Variational Calculus and its Applications in Control Theory and Nano mechanics

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Lecture – 05: Introduction –Euler Lagrange Equations Part-5

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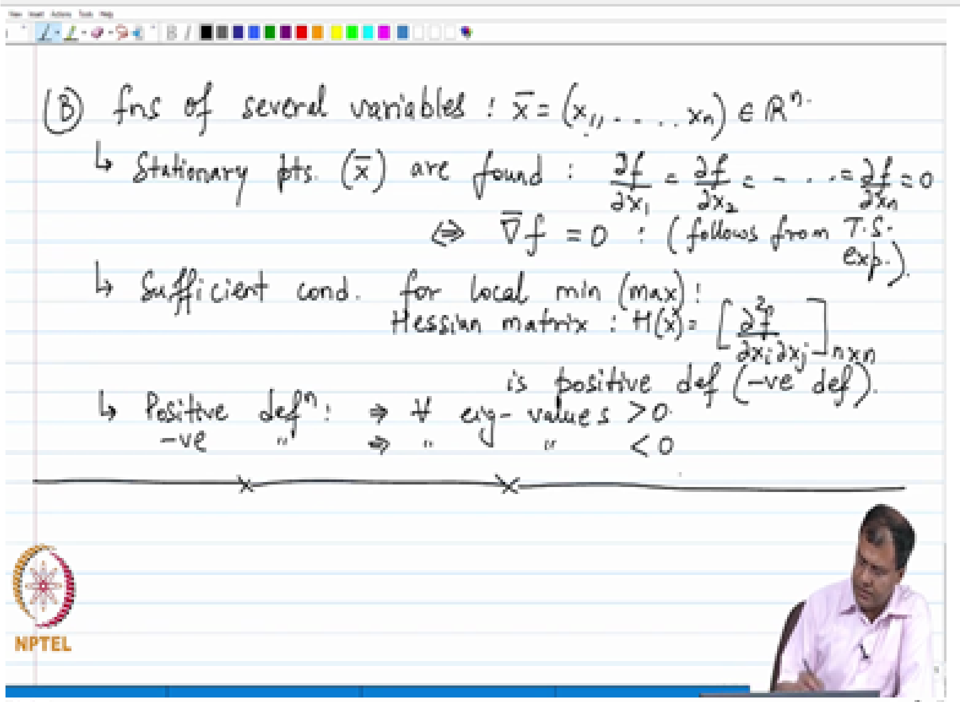

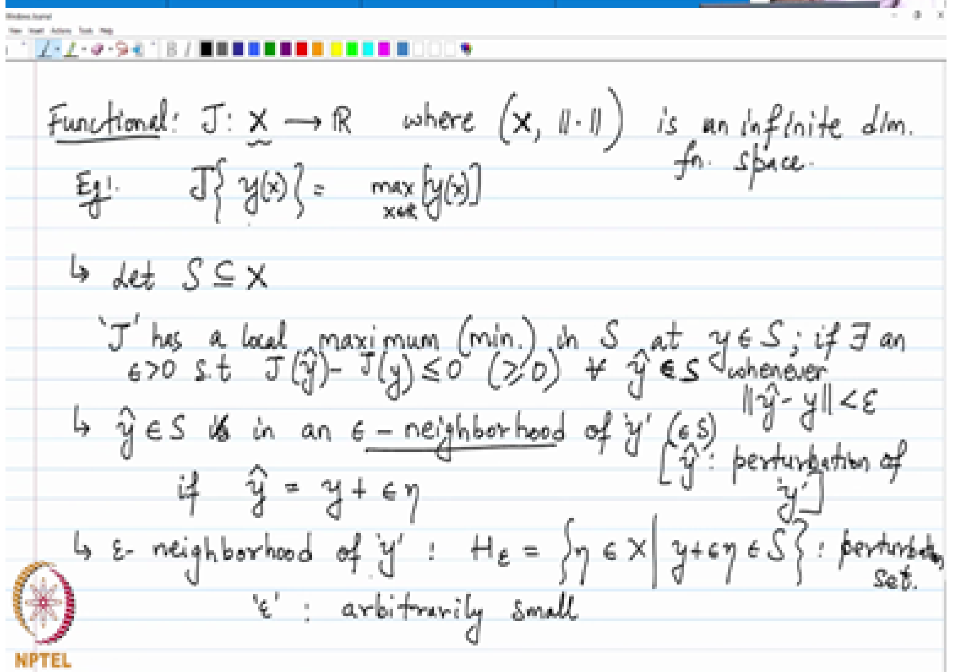

The revision for the finite dimensional calculus is over. now, we are at a stage where we can go to the description of the extremization of functionals. So, let us now slowly start by introducing what is a functional.So, in my first lecture, I had briefly said that functional is a function of a function or in pure mathematical language, a functional is a map from the set of all functions. So, this is the set of all functions X to the set of real numbers, where this time my vector space is X, the norm, where this particular norm is the function norm, which I am going to describe in a minute.

So, this particular vector space is an infinite dimensional vector space or I would say function space. So, before I move ahead, let me quickly give some examples. A very simple example could be J(y) where y is a function of x, the maximum of y, evaluated for all values of x in R. So, we can see that the maximum of the function y(x) will be a real number. So, $J : X \to R$ and hence a functional. So, Let us further say that $S \subset X$.let us further say then, J has a local, so note that my description of a local minima or local maxima or in general local extrema will follow in a parallel manner to the description like we did in the description of the finite dimensional calculus.

let J has a local maxima or a local minima in S. And let Y be a function in the subset space S of functions. If there exists an epsilon, so I am trying to describe the definition of the local maxima of this functional J. So, that can be done $\exists\ \epsilon > 0\ \ s.t\ \ J(\hat{y}) - J(y) \leqslant 0\ \ \forall\ \ \hat{y} \in S$ whenever $\|\hat{y} - y\| < \epsilon$
For local minima $\exists\ \epsilon > 0\ \ s.t\ \ J(\hat{y}) - J(y) \geqslant 0\ \ \forall\ \ \hat{y} \in S$ whenever $\|\hat{y} - y\| < \epsilon$.

So, what I have just said is that suppose $\hat{y} \in S$ it is in an ϵ - neighborhood. What I have just described above is the space of all functions in the ϵ -neighborhood of y. All of these functions, they belong to the subset S. We call this $\hat{y}$, since $\hat{y}$ is in the ϵ - neighborhood of y, we call this $\hat{y}$ the perturbation of y.

If I write $\hat{y} = y + \epsilon\eta$ then all these definitions are equivalent, If y hat is the perturbation of y, if y hat can be written in the form of this following relation $\hat{y} = y + \epsilon\eta$ where I describe my functions η to be the space of perturbation, to be coming from the space of perturbation functions with 0's at the end points.

So, I describe my ϵ - neighborhood set, ϵ - neighborhood of the function y. I describe this ϵ - neighborhood set as $H_\epsilon = \{\eta \in X | y + \epsilon\eta \in S\}$ So, this is my set of all perturbation functions. I call this as my perturbation set.

And I can always make ϵ to be arbitrarily small. So, I can always modulate my perturbation such that it is closer or farther away from the function under consideration. Now with all this basic definition of a functional, let us start describing the result, that is the goal of this lecture. So, what we have is the following:

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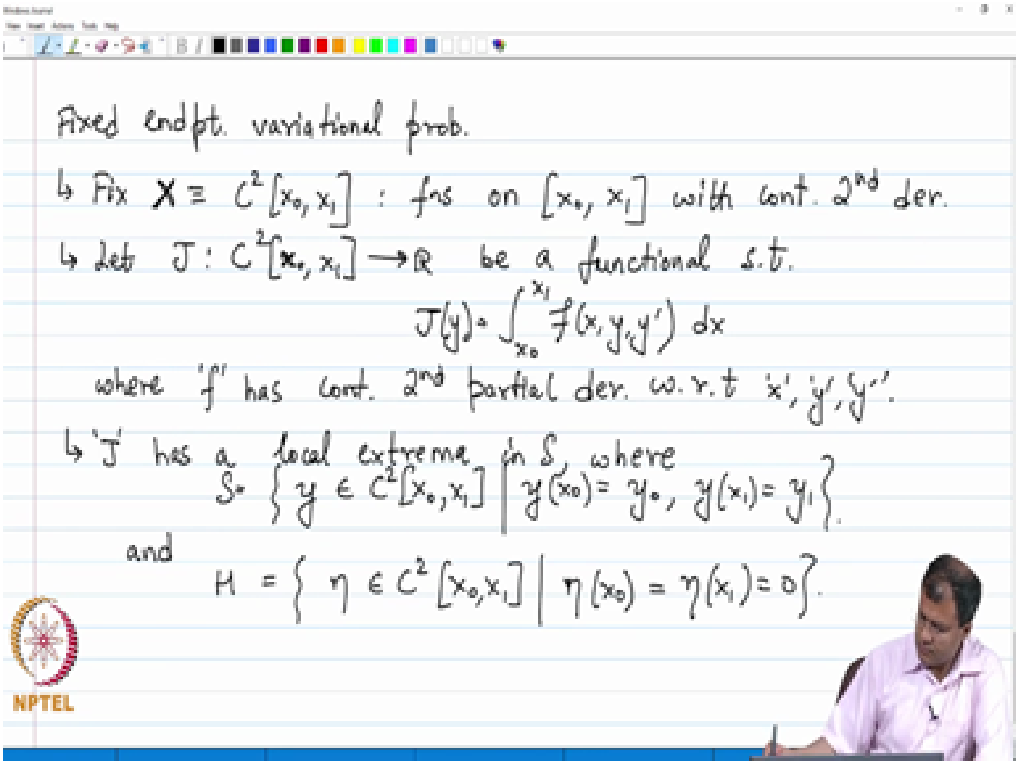

So, I am going to describe the simplest possible variational problem, i.e the fixed endpoint variational problem. So, my boundary condition is such that the values are fixed at the boundaries. So, let me say that my function space X is space of all continuously differentiable functions up to second order inside $[x_o, x_1]$.

$J : C^2[x_o, x_1] \to R$ be a functional s.t

$$J(y) = \int_{x_o}^{x_1} f(x, y, y^{'})dx$$

where f has continuous second partial derivatives with respect to all the variables involved in the argument of this function, i.e w.r.t x, y, $y^{'}$.

Or I say that J has a local extrema in S, where

$$S = \{y \in C^2[x_o, x_1] \mid y(x_o) = y_0, y(x_1) = y_1\}$$

and

$$H = \{\eta \in C^2[x_o, x_1] \mid \eta(x_o) = \eta(x_1) = 0\}$$

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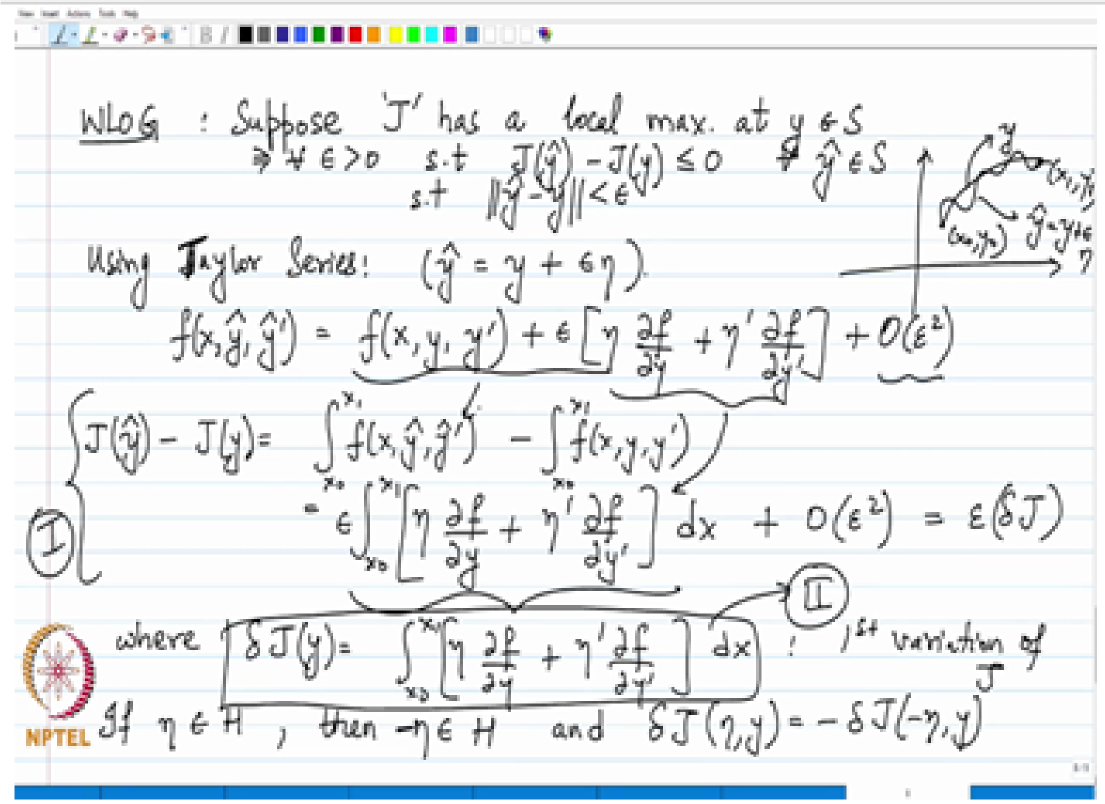

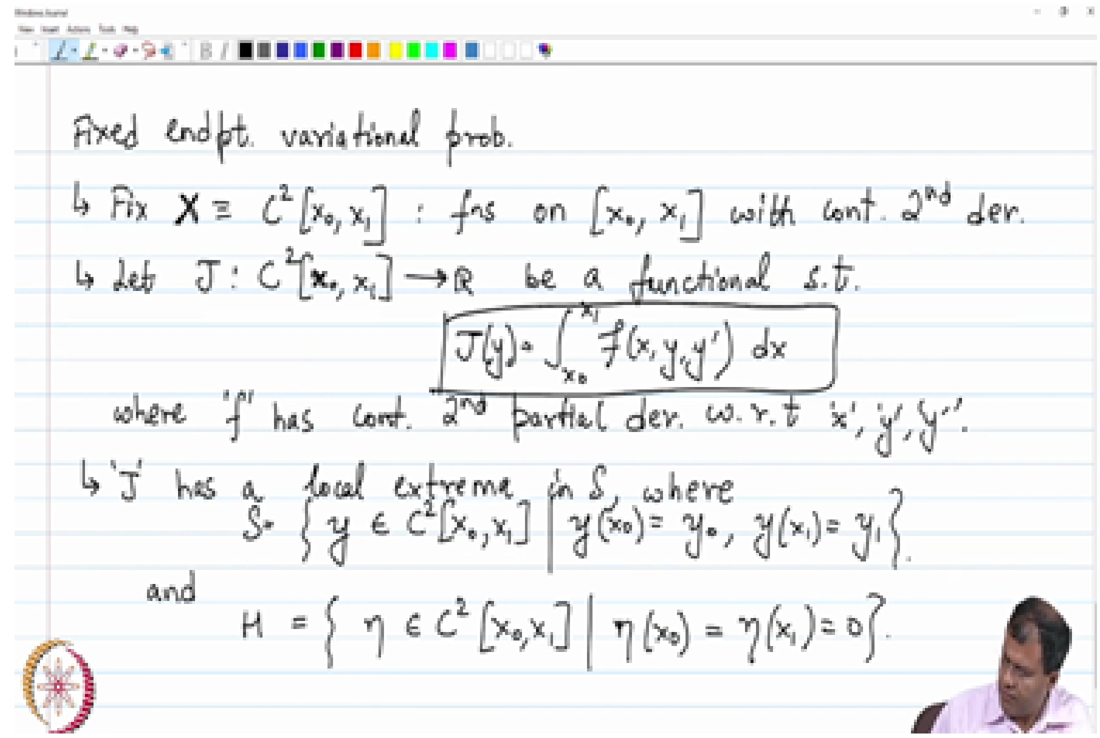

So, then we can further assume without loss of generality that suppose J has a local maxima at $y \in S$,

$$\Rightarrow \quad \forall\ \epsilon > 0\ \ s.t\ \ J(\hat{y} - J(y)) \leqslant 0\ \ s.t\ \ \|\hat{y} - y\| < \epsilon$$

So, what I am saying is if I have to draw this, all these statements pictorically, I am talking about a function starting from (x_0, y_0) to (x_1, y_1) and which can be perturbed using our perturbation function

η. So,($\hat{y} = y + \epsilon\eta$), So, suppose we have a local maxima at y. Then it must be that for all ϵ, all $\hat{y}$ in the ϵ - neighbourhood we must satisfy this inequality.

For all $\hat{y}$ such that, as I said $\hat{y}$ lies in the ϵ - neighbourhood of y. And, so that is the definition of the local maxima. So, now let us expand our function f using Taylor Series expansion.

$$f(x, \hat{y}, \hat{y'}) = f(x, y, y') + \epsilon\left[\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right] + O(\epsilon^2)$$

$$J(\hat{y}) - J(y) = \int_{x_o}^{x_1} f(x, \hat{y}, \hat{y'}) - \int_{x_o}^{x_1} f(x, y, y') = \epsilon\int_{x_o}^{x_1}\left[\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right]dx + O(\epsilon^2) = \epsilon(\delta J) \qquad \textbf{I}$$

Where $\quad \delta J(y) = \int_{x_o}^{x_1}\left[\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right]dx$ (Ist version of J) $\qquad$ **II**

so far we have come to a point where we have described the first variation of the functional J. Notice that if my perturbation is η and I replace η by $-\eta$, then $-\eta$ also is in the perturbation set. i.e If $\eta \in H$, then $-\eta \in H$ and $\delta J(\eta, y) = -\delta J(-\eta, y)$

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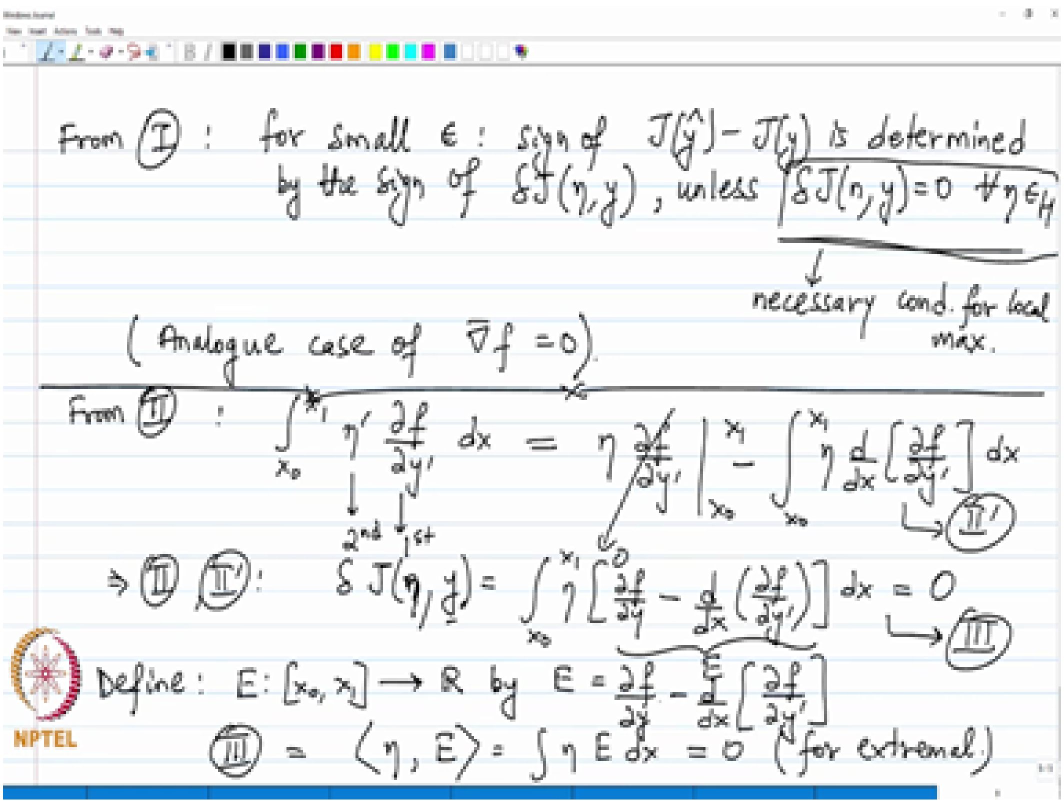

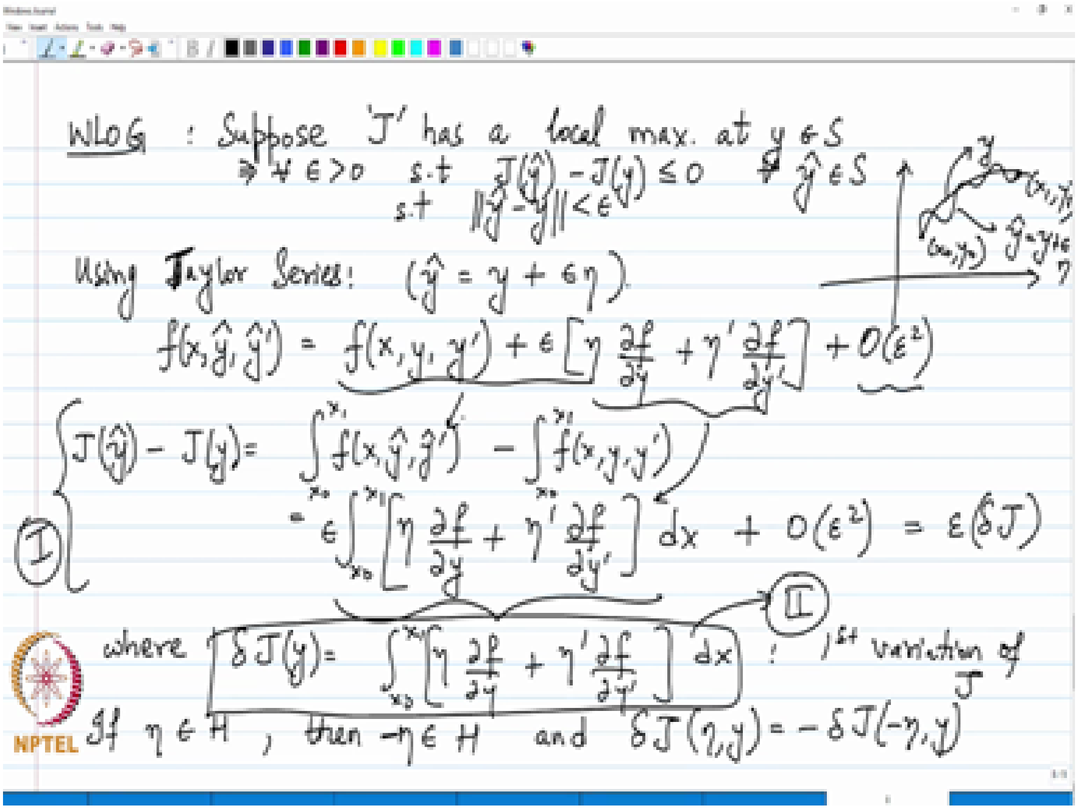

From **I**, we see that if ϵ is small enough, then the sign of $J(\hat{y}) - J(y)$ is determine by the Sign of $\delta J(\eta, y)$ is positive or negative

Now, we want that this particular variation should be sign independent, also should not depend on ϵ. And that is only possible when $\delta J(\eta, y) = 0 \ \ \forall \, \eta \in H$ (necessary condition for local max).

Otherwise, if the variation is sign dependent, we will not be able to find the extremal. Again, a result which follows in parallel to our finite dimensional calculus case. So, so far we have arrived at a point that to find the extremal we must have the first variation equal to 0. I call this as the necessary condition for finding the extremal, necessary condition for local max or local min. I am talking about max at this stage. Well, the similar argument holds for local min as well. And this is an analogous case of $\bar{\bigtriangledown} f = 0$, that we did in the finite dimensional calculus case.

So From the expression **II**, I want to change this second term using integration by parts so that even this term has a particular factor eta. And I want to pull out this perturbation function eta out of the integral and show some results, which is independent of this perturbation function eta.

so from **II**

$$\int_{x_o}^{x_1} \eta^{'} \frac{\partial f}{\partial y'} dx = \eta \frac{\partial f}{\partial y'}|_{X_o}^{x_1} - \int_{x_o}^{x_1} \eta \frac{d}{dx}\left[\frac{\partial f}{\partial y'}\right] dx \qquad \mathbf{II}^{'}$$

Now, we also know that the perturbation functions eta are such that they vanish at the boundary. So, which means that the first term is going to be 0. It vanishes at the boundary. And we arrive that this particular quantity is the negative of this quantity.

$$\textit{From } \mathbf{II} \textit{ and } \quad \mathbf{II}^{'} \delta J(\eta, y) = \int_{x_o}^{x_1} \eta \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right] dx = 0 \qquad \mathbf{III}$$

Now, intuitively all the students can see that, what we are trying to do is we are trying to figure out a condition for the existence of the extremal value y. And we have found an integral condition and from here what we will show that since this holds for all for any arbitrary η, this must be true for the case when, only when this particular integrand inside this bracket is 0. So, the integral being 0 leads to the fact that this quantity inside this bracket must be equal to 0. And that is our goal.

Define $\quad E[x_o, x_1] \rightarrow R$ by $\quad \mathrm{E} = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$

By **III**, $\quad \langle \eta, E \rangle = \int \eta E dx = 0.$

So, what now I am going to show from this point onwards that this integral being 0 will lead to the fact that E must be equal to 0. So, for that I need to show 2 small results. So, the idea before I move on further, let me just state the basic idea here. To show that this function E = 0, we will show by contradiction.

We will assume that, let us say E $\neq$ 0 although the integral is 0. And we will arrive at a particular contradiction, namely we will figure out an η, which is a perturbation function, which is non-zero and that perturbation leads to a contradiction. So, the goal is to find a perturbation, particular perturbation eta, which leads to proving of the result via contradiction.