Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 48
Noether's Theorem, Introduction to Second Variation Part 6

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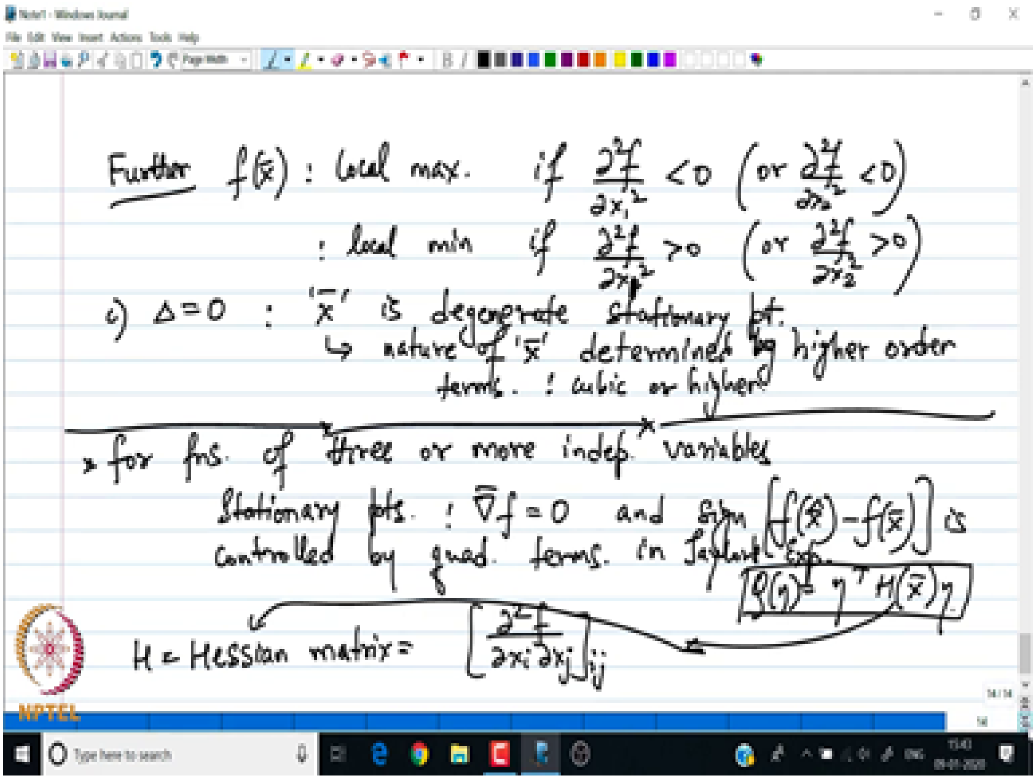

Further, $f(\bar{x})$ is a local maximum if either $\frac{\partial^2 f}{\partial x_1^2} < 0$ or $\frac{\partial^2 f}{\partial x_2^2} < 0$. And it is a local minimum, if either $\frac{\partial^2 f}{\partial x_1^2} > 0$ or $\frac{\partial^2 f}{\partial x_2^2} > 0$. So, that concludes our second derivative test. There is one final statement suppose the discriminant $\Delta = 0$, then our second derivative test fails.So $\bar{x}$ is a degenerate stationary point and the nature this point $\bar{x}$ is determined by higher order terms. Let us say, cubic terms and so on, because no longer we are able to determine the nature of the extremal just by looking at the quadratic term. So, we can continue our discussion for functions of several independent variable.

I am just going to briefly state the result. So, for functions of three or more, three or more independent variables I see that my stationary points are found by $\bar{\nabla} f = 0$ and the sign of $f(\hat{\bar{x}}) - f(\bar{x})$ is controlled by quadratic terms in the Taylor's expansion $Q(\eta) = \eta^T H(\bar{x}) \eta$. So the quadratic terms are governed by this, this product. The sign of this product will tell us the nature of the extremal, where is the so-called Hessian matrix.

And I am sure students are familiar with this matrix. This is of the form $H = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{ij}$.

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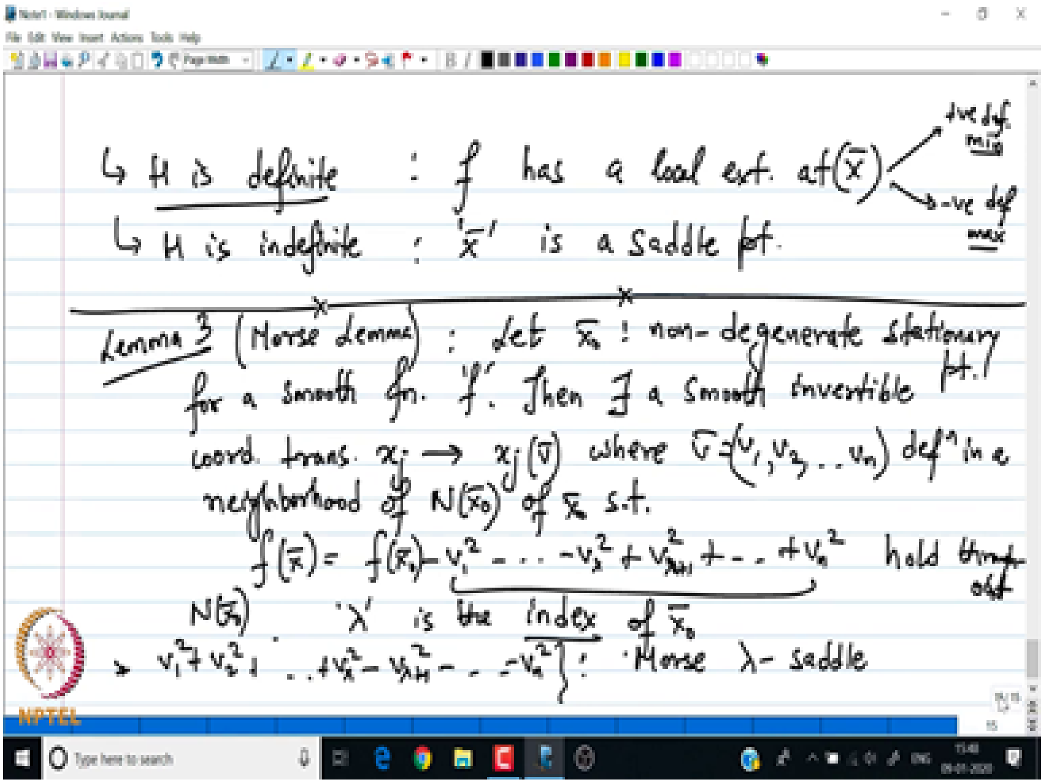

So, suppose Hessian matrix H is definite matrix then the function f has a local extrema at $(\bar{x})$. And suppose H is indefinite matrix, then$(\bar{x})$ is a saddle point, and further we can classify the first statement namely if H is positive definite, then we expect that the local extrema is minima. And if it is negative definite, then the local extrema is the maxima.

So, we can write down these further results. So then, I am going to end my discussion on this finite dimensional calculus by providing two results, which are of extreme importance, specially when we discuss our optimization of functional calculus. The first result is in the form of the Morse Lemma which tells us how to distinguish between the saddle points, the local maxima and the local minima in the neighborhood of extrema.

This is Lemma 3 . Lemma 2 was introduced in lecture 2. So the Morse Lemma says the following: Suppose $\bar{x_0}$ is a non-degenerate stationary point for a smooth function f, then there exists a smooth invertible coordinate transformation $x_j \to x_j(\bar{v})$ where $\bar{v} = (v_1, ..., v_n)$ is defined in a neighborhood $N(\bar{x_0})$ of $\bar{x_0}$ such that

$$f(\bar{x}) = f(\bar{x_0}) - v_1^2 - ... - v_\lambda^2 + v_{\lambda+1}^2 + ... + v_n^2$$

holds throughout $N(\bar{x_0})$, where λ is the index of the point $\bar{x_0}$. Now, what is the significance of λ?

Consider $v_1^2 + ... + v_\lambda^2 - v_{\lambda+1}^2 + ... - v_n^2$ we call this as the Morse $\lambda-$ saddle. Now, specifically if $\lambda = n$, which means that all the terms are being subtracted v_1 to v_n, then what do I expect? I expect that $\bar{x_0}$ is a local max. Any function value at any point other than $\bar{x_0}$ in the neighborhood of $\bar{x_0}$ will be lower than the functional value at $\bar{x_0}$.

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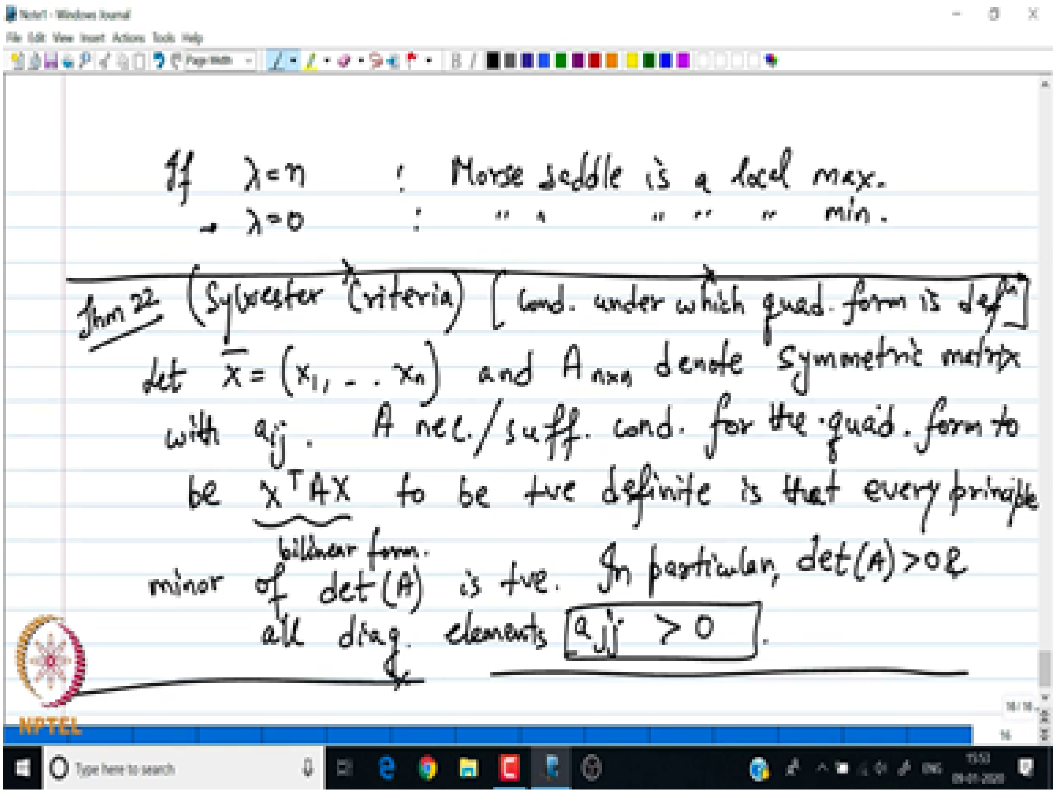

So the conclusion here is if $\lambda = n$, then I see that Morse saddle is , a local maxima. And on the other hand, if $\lambda = 0$ then Morse saddle is a local minima. The functional value at $\bar{x_0}$ is the lowest possible value among all such point x in the neighborhood of $\bar{x_0}$ in the second case. So, then I end my discussion by now stating a result in the form of a theorem.

The theorem is known as the Sylvester criteria. So essentially this criteria tells us about the condition for which a matrix, let us say the Hessian matrix is, when is it positive definite or when is it negative definite.The conditions under which the quadratic form is definite.

The statement says, let $\bar{X} = (x_1, x_2, ..., x_n)$ and let $A_{n\times n} = [a_{ij}]_{n\times n}$ denote a symmetric matrix. So the Sylvester criteria says that a necessary as well as sufficient condition n for the quadratic form $X^T AX$ to be positive definite, is that every principal minor of the $det\,(A)$ is positive. In particular,$det\,(A) > 0$ itself is positive and, all diagonal elements $a_{jj} > 0$. So that is a Sylvester criteria for finding the positive or the negative definiteness of the matrix and we now have the sufficient background to look at the optimization of functionals.

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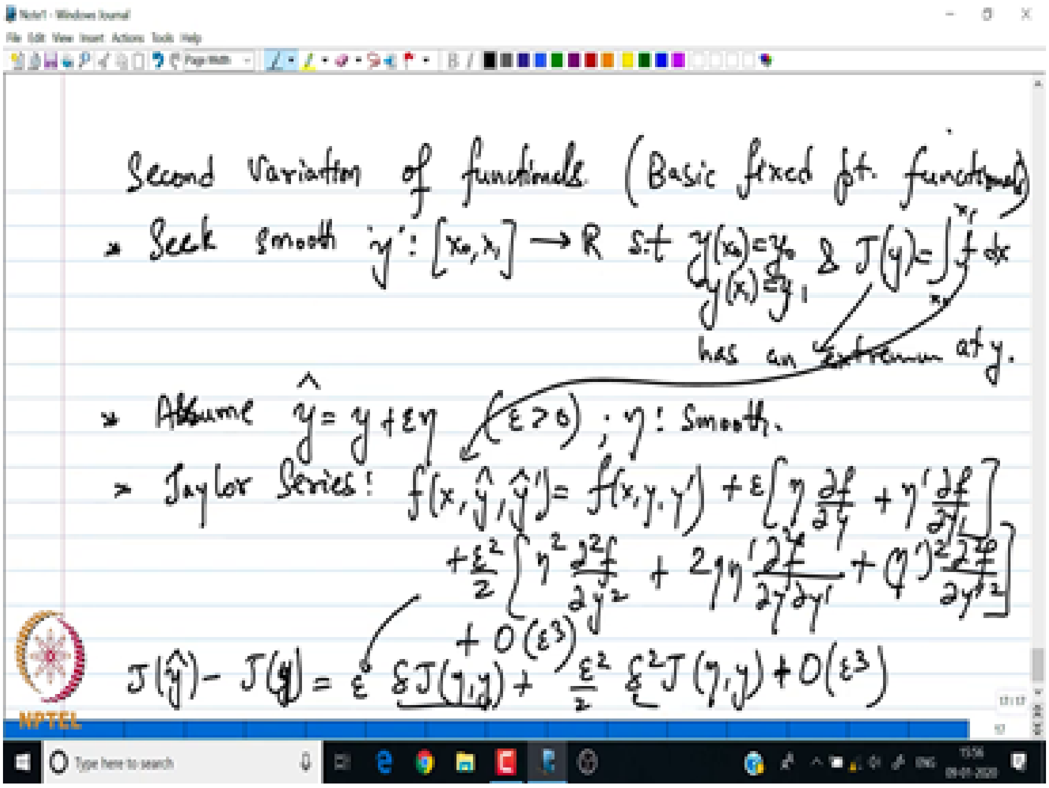

So let me now briefly start with the basic definition. So the major description will come in our next lecture. I am going to look at the basic fixed point functionals, well, functionals. we are seeking smooth functions $y : [x_0, x_1] \to \mathbb{R}$ such that $y(x_0) = y_0 \; and \; y(x_1) = y_1$ and such that the extremal $J(y) = \int_{x_0}^{x_1} f dx$ has an extremum at y.

We continue our discussion with the standard perturbation argument. So assume a perturbation of the form

$$\hat{y} = y + \epsilon\eta$$

where $\epsilon > 0$ e and η is smooth function and then use Taylor series integrand of f.We get

$$f(x, \hat{y}, \hat{y}') = f(x, y, y') + \epsilon\left[\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right] + \epsilon^2\left[\eta^2\frac{\partial^2 f}{\partial y^2} + 2\eta\eta'\frac{\partial^2 f}{\partial y\partial y'} + (\eta')^2\frac{\partial^2 f}{\partial y^2}\right] + O\left(\epsilon^3\right)$$

So variation in the functional is

$$J(\hat{y}) - J(y) = \epsilon\delta J(\eta, y) + \frac{\epsilon^2}{2}\delta^2 J(\eta, y) + O\left(\epsilon^3\right)$$

Notice that we have only written up to second order. I know that this is my standard first variation. We have done a lot of problems on the first variation, so we do not worry about it.

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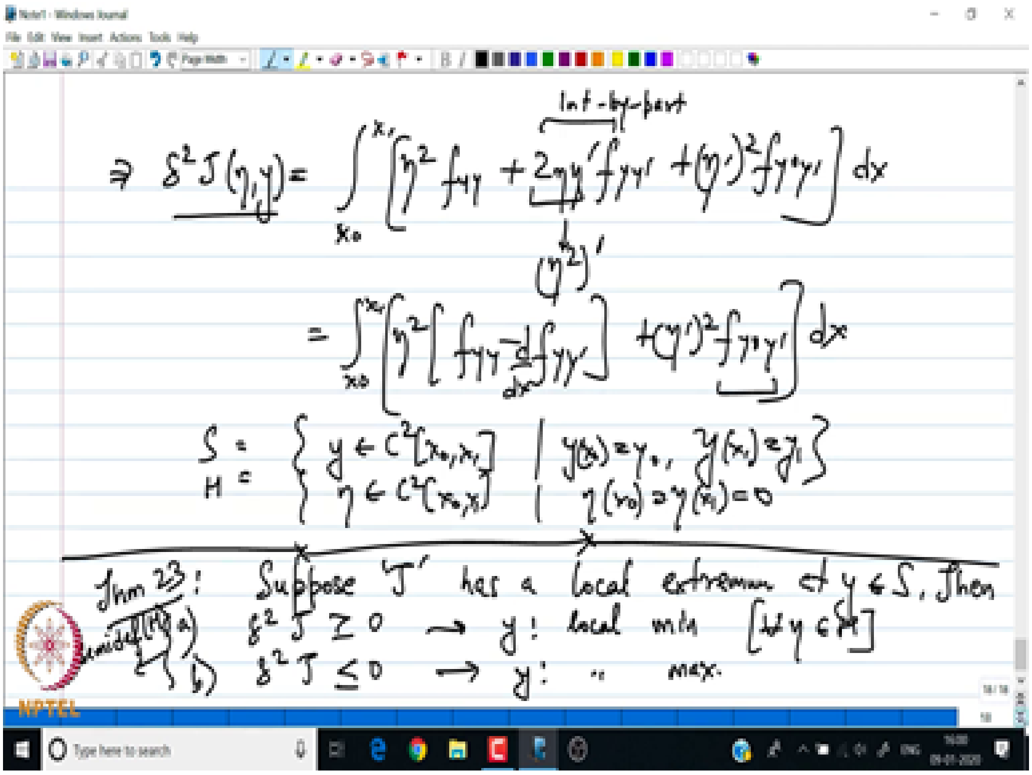

Let us now look at the second variation. So second variation of this functional is of the form

$$\delta^2 J\left(\eta, y\right) = \int_{x_0}^{x_1} \left[\eta^2 f_{yy} + 2\eta\eta' f_{yy'} + \left(\eta'\right)^2 f_{y'y'}\right] dx$$

We integrate by parts. And we get

$$\delta^2 J\left(\eta, y\right) = \int_{x_0}^{x_1} \left[\eta^2 \left[f_{yy} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f_{yy'}\right)\right] + \left(\eta'\right)^2 f_{y'y'}\right] dx$$

Then we will show that the sign of this second variation completely depends on the sign of this quantity. Let me also quickly introduce sets S and H.

$$S = \left\{y \in C^2\left[x_0, x_1\right] : y\left(x_0\right) = y_0 \; and \; y\left(x_1\right) = y_1\right\}$$
$$H = \left\{\eta \in C^2\left[x_0, x_1\right] : \eta\left(x_0\right) = \eta\left(x_1\right) = 0\right\}$$

Now I need to connect the sign of the second variation with the location of the maxima or the minima. The next result, theorem 23, exactly does that.

Suppose J has a local extremum at $y \in S$, then if $\delta^2 J \geq 0$ then y is a local minimum $\forall \eta \in H$ and if $\delta^2 J \leq 0$ then y is a local maximum $\forall \eta \in H$

So, that is the relation between max and min and the sign of the second variation. When we eventually calculate the second derivative we have to look at the definiteness or the semi definiteness of Hessian matrix, which is involved in this setup.

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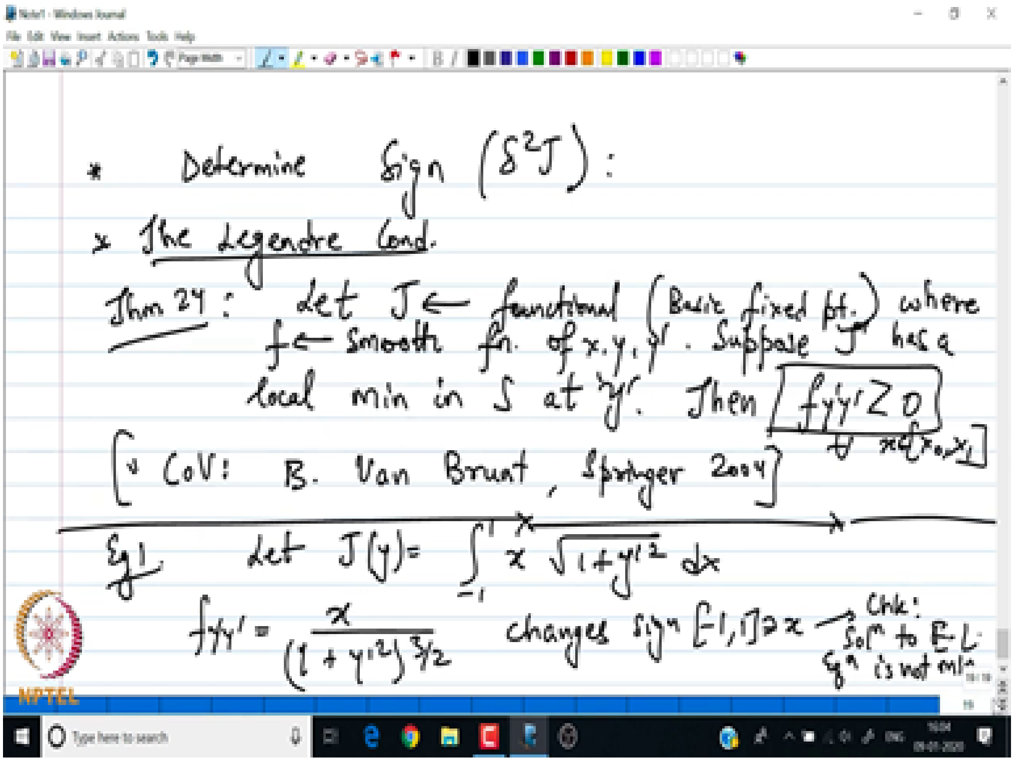

Now, I am going to finally end this topic here by stating one of the very vital results known as the Legendre condition. So, this is the first result that I am going to state in order to determine the sign of the second variation. So, the idea is to determine the sign of $\delta^2 J$. Because once we know the sign, we can immediately say whether the extremum involved is maxima or minima.

The Legendre condition exactly does that. The Legendre condition says the following : Let us assume that J is a functional which is the basic fixed point functional that we have began with, and integrand f is a smooth function. When I say smooth, this is continuously differentiable up to second order function of (x, y, y') .

Suppose, suppose J hasa local minimum in S at y, then $f_{y'y'} \geq 0 \ \forall x \in [x_0, x_1]$. So $f_{y'y'}$ is a crucial quantity whose sign needs to be checked to determine whether we have found a local minimum or not. Again, the proof is not going to be shown here.

And for the complete proof, I am going to refer the following text, Calculus of Variation by Bruce van Brunt that we are also following in this course. This is the book published by Springer 2004. Let us quickly look at an example of the application of Legendre condition. Consider a functional of the form

$$J(y) = \int_{-1}^{1} x\sqrt{1+y'^2}dx$$

We see that $f_{y'y'}$ is given by the following:

$$f_{y'y'} = \frac{x}{(1+y'^2)^{\frac{3}{2}}}$$

Now $f_{y'y'}$ changes sign in the interval $x \in [-1, 1]$. The denominator is always positive, but the numerator changes sign and students are asked to check the solution to the Euler-Lagrange equation for this functional is not a minimum. And this Legendre condition guarantees because the sign changes.

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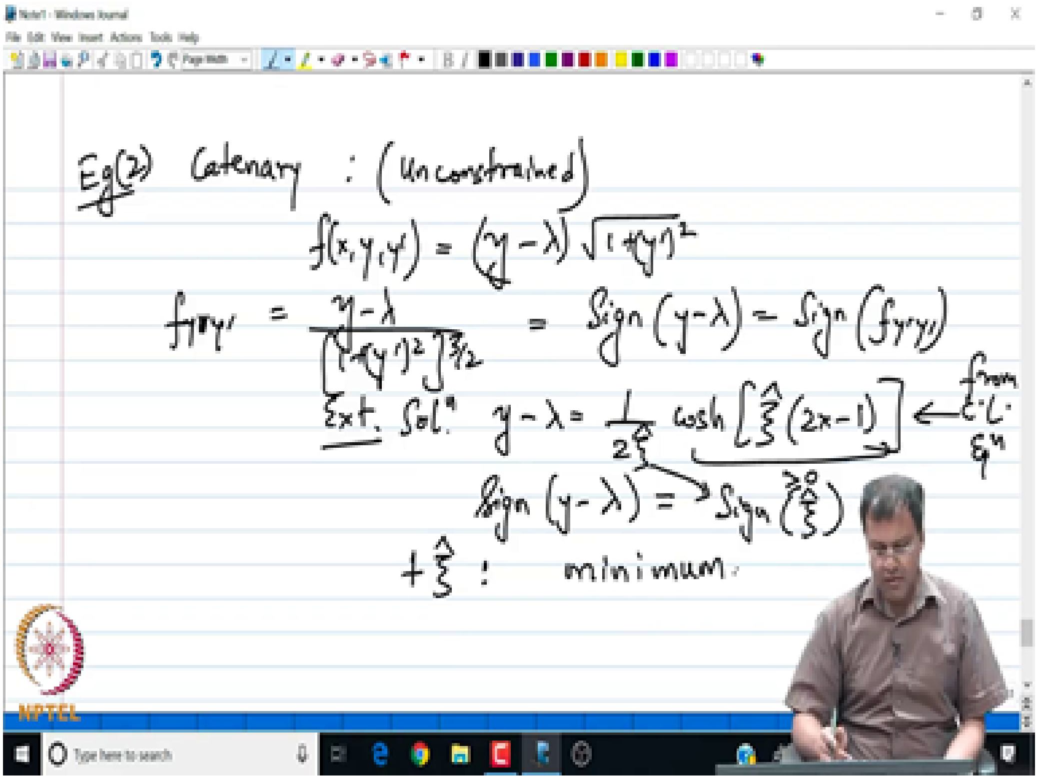

We will end this discussion by giving one more example of the Catenary. So recall, in our Catenary problem, we will look at the unconstrained problem. So, in the Catenary problem

$$f(x, y, y') = (y - \lambda)\sqrt{1 + (y')^2}$$

And let me quickly evaluate this Legendre derivative

$$f_{y'y'} = \frac{y-\lambda}{(1+y'^2)^{\frac{3}{2}}}$$

Notice that the sign of $y-\lambda$ determines the sign of $f_{y'y'}$ and if people recall that $y-\lambda$ being the extremal, they recall that the solution in this case was

$$y - \lambda = \frac{1}{2\hat{\xi}}\cosh\left[\hat{\xi}(2x - 1)\right].$$

So this is extremal solution coming from Euler-Lagrange equation. Since cosh is always non-negative, so the sign of $y - \lambda$ is completely determined by the sign of $\hat{\xi}$. So students are asked to check that the positive root gives us the minima.

And that is confirmed by the Legendre condition.In the next lecture, we will begin with that Legendre condition is not going to be sufficient in determining whether a extrema is maxima or minima.