Variational Calculus and its Applications in Control Theory and Nano mechanics
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Lecture – 49
Conjugate Points / Jacobi Accessory Equations / Introduction to Optimal Control Theory
Part 1

In today's lecture, I am going to continue my discussion on the analysis of second variation of the functional, namely given an extremal y of the functional, can we determine the nature of the extremal ?

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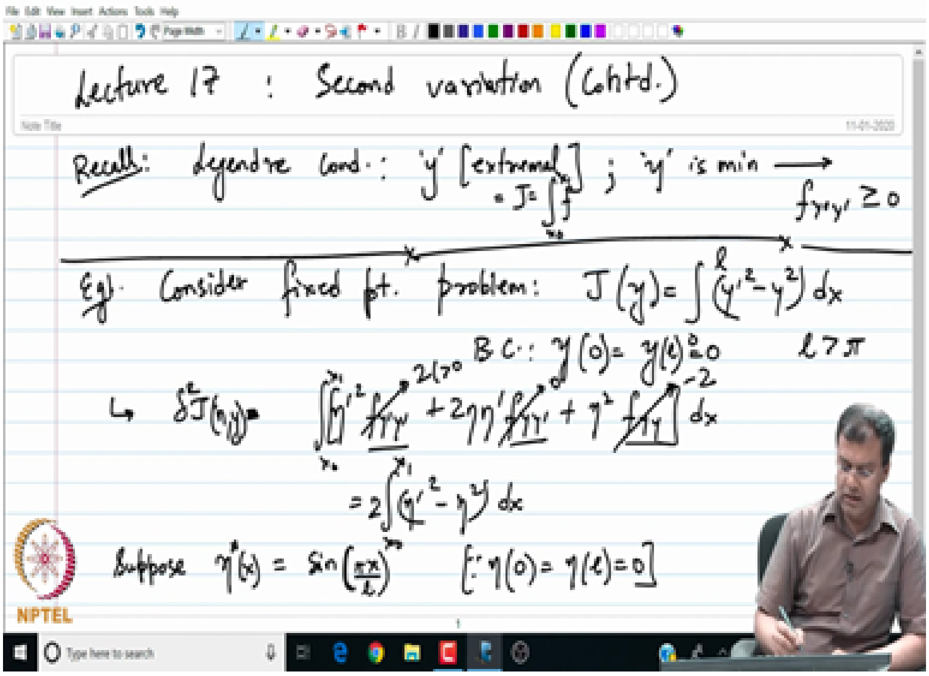

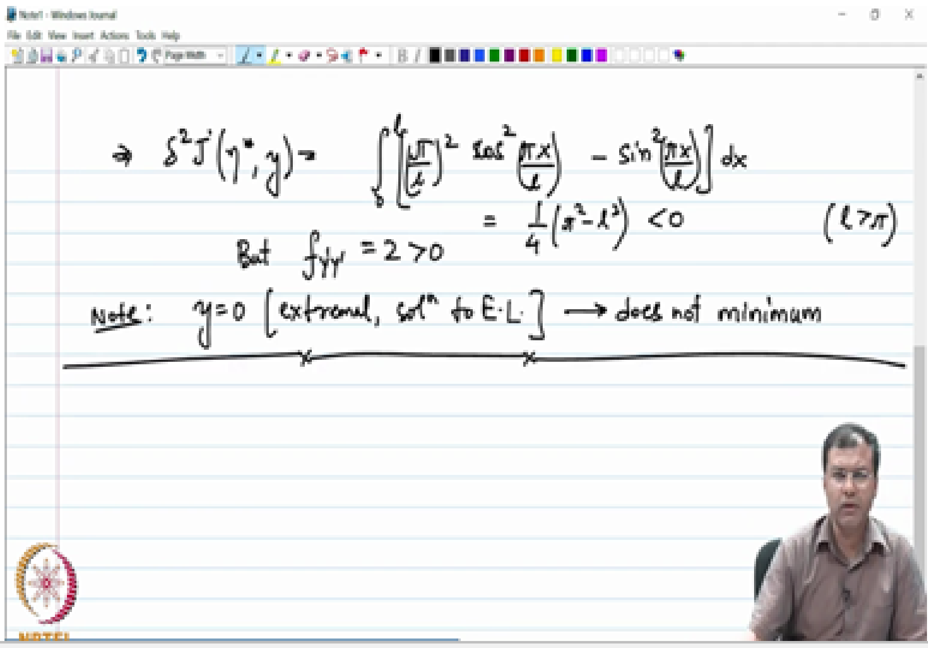

So, this is a discussion on second variation, continued. In last lecture, we ended our lecture on the introduction to the Legendre Condition. So, recall the Legendre condition, that if if y is an extremal of $J = \int_{x_0}^{x_1} f dx$ then y being minimum leads to the condition that $f_{y'y'} \geq 0$.

We saw that this was a good starting point to look at the nature of the extrema. But we will also see that this is not a complete condition. And hence later on we will develop a complete condition to determine the nature of the extremal. So, let us continue our discussion on the Legendre condition. So, the first example that I have is consider the fixed point where I have to determine the nature of the extremal.

$$J(y) = \int_0^l \left(y'^2 - y^2\right) dx$$

And the boundary conditions are : $y(0) = y(l) = 0$ Further I am also given that $l > \pi$. So, if we were to look at the second variation, the second variation is

$$\delta^2 J(\eta, y) = \int_{x_0}^{x_1} \left[(\eta')^2 f_{y'y'} + 2\eta\eta' f_{yy'} + \eta^2 f_{yy}\right] dx$$

So, once we plug in all these partial derivatives we get

$$\delta^2 J(\eta, y) = 2\int_{x_0}^{x_1} \left(\eta'^2 - \eta^2\right) dx$$

Now suppose $\eta^\star(x) = \sin\left(\frac{\pi x}{l}\right)$. Now the reason I have taken this sort of a function is because $\eta(0) = \eta(l) = 0$. So, this is a good choice for our perturbation function. And for this particular $\eta^\star(x)$ the second variation is

$$\delta^2 J(\eta^\star, y) = \int_0^l \left[\left(\frac{\pi}{l}\right)^2 \cos^2\left(\frac{\pi x}{l}\right) - \sin^2\left(\frac{\pi x}{l}\right)\right] dx$$

Once I integrate, this turns out to be

$$\delta^2 J(\eta^\star, y) = \frac{1}{4}\left(\pi^2 - l^2\right)$$

Now notice that we had chosen $l > \pi$. For this condition,

$$\delta^2 J(\eta^\star, y) = \frac{1}{4}\left(\pi^2 - l^2\right) < 0$$

So, the second variation of the functional is negative, but $f_{y'y'} = 2 > 0$. So certainly, the Legendre condition holds but we are getting the second variation to be negative definite.

Now note that $y = 0$ which is an extremal or the solution to the Eular-Lagrange this does not give the minimum. So, this is the implication of , the fall out of the Legendre condition here, which means that although we getting the second variation negative, but the Legendre condition holds. And the implication being that we have one solution to the Euler-Lagrange which is not the minimum.

So, the moral here is that Legendre condition is mainly a pointwise restriction. It is not a restriction which is global in character Or which holds for all points x in the entire interval under consideration. So, we are now going to derive a result which is global in character, but which also utilises the advantage or the usefulness of Legendre condition.The condition is called Jacobi Necessary Condition.

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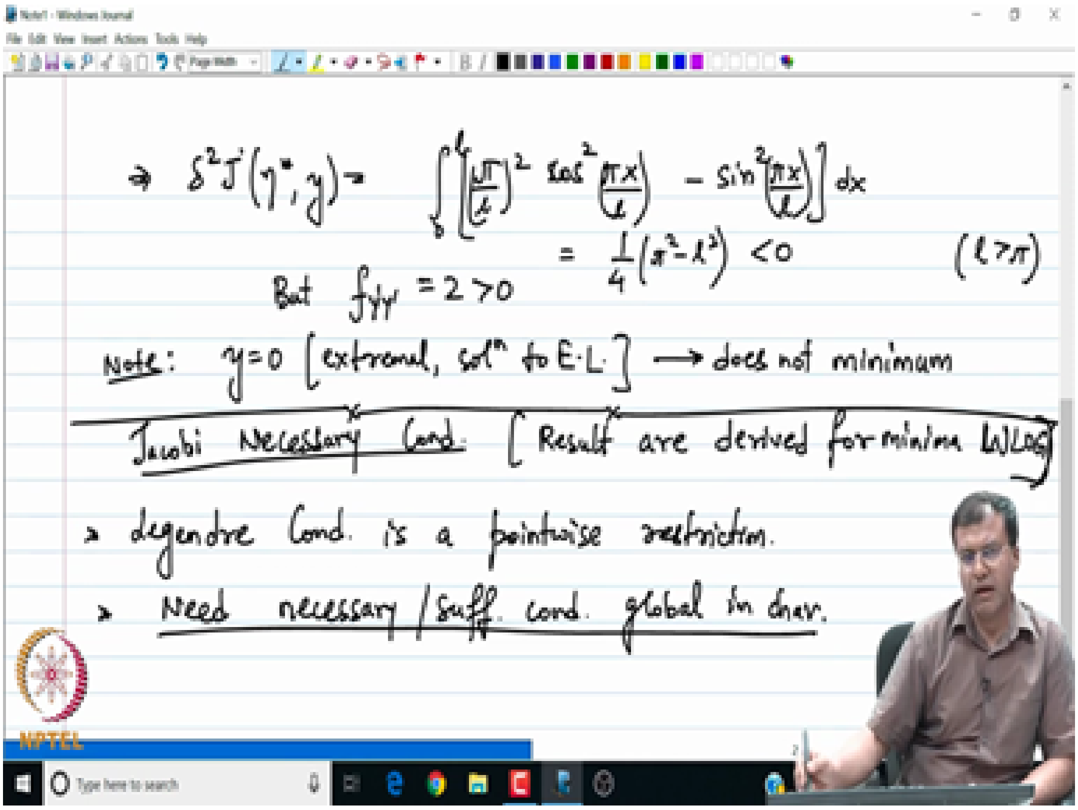

As I just said, throughout our discussion I am going to only talk about deriving the minima from the extrema. Results are derived for minima. This is without loss of generality. Similar set of arguments will hold for maxima with the signs reversed. So, the Legendre condition, as I just said, is a point wise restriction.

We saw that depending on the values of x attained from the interval, $f_{y'y'}$ will either be positive or negative. And even if it is positive, does not guarantee that we are going to get a positive definite second variation. So, Legendre condition is definitely not the complete solution. So we need necessary as well as sufficient conditions which is global in character which means that the condition should hold $\forall x \in [x_0, x_1]$

So, the goal is as follows we need a global necessary and sufficient condition, and the starting point is our Morse lemma. Namely we are going to appeal to the Morse 0−saddle extended to the infinite dimensional case. We have looked at the Morse lemma for the finite dimensional case.

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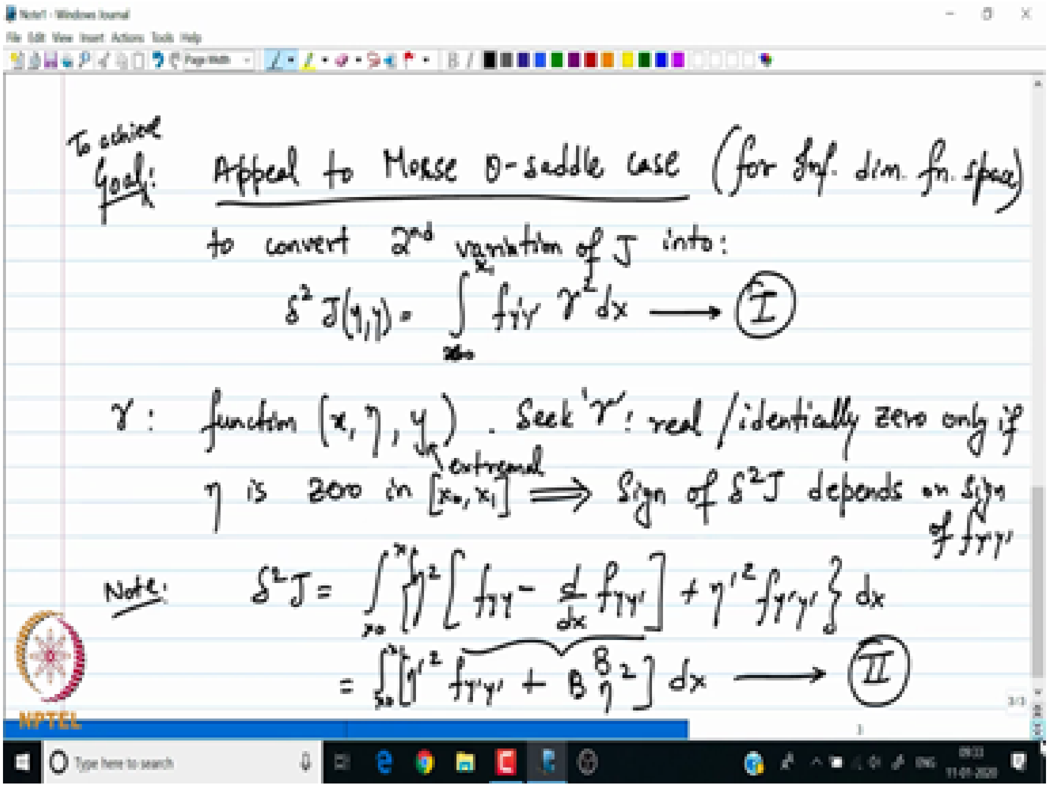

We are going to achieve our goal via Morse 0− saddle case which is for the infinite dimensional function space. That is, by appealing to Morse 0− saddle case we are going to convert our second variation, our second variation of J into a following form :

$$\delta^2 J\left(\eta, y\right) = \int_{x_0}^{x_1} f_{y'y'}\gamma^2 dx \qquad \textbf{(1)}$$

where γ is a real valued function, which means that the sign of the second variation will be completely determined by the sign of $f_{y'y'}$ provided we are able to reduce to form (1).Then also some another property of γ that we need is it is a function (x, η, y), y being our extremal.

We seek γ real and identically 0. only if $\eta = 0$ in $[x_0, x_1]$. If we are able to find such a γ which is real valued and vanishes for the case when eta vanishes, then we are guaranteeing that the sign of $\delta^2 J$ depends completely on the sign of $f_{y'y'}$. So let us now look at what is our second variation.

Our second variation is

$$\delta^2 J = \int_{x_0}^{x_1} \left[\eta^2 \left[f_{yy} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f_{yy'}\right)\right] + \left(\eta'\right)^2 f_{y'y'}\right] dx$$

Now let me call $\left[f_{yy} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f_{yy'}\right)\right] = B$.Then second variation is as follows:

$$\delta^2 J = \int_{x_0}^{x_1} \left[\left(\eta'\right)^2 f_{y'y'} + B\eta^2\right] dx \qquad \textbf{(2)}$$

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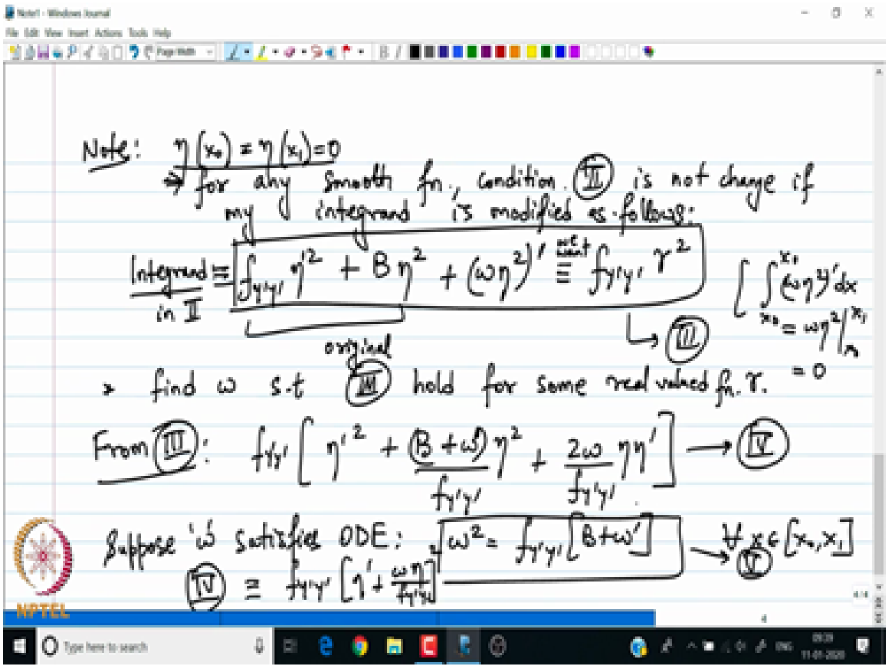

Note that $\eta\left(x_0\right) = \eta\left(x_1\right) = 0$. This observation implies for any smooth function the condition (2), is not changed if integrand is modified as follows:

$$\text{Integrand in (2)} \equiv \quad f_{y'y'}\left(\eta'\right)^2 + B\eta^2 + \left(\omega\eta^2\right)' \equiv f_{y'y'}\gamma^2 \qquad \textbf{(3)}$$

Note that $\int_{x_0}^{x_1} \left(\omega\eta^2\right)' dx = \omega\eta^2\Big|_{x_0}^{x_1} = 0$

We have to find ω such that (3) holds for some real valued function γ.So, this is our goal.From (3) we get the following :

$$f_{y'y'}\left[\eta'^2 + \frac{(B+\omega')}{f_{y'y'}}\eta^2 + \frac{2\omega}{f_{y'y'}}\eta\eta'\right] \qquad \textbf{(4)}$$

So, all we need to do is, the quantity inside the bracket that we have to reduce it into a perfect square of some function. So that is possible if the following holds. Suppose ω satisfies the ODE given by

$$\omega^2 = f_{y'y'}\left[B + \omega'\right] \quad \forall x \in \left[x_0, x_1\right] \qquad \textbf{(5)}$$

So, if (5) holds, then (4) $\equiv f_{y'y'}\left[\eta' + \frac{\omega\eta}{f_{y'y'}}\right]^2$ So, my obvious choice for γ in that case will be the quantity inside the bracket. So if I find a ω such that ω satisfies the ordinary differential equation, which is a first order non-linear equation, then I am guaranteed that I can get a γ such that I can reduce the quantity inside the bracket given by (4) into a perfect square.

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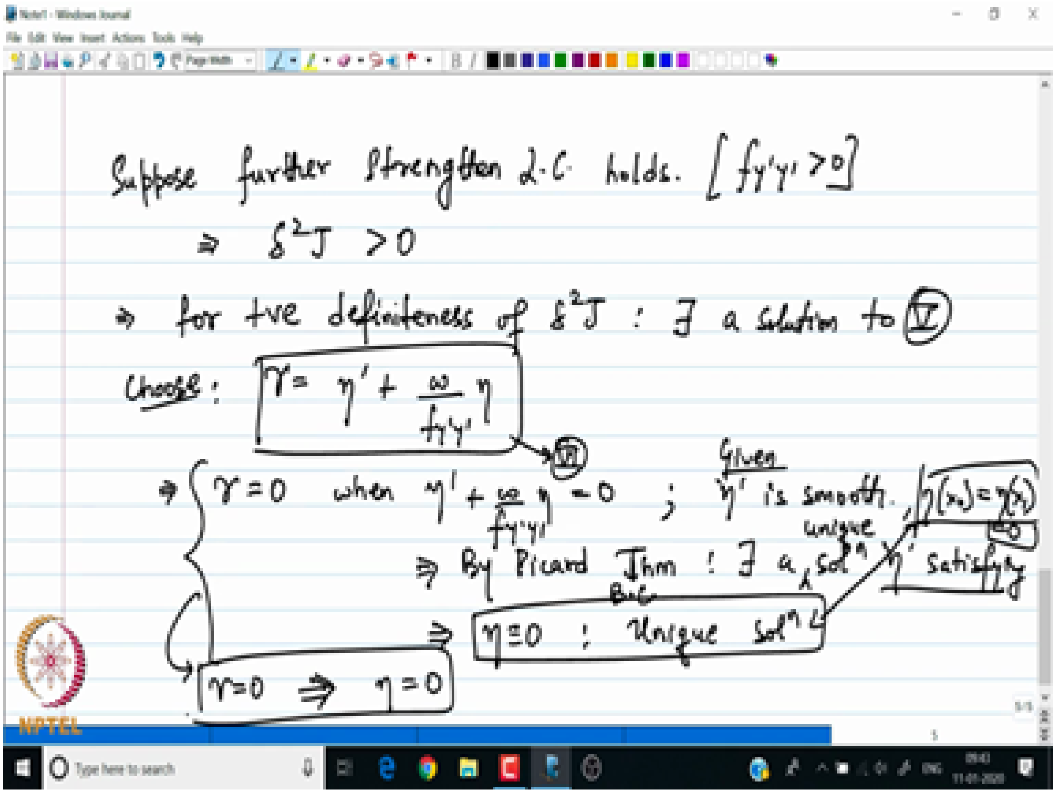

Suppose further the strengthened Legendre condition holds that is $f_{y'y'} > 0$ Then $\delta^2 J > 0$ So,the conclusion is the following: For our positive definiteness of $\delta^2 J$ I need that there is a solution to (5).

Let me choose

$$\gamma = \eta' + \frac{\omega}{f_{y'y'}}\eta \qquad \textbf{(6)}$$

Now,

$$\gamma = 0 \text{ When } \eta' + \frac{\omega}{f_{y'y'}}\eta = 0 \;;$$

η is a smooth function which means that it contains all its derivatives at least the first derivative as well as the function itself is continuous. Also η vanishes at the boundary. Then,by Picard's Existence theorem there exists a unique solution η satisfyingthe boundary condition $\eta(x_0) = \eta(x_1) = 0$ Now I can see that one solution to the ODE given by the expression in (6) is $\eta = 0$. Since η is a unique solution, then $\eta \equiv 0$ is our unique solution which satisfies the boundary condition as well.

The conclusion here is that $\gamma = 0$ only if $\eta = 0$. And we know that γ is a real valued function.Let me summarise our discussion in the form of a small result. Let me call this as a lemma. So, the result is as follows:

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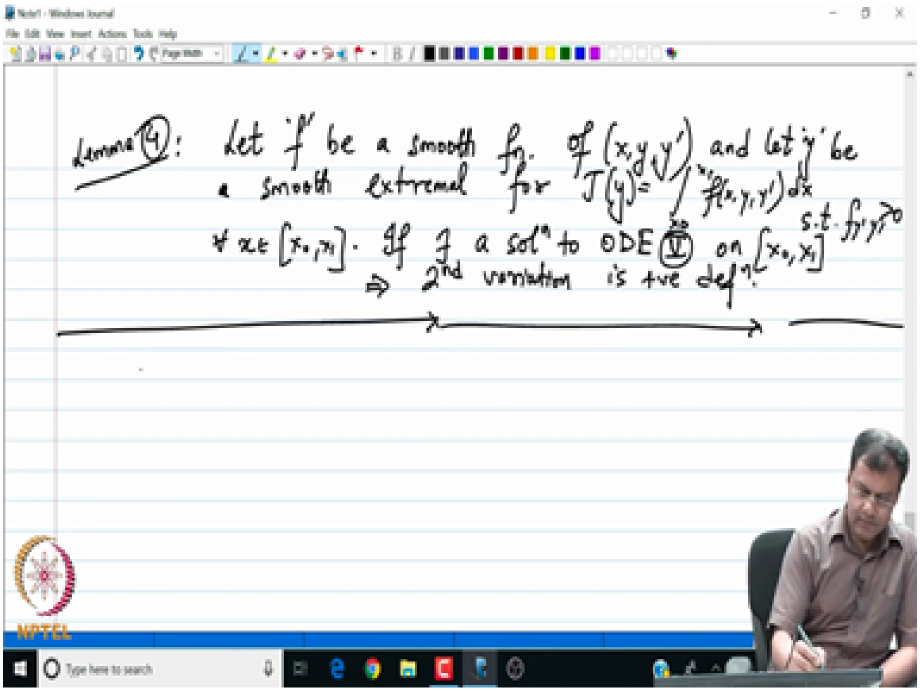

Let f be a smooth function of (x, y, y'). Let y be a smooth extremal for $J(y) = \int_{x_0}^{x_1} f(x, y, y')\, dx$ such that my strengthened Legendre condition $f_{y'y'}.0$ holds $\forall x \in [x_0, x_1]$. If there exists a solution to ODE (5) on $[x_0, x_1]$ it implies that second variationis positive definite.So, all I have to do is to find the solution to the ODE that we had assumed and provided that the solution exists.Let us now focus on the solution to the ODE.

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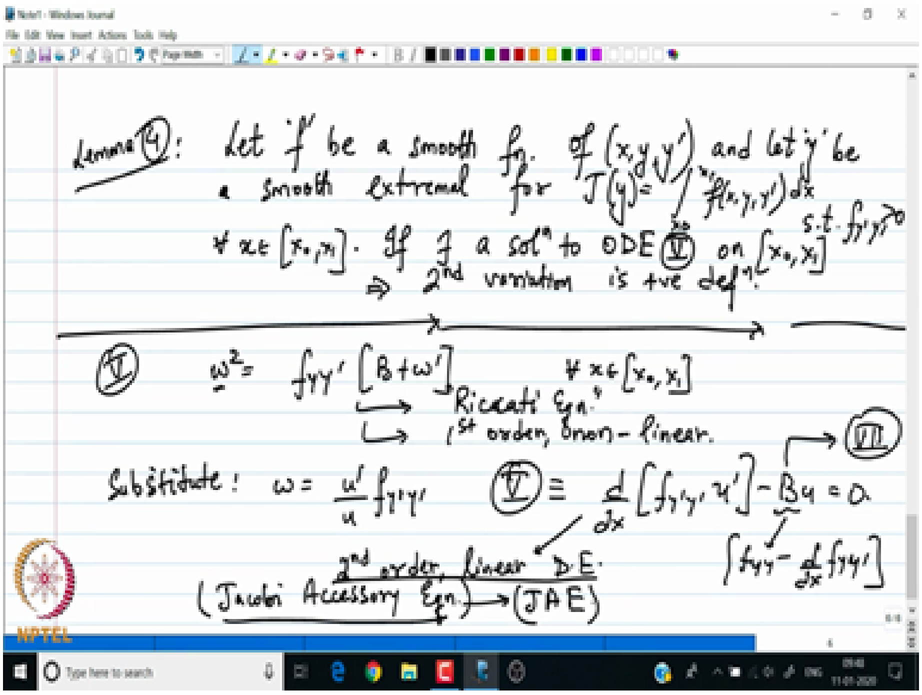

The ODE (5) is

$$\omega^2 = f_{y'y'}\left[B + \omega'\right] \quad \forall x \in [x_0, x_1]$$

Notice that this ODE is in the form of a famous the Riccati equation. And secondly this is also a first order non-linear ODE. So, the first step that we will do is, we are going to change it into a linear ODE in order to satisfactorily find the solution to it.

So,we substitute $\omega = \frac{u'}{u} f_{y'y'}$ So, in this case (5) is identical to the following:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f_{y'y'}u'\right] - Bu = 0 \qquad \textbf{(7)}$$

where $B = f_{[yy]} - \frac{\mathrm{d}}{\mathrm{d}x} f_{yy'}$ So, note that this is a second order linear differential equation and also known as the Jacobi Accessory equation orin short term note it as JAE. So, from now on JAE will denote the equation (7) or the Jacobi Accessory equation.

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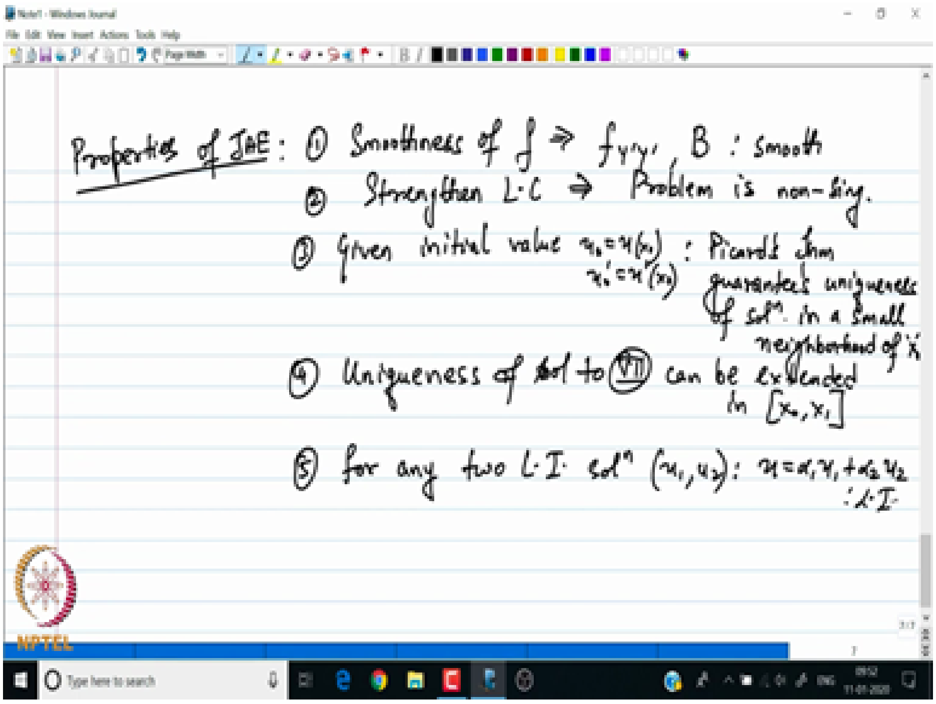

So, some of the properties of JAE are as follows:

(1) The smoothness of f implies that $f_{y'y'}$ and B are also smooth.

(2) The strengthened Legendre condition implies problem is non-singular.

(3) Given initial values $u_0 = u\left(x_0\right); u_0' = u'\left(x_0\right)$ Picard's theorem guarantees uniqueness of solution in a small neighbourhood x_0.

(4) The uniqueness of solution to equation (7) can be extended in $[x_0, x_1]$

(5) for any two linearly independent solutions u_1, u_2 I see that the solution $u = \alpha_1 u_1 + \alpha_2 u_2$ is a linearly independent solution of the problem.

So, it is in terms of the solution to the Jacobi accessory equation.