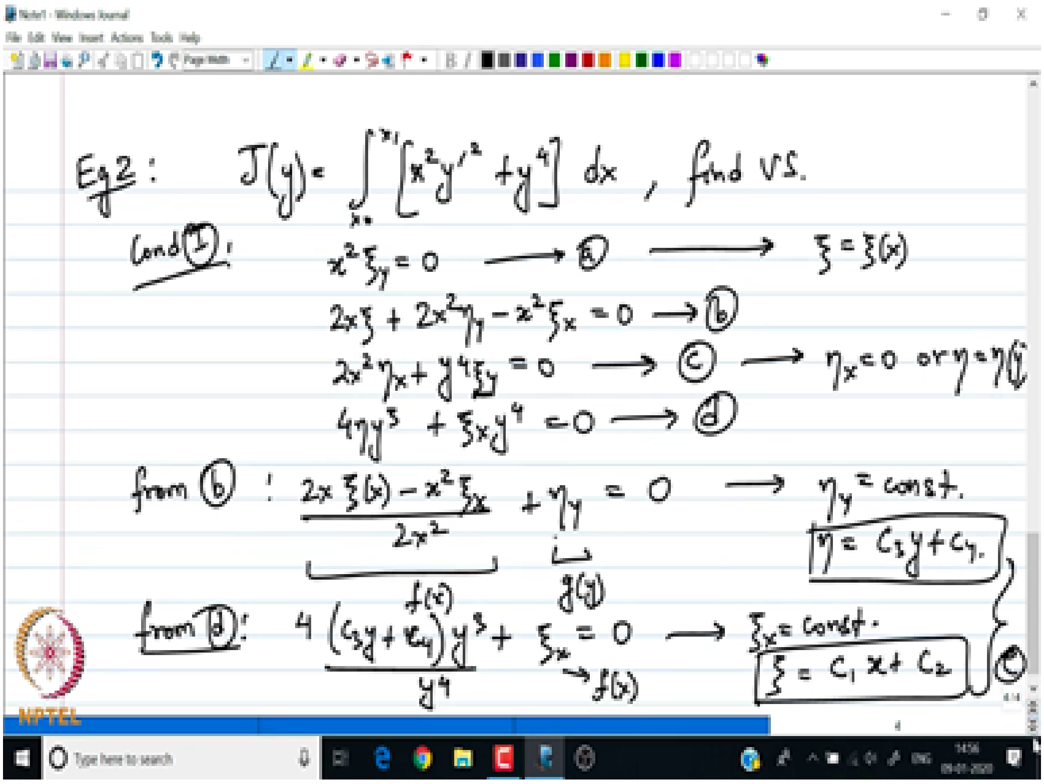

The whole idea of finding infinitesimal generators by solving condition (1) may not yield any result. So let us look at the case. The second example involves this functional which is

$$J\left(y\right) = \int_{x_0}^{x_1}\left[x^2y'^2 + y^4\right]dx$$

We need to find the variational symmetries

So using condition (1) the coefficients of various powers of y' and so these are:

$$x^2\xi_y = 0 \qquad \textbf{(a)}$$

$$2x\xi + 2x^2\eta_y - x^2\xi_x = 0 \qquad \textbf{(b)}$$

$$2x^2\eta_x + y^4\xi_y = 0 \qquad \textbf{(c)}$$

$$4\eta y^3 + \xi_x y^4 = 0 \qquad \textbf{(d)}$$

Students should check thatwe get all these conditions by equating the coefficients equal to 0, the coefficients of the independent variable y'.

From here, I can immediately conclude from first relation that $\xi = \xi\left(x\right)$ is a function of x and , from condition (c) we see that since $\xi_y = 0$ as ξ is a function of x only. So from this condition, I can see that $\eta_x = 0 \; or \; \eta = \eta\left(y\right)$.

So then we can use condition (b) and (d).From condition (b), notice that ξ is a function of x. So I get

$$\frac{2x\xi(x) - x^2\xi_x}{2x^2} + \eta_y = 0$$

And again, I see that this is a function purely of x and this is equated to negative of η_y and this is only possible when η_y is a constant.So η_y is a constant or η is a straight line. Let us call this as

$$\eta = c_3y + c_4$$

Now, we have found one variational symmetry. Let us now plug it back. So, from the last expression (d) I get

$$\frac{4(c_3y+c_4)y^3}{y^4} + \xi_x = 0$$

So from here I get ξ_x is again a constant. And which means I can find that my ξ is a straight line, so $\xi = c_1x + c_2$. So I have found my variational symmetry in terms of family of 4 constants.

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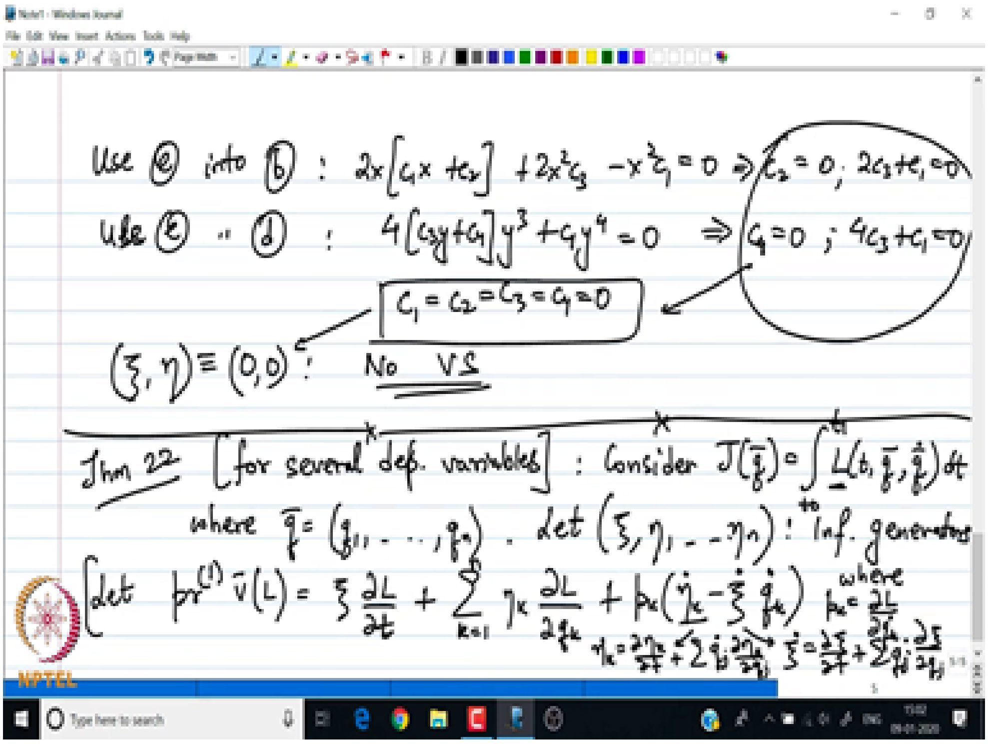

But let us see what happens. So let me call these two relations as my relations (e). So if I plug relation (e) into (b),I see the following.

$$2x\left[c_1x + c_2\right] + 2x^2c_3 - x^2c_1 = 0$$

And from here, I am going to, equating the various powers of x and we get two relations $c_2 = 0 \; and \; 2c_3 + c_1 = 0$

And then we again use (e) into the relation (e). I get the following.

$$4\left[c_3y + c_4\right]y^3 + c_1y^4 = 0$$

From here I get relations

$$c_4 = 0 \;\; ; \;\; 4c_3 + c_1 = 0$$

So, if I were to equate all these 4 equationsI am going to get only one unique solution

$$c_1 = c_2 = c_3 = c_4 = 0.$$

In other words, I get that my infinitesimal generators$(\xi, \eta) \equiv (0, 0)$ are identically 0.

The conclusion is that there is no non-trivial or in fact there is no variational symmetry in this example. So, variational symmetries may not exist. So they exist if and only if they satisfy the conditions that I have shown. So, variational symmetries may not exist for a functional. It all dependson it is variationally invariant or not, the functional itself.

So, I am going to end the discussion on finding variational symmetries by extending the result of condition (1) to functions of several dependent variables and then look at an example. So, then let me state another result in the form of a theorem.This is for several dependent variables. Let us consider

$$J\left(\bar{q}\right)=\int_{t_0}^{t_1} L\left(t,\bar{q},\dot{\bar{q}}\right) \qquad where \quad \bar{q}=\left(q_1,...,q_n\right)$$

And let me consider my variational symmetry as the infinitesimal generator as follows $(\xi, \eta_1, ..., \eta_n)$. So these are my infinitesimal generators if and only if certain condition holds. So before that, let me introduce the function known as the first order prolongation operator of this Lagrangian

$$pr^{(1)}\bar{v}\left(L\right)=\xi\frac{\partial L}{\partial t}+\sum_{k=1}^{n}\eta_k\frac{\partial L}{\partial q_k}+p_k\left(\dot{\eta_k}-\dot{\xi}\dot{q_k}\right) \qquad \text{where}$$

$$p_k=\frac{\partial L}{\partial \dot{q_k}}$$

$$\dot{\xi}=\frac{\partial \xi}{\partial t}+\sum \dot{q_j}\frac{\partial \xi_k}{\partial q_j}$$

$$\dot{\eta_k}=\frac{\partial \eta_k}{\partial t}\sum \dot{q_j}\frac{\partial \eta_k}{\partial q_j}$$

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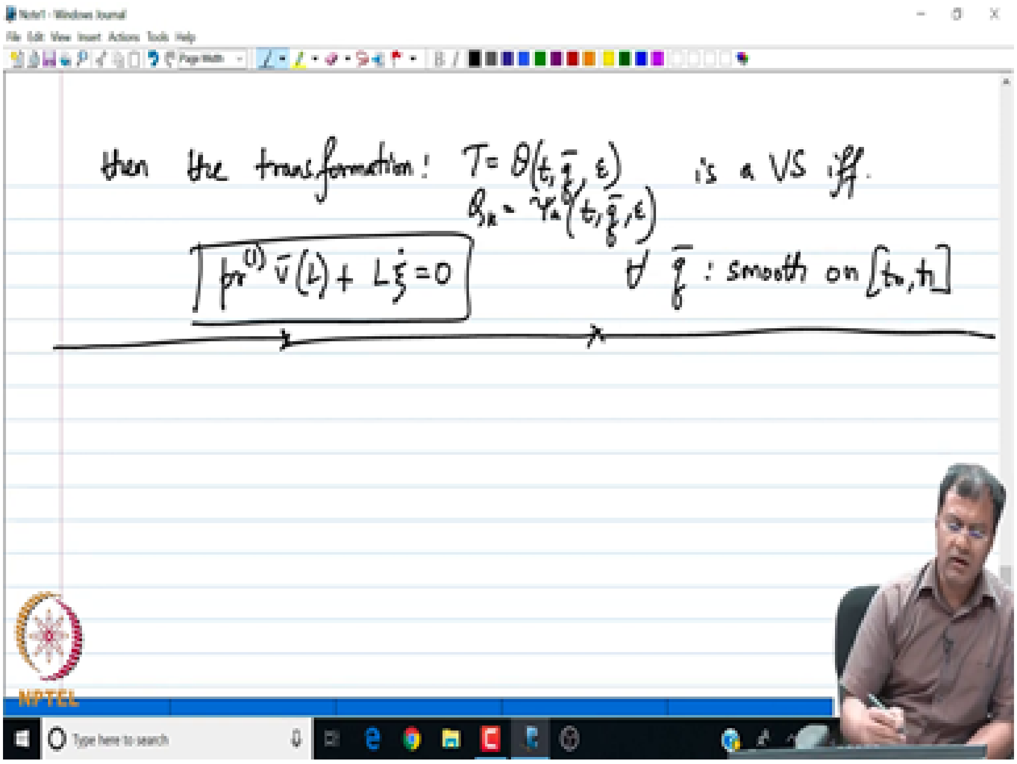

So, the result says that the transformation

$$T=\theta\left(t,\bar{q},\epsilon\right)$$

$$Q_k=\psi_k\left(t,\bar{q},\epsilon\right)$$

is a variational symmetry if and only if I have

$$pr^{(1)}\bar{v}\left(L\right)+L\dot{\xi}=0 \qquad \textbf{(1')}$$

This is the multi variable version of condition (1). So this is, for all the extremal $\bar{q}$ which is a smooth function on the interval $[t_0, t_1]$. So let us look at an application of this result.